

THEORY AND APPLICATION
OF MODAL CONTROL

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ABSTRACT

A theory of mode controllability and observability is developed for the class of time-invariant linear dynamical systems. These concepts are shown to be alternate characterizations of Kalman's concepts of (state) controllability and observability. With the aid of these properties conditions for the existence and uniqueness of modal controllers are established.

A general technique for modal control is developed. The property of modal decomposition allows a powerful recursive design procedure to be formulated which allows a control system to be designed in an evolutionary manner. This procedure is recursive in the sense that a selected number of modes are moved to desired locations at each iteration while the others remain fixed. State decomposition techniques are also presented which ease the computational requirements associated with the design of the controller. This technique is compared to other methods for shifting modes.

A method is given for reducing the number of states that are required by the control law. The design of an observer to reconstruct inaccessible states is shown to be the dual of the modal control problem.

The location of the modes is discussed with respect to

transient response, modelling, sensitivity, a quadratic performance index, and the constrained gain problem.

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LIST OF COMMON SYMBOLS

Matrices

| | |
|-----------------------------------|--|
| $A, \bar{A}, \hat{A}, J, \Lambda$ | System matrix in various representations. |
| C, \bar{C}, \hat{C}, P^T | Actuating matrix in various representations. |
| F, \bar{F}, \hat{F}, G | Feedback matrix in various representations. |
| H, \bar{H}, \hat{H}, W | Output matrix in various representations. |
| U | Nonsingular linear transformation that takes A to Jordan Canonical Form. |
| V | $[U^{-1}]^T$. |

Vectors

| | |
|-------|---|
| c_i | i^{th} column of C |
| f_i | i^{th} column of F |
| g_i | i^{th} column of G |
| m | Manipulated input (control) vector |
| p_i | i^{th} column of P |
| u_i | i^{th} column of U , eigenvector of A when Jordan Canonical Form of A is diagonal. |
| v_i | i^{th} column of V . |

Scalars

| | |
|----------|------------------------------------|
| j | $\sqrt{-1}$ |
| m_k | k^{th} component of m |
| p_{ki} | k^{th} component of p_i |

Scalars (cont'd)

| | |
|---------------|--|
| u_{ki} | k^{th} component of u_i |
| v_{ki} | k^{th} component of v_i |
| α_{ik} | $\langle p_i, g_k \rangle$ |
| γ_i | desired value for i^{th} eigenvalue |
| λ_i | i^{th} eigenvalue of A |

Notation

| | |
|-------------------|---|
| adj[·] | Adjoint (adjugate) of [·] |
| det[·] | Determinant of [·] |
| c.c. | Completely controllable |
| c.c.p. | Complex conjugate pair (of modes) |
| c.o. | Completely observable |
| c.s.c. | Completely (state) controllable |
| c.s.o. | Completely (state) observable |
| c.m.c. | Completely (mode) controllable |
| c.m.o.d. | Completely (mode) observable (restricted to systems with distinct modes) |
| Im[·] | Imaginary part of [·] |
| M.S.T. | The Mode Shifting Technique developed in this report |
| Re[·] | Real part of [·] |
| r.p. | Real pair (of modes) |
| s.v. | State variables |
| [·] ⁻¹ | Inverse of [·] |

| | |
|---|--|
| $[\cdot]^T$ | Transpose of $[\cdot]$ |
| $([\cdot])_k$ | The first k rows of $[\cdot]$ |
| $[\cdot]^k$ | $[\cdot][\cdot] \cdots [\cdot]$ (k times) |
| $[\cdot]^{(k)}$ | $[\cdot]$ after k iterations |
| $\dot{[\cdot]}$ | $\frac{d}{dt} [\cdot]$ |
| $[\cdot]^*$ | Complex conjugate of $[\cdot]$ |
| $[\cdot]'$ | Real part of $[\cdot]$ |
| $[\cdot]''$ | Imaginary part of $[\cdot]$ |
| $\langle [1], [2] \rangle$ | Scalar product of the vectors $[1]$ and $[2]$, $[1]^T [2] = [2]^T [1]$. |
| $P_i(\lambda_1, \lambda_2, \dots, \lambda_n)$ | The sum of the products, taken i at a time, of the elements from the set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. |
| $P_i(\lambda)$ | A shortened notation used when the set of λ 's under discussion is clear. $P_i(\lambda)$ is defined to be unity when $i = 0$, and zero when i exceeds the number of elements in the set of λ 's . |
| $P_i(\lambda \lambda_k)$ | $P_i(\lambda)$ with $\lambda_k = 0$. |
| $P_i(\lambda \lambda_k, \lambda_\ell)$ | $P_i(\lambda)$ with $\lambda_k = \lambda_\ell = 0$. |

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CHAPTER I

INTRODUCTION

A. Motivation

In the control of large multivariable plants problems arise from the dimensionality of the plant, and interactions among its terminal variables. To alleviate the problems which accompany these plants a theory of multilevel systems is proposed by Mesarovic [23] as a conceptual framework for the design of control systems.

The structure of a multilevel system is not unlike the pyramid-like organizational structure of a large corporation. Such a structure is useful because it permits large difficult problems to be decomposed into smaller easier to solve subproblems, the solutions of which are efficiently coordinated by a hierarchy of decision makers (controllers). A more detailed discussion of multilevel control can be found in Mesarovic et al. [24].

Keeping within the framework of multilevel theory Pearson [29,30] suggests the use of 'model filtering' and 'goal filtering' to aid in the design of multilevel control systems. These concepts can roughly be described in the following manner.

Each level isolates certain dominant aspects of the system by using control to suppress the remainder in such a way that the former constitute a fairly realistic model of the situation. If applied several times this is essentially a recursive simplification technique in which a successively simplified view of the actual system is communicated upward from level to level. Two means of control of the system are introduced in the design: A pre-defined model, which in essence defines the dominant attributes of the system relative to each level; and the goal of the local controller, which allows a trade-off between the amount of simplification achieved and the effort required to do so.

The system should be designed in a way which results in an optimum distribution of effort over the hierarchy. It is likely that the hierarchy will have a layer-like structure similar to the one proposed by Lefkowitz [20] consisting of the four layers: direct, optimizing, adaptive, and self-organizing control. Thus, it appears that the goal filtering should be down the hierarchy so that the most sophisticated part of the problem remains at the highest level where it is solved in relation to the most abstracted view of the system.

In attempting to develop a formal theory of design using the simplification concepts of model and goal filtering it becomes apparent that existing control techniques must be extensively

modified or new techniques developed. The classical approach while intuitively appealing is difficult to apply to large multi-variable systems. For well defined problems optimal control theory is conceptually simple to apply, but suffers from several drawbacks. One problem arises from the fact that in some situations it is difficult to specify a meaningful performance index that is analytically tractable. Another problem stems from the difficulty in manipulating the performance index to create desired changes in the response characteristics of the system or structure of the controller.

To gain insight into the control of large multivariable systems attention is focussed on the class of time-invariant linear dynamical systems. For this class of systems Rosenbrock [32] suggests the use of modal control as a design aid. Modal control can be defined as the changing of the system's modes (i.e. the eigenvalues of the system matrix) to help achieve desired control objectives.

It is believed that the technique of modal control will help fill the gap between the intuitively appealing classical control methods, and the powerful techniques of linear optimal control. This study is concerned with developing a theory of modal control that hopefully will provide a framework for future research in multilevel control. A statement of the problems considered in

this study and the major results obtained are presented in the following two sections. A summary of the contents of this report along with proposed extensions may be found in Chapter 9.

B. Problem Statement

Consider the time-invariant linear dynamical system represented by

$$\dot{x}(t) = Ax(t) + Cm(t)$$

$$y(t) = Hx(t)$$

where $x(t)$, $m(t)$, and $y(t)$ are the system state, control, and output vectors respectively. The real constant matrices A , C , and H are of dimension $(n \times n)$, $(n \times r)$, and $(e \times n)$ respectively. Let $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_n\}$ be the spectrum, i.e. the set of eigenvalues, of the system matrix A .

Let $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ be an arbitrary set of n complex numbers such that any γ_i with $\text{Im}(\gamma_i) \neq 0$ appears in Γ in a conjugate pair.

The basic problem, which has several parts, may be formulated in the following manner.

(i) Under what conditions does there exist a real feedback control (modal controller) $m = Fx$ such that the spectrum of the closed loop system matrix $[A + CF]$ has a spectrum given by the

set Γ .

(ii) If a modal controller exists is it unique?

(iii) Obtain an algorithm for specifying the modal controller, i.e. determine the required feedback matrix F . This method should be efficient and capable of taking advantage of any nice structural properties of the system under consideration.

(iv) Is measurement of the complete state vector necessary to achieve a desired distribution of modes? That is, can a modal control law $m = Fx$ be found in which F has a set of zero columns?

(v) Is modal control possible when the state variables are inaccessible? If so, how may it be accomplished?

In the framework of this basic problem statement it is also desired to find what the implications of the theory are to various aspects of controlling large time-invariant linear dynamical systems. In particular some of the aspects examined are:

(vi) The constrained gain problem.

(vii) Modelling.

(viii) Synthesis of transfer function matrices.

(ix) Specification of the spectrum Γ .

C. Nature of Results

The major contributions of this thesis are listed below to correspond to the problem statements above.

(i) The concepts of mode controllability and observability are introduced in Section (3.A). The major result related to the question of the existence of modal controllers is Theorem (3.4) which states: a time-invariant linear dynamical system S is completely (mode) controllable if and only if it is completely (state) controllable. This theorem is essential to the theory developed in later sections.

(ii) Section (3.B) is concerned with the uniqueness of modal controllers. It is shown that in general a unique modal controller exists only in the case of a single-input completely controllable system.

(iii) A general technique for modal control is developed in Chapters 4,5, and 6. Chapter 7 compares this technique with the state of the art. An efficient recursive mode shifting procedure is derived which allows an arbitrary number of (controllable) modes to be moved to desired locations at each iteration while the other modes remain fixed. It is shown how modal and state decomposition lead to a reduction in the effort required to determine the modal controller, i.e. the matrix F . The use of an

approximate method for decomposing weakly coupled systems is also discussed.

(iv) The measurement of the complete state vector is not always necessary for modal control. The number of required states to be measured may be reduced by up to $r-1$, where r is the number of components in the control vector. Proposition 5.85 states the conditions for the reduction of state measurements.

(v) When the states are inaccessible it is shown in Section (8.A) that an asymptotic state estimator (observer) may be employed to reconstruct the state vector. The introduction of the observer just has the effect of adding additional modes (those corresponding to the observer) to the overall system. If the plant is completely observable, then the problem of designing an observer having arbitrarily specified modes is a dual of the modal control problem for the plant.

(vi) A heuristic for shifting the modes of a system to alter the elements of the feedback matrix F is presented in Section (8.C.5).

(vii) The relation of the mode shifting technique developed in this thesis to a class of models based on a dominant mode approximation is investigated. An expression for the instantaneous error in the approximate state variables is derived along with suggestions for reducing it.

(viii) Use of the spectral characterization of systems enables the transfer function matrices of a special class of systems to be easily investigated (Section (3.C)). This class of systems has the property that mode shifting may be accomplished while the zeros of the associated transfer function matrix remain invariant. All single-input systems are members of this class.

(ix) In Section (8.3) guides are given for locating the modes with respect to stabilization, classical transient criteria, and sensitivity. The question of optimally locating the modes with respect to a quadratic performance index is also considered. It is shown that for a class of systems the optimal mode locations (which are obtained in a direct manner) lead to a unique modal controller, which is then optimal.

CHAPTER II

MATHEMATICAL DESCRIPTION OF THE SYSTEM

The system under consideration is the time-invariant linear dynamical system characterized by

$$\dot{x} = Ax + Cm \quad (2.1a)$$

$$y = Hx \quad (2.1b)$$

where $x = x(t)$, $m = m(t)$, and $y = y(t)$ are the state, manipulated (control) input, and output vectors respectively. The real constant matrices A , C , and H are of dimension $(n \times n)$, $(n \times r)$, and $(e \times n)$ respectively.

In the sequel it is convenient to also use the following representations.

Jordan Canonical Form

The Jordan canonical representation of (2.1) is obtained via the nonsingular transformation of state $x = Uz$.

$$\dot{z} = Jz + P^T m \quad , \quad (P^T = U^{-1}C) \quad (2.2a)$$

$$y = Wz \quad , \quad (W = HU) \quad (2.2b)$$

$$J = U^{-1}AU = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_v \end{bmatrix}, \quad J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix}$$

Distinct Diagonal Form

Since most of the sequel concentrates on systems with distinct eigenvalues a slightly different notation is reserved for this case.

$$\dot{z} = \Lambda z + P^T m, \quad (P^T = V^T C) \quad (2.3a)$$

$$y = Wz, \quad (W = HU) \quad (2.3b)$$

$$\Lambda = V^T A U = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}, \quad \lambda_i \neq \lambda_k$$

In this case the columns of U are the eigenvectors of A , and $V^T = U^{-1}$. Appendix A reviews the relevant properties of the u_i and v_i .

Controllable Form

Kalman [18] shows that a nonsingular transformation of state $x = N \bar{x}$ exists which takes (2.1) to the form

$$\dot{\bar{x}} = \bar{A} \bar{x} + \bar{C}m \quad , \quad (\bar{C} = N^{-1}C) \quad (2.4a)$$

$$y = \bar{H} \bar{x} \quad , \quad (\bar{H} = HN) \quad (2.4b)$$

$$\bar{A} = N^{-1}AN = \left[\begin{array}{c|c} \bar{A}_{11} & \bar{A}_{12} \\ \hline 0 & \bar{A}_{22} \end{array} \right] , \quad \bar{C} = \left[\begin{array}{c} \bar{C}_{11} \\ \hline 0 \end{array} \right] .$$

The dimensions of \bar{A}_{11} , \bar{A}_{12} , \bar{A}_{22} , and \bar{C}_{11} are $(n_c \times n_c)$, $(n_c \times n - n_c)$, $(n - n_c \times n - n_c)$, and $(n_c \times r)$ respectively.

In addition the rank of the $(n_c \times m_c)$ matrix $[\bar{C}_{11}, \bar{A}_{11} \bar{C}_{11}, \dots, \bar{A}^{n_c-1} \bar{C}_{11}]$ is equal to $n_c \leq n$.

Companion Matrix Form (r = 1)

If the representation (2.1) is completely controllable and $r = 1$, then a nonsingular transformation of state $x = T \hat{x}$ exists which takes it to the companion matrix (phase-variable) form

$$\dot{\hat{x}} = \hat{A} \hat{x} + \hat{c} m \quad , \quad (\hat{c} = T^{-1}C) \quad (2.5a)$$

$$y = \hat{H} \hat{x} \quad , \quad (\hat{H} = HT) \quad (2.5b)$$

$$\hat{A} = T^{-1}AT = \left[\begin{array}{cccc} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{array} \right] , \quad \hat{c} = \left[\begin{array}{c} 0 \\ \vdots \\ 0 \\ 1 \end{array} \right] .$$

Companion Matrix Form ($r \geq 1$)

Recently the companion matrix form has been generalized by Anderson and Luenberger [1] for completely controllable multi-input systems. Unfortunately the form is not unique if $r > 1$, but nevertheless a useful structure is obtained for each derived non-singular transformation of state $x = T \hat{x}$.

$$\dot{\hat{x}} = \hat{A} \hat{x} + \hat{C}m \quad , \quad (\hat{C} = T^{-1}C) \quad (2.6a)$$

$$y = \hat{H} \hat{x} \quad , \quad (\hat{H} = HT) \quad (2.6b)$$

$$\hat{A} = T^{-1}AT = \left[\begin{array}{c|ccc} \hat{A}_{\xi\xi} & & & \hat{A}_{\xi 1} \\ \hline \vdots & & & \vdots \\ \hline \hat{A}_{1\xi} & \dots & & \hat{A}_{11} \end{array} \right] , \quad \hat{C} = [\hat{c}_1 \vdots \hat{c}_2 \vdots \dots \vdots \hat{c}_r] .$$

Note the labeling of the submatrices \hat{A}_{ik} .

$\xi \leq r$

$$\hat{A}_{ik} = \begin{cases} \text{a zero matrix} & (i > k) \\ \text{a companion matrix} & (i = k) \\ \text{a matrix of zeros except possibly for the first} & \\ \text{column} & (i < k) \end{cases}$$

\hat{c}_i is of arbitrary form except when $\xi - i \geq 0$. In this case the component of \hat{c}_i in the row corresponding to the last row of

A_{ii} is unity, while the other components of \hat{c}_i are zero.

Transfer Function Matrix

With the aid of Laplace transform theory it is easy to show that the transfer function matrix $T(s)$ corresponding to system (2.1) is

$$T(s) = \begin{bmatrix} \frac{y_1}{m_1}(s) & \dots & \frac{y_1}{m_r}(s) \\ \vdots & & \vdots \\ \frac{y_e}{m_1}(s) & \dots & \frac{y_e}{m_r}(s) \end{bmatrix} = H[sI-A]^{-1}C = \frac{H[\text{adj}(sI-A)]C}{\det[sI-A]} \quad (2.7)$$

If the system (2.1) is not completely controllable and completely observable, then some of the zeros will cancel some of the modes to create an effective transfer function matrix with a number of poles less than the order of the system. Note that in this report the words "mode" and "eigenvalue" are used synonymously.

CHAPTER III

PROPERTIES OF THE SYSTEM

A. Characterization of Controllability and Observability

In order to consider modal control on a sound theoretical foundation it is necessary to examine the properties of controllability and observability for time-invariant linear dynamical systems from a modal point of view. The main result of this section is an alternate characterization of these fundamental concepts.

The familiar state dependent definitions of controllability and observability, which were introduced by Kalman [16], are first presented for comparison.

Definition 3.1 A state $x(o)$ is said to be controllable if there exists a control m defined over a finite interval $0 \leq t \leq T$ such that $x(T) = 0$. If every state $x(o)$ is controllable the system is said to be completely (state) controllable or c.s.c.

Definition 3.2 A state $x(o)$ is said to be observable if there exists a finite time $T > 0$ such that knowledge of the zero-input response defined over $0 \leq t \leq T$ is sufficient to determine $x(o)$. If every state $x(o)$ is observable the system is said to

be completely (state) observable or c.s.o.

A concept closely related to state controllability involves the controllability of the modes.

Definition 3.3 A system is said to be completely mode controllable, or c.m.c., if there exists a real control m which allows the modes of the system to be moved to an arbitrary set of complex numbers $\Gamma = \{\gamma_1, \gamma_2, \dots, \gamma_n\}$ such that any γ_i with $\text{Im}(\gamma_i) \neq 0$ appears in Γ in a conjugate pair.

The link connecting state and mode controllability is provided by the basic theorem.

Theorem 3.4 A time-invariant linear dynamical system S is c.m.c. if and only if it is c.s.c.

The proof of this theorem will proceed via several lemmas.

Lemma 3.5 A time-invariant linear dynamical system S is c.s.c. if and only if there exists no representation of S having $\dot{x}_\ell = \lambda x_\ell$ as one of its component state equations.

Proof: (\implies) The proof is by contradiction. Assume that S is c.s.c. and such a representation exists. Then x_ℓ , the ℓ^{th} component of x , is not controllable, and hence this representation of S is not c.s.c. But controllability is invariant under nonsingular linear state transformations. Therefore S itself is not c.s.c. contradicting the initial assumption.

(\Leftarrow) It is sufficient to prove that if S is not c.s.c., then a representation S of S exists having a component state equation $\dot{x}_\ell = \lambda x_\ell$. From Kalman's canonical structure theorem [18], S may be represented in the form given by (2.4), where n_c is the dimension of the controllable state space. Thus if S is not c.s.c. \bar{A}_{22} is an $n_u \times n_u$ matrix where $n_u \geq 1$.

Consider the nonsingular linear transformation given by $x = T \bar{x}$

$$T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}, \text{ where } T_{11} \text{ is a}$$

nonsingular matrix of appropriate order, and T_{22} is selected such that $T_{22}^{-1} \bar{A}_{22} T_{22} = J[\bar{A}_{22}]$, i.e. the Jordan canonical form of \bar{A}_{22} .

The new system representation is now given by,

$$S : \dot{x} = Ax + Cm$$

$$\text{where } A = \begin{bmatrix} T_{11}^{-1} \bar{A}_{11} T_{11} & T_{11}^{-1} \bar{A}_{12} T_{22} \\ 0 & J[\bar{A}_{22}] \end{bmatrix} \text{ and } C = \begin{bmatrix} T_{11}^{-1} \bar{C}_{11} \\ 0 \end{bmatrix}.$$

From inspection of S it is clear that this representation has at least one component state equation of the form $\dot{x}_\ell = \lambda x_\ell$. This

proves the lemma.

Comment: Note that λ is actually a mode of the system S .

For the next lemma it is convenient to consider the Jordan canonical form of the system S , given by

$$\dot{z} = Jz + P^T m \quad (3.6a)$$

$$y = Wz \quad (3.6b)$$

where $J = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_v \end{bmatrix}$ and $P^T = \begin{bmatrix} p_{1,1}^T \\ \vdots \\ p_{1,n_1}^T \\ \vdots \\ p_{v,1}^T \\ \vdots \\ p_{v,n_v}^T \end{bmatrix}$.

Note that the row vectors of P^T are numbered to correspond to the Jordan blocks. Each block J_i is of dimension n_i ($i = 1, 2, \dots, v$).

Lemma 3.7 The system represented by (3.6) is c.s.c. (c.s.o) if and only if all rows of P^T (columns of W) corresponding to the last row (first column) of Jordan blocks containing the same mode are linearly independent.

Proof: It is sufficient to prove the controllability part of the

lemma. The observability part follows immediately from Kalman's theory of duality.

(\implies) It will be shown that the existence of a set of dependent rows in P^{π_1} , corresponding to the last row of Jordan blocks containing the same mode, implies that the system is not c.s.c. For notational simplicity only, assume that $\lambda_1 = \lambda_i$ ($i = 1, 2, \dots, k \leq v$) and that the dependent rows in question correspond to Jordan blocks containing λ_1 . Then there exists a non-trivial set of scalars, δ_i ($i = 1, 2, \dots, k$), such that

$$\sum_{i=1}^k \delta_i p_{i, n_i} = 0.$$

Perform a nonsingular transformation of state $x = Nz$ such that one of the new state variables, say x_ℓ , is defined by the relation

$$x_\ell = \sum_{i=1}^k \delta_i z_i.$$

This new state variable satisfies the equation $\dot{x}_\ell = \lambda x_\ell$. As a consequence of Lemma 3.5 it follows that the system is not c.s.c.

(\impliedby) It is sufficient to show that a system which is not c.s.c. must have a set of p_{i, n_i} corresponding to the same mode which are linearly dependent. Assume that the system is not c.s.c. From Lemma 3.5 it follows that a representation of the system exists

having $\dot{x}_\ell = \lambda x_\ell$ as one of its component state equations, where λ is a mode of the system. For simplicity of notation let $\lambda = \lambda_1$, where the first k Jordan blocks are the only blocks containing the mode λ_1 . Then x_ℓ must be a non-trivial linear combination of the states corresponding to the first k Jordan blocks, i.e.

$$x_\ell = \sum_{i=1}^{n_1} \delta_{1,i} z_{1,i} + \dots + \sum_{i=1}^{n_k} \delta_{k,i} z_{k,i},$$

where at least one of the δ 's is different from zero. The state equation for x_ℓ is

$$\begin{aligned} \dot{x}_\ell = \lambda x_\ell + & \sum_{i=1}^{n_1-1} \delta_{1,i} z_{1,i+1} + \dots + \sum_{i=1}^{n_k-1} \delta_{k,i} z_{k,i+1} \\ & + \left[\sum_{i=1}^{n_1} \delta_{1,i} p_{1,i}^T + \dots + \sum_{i=1}^{n_k} \delta_{k,i} p_{k,i}^T \right] m. \end{aligned}$$

This implies that

$$\begin{aligned} \delta_{1,i} &= 0, \quad i = 1, 2, \dots, n_1-1, \\ &\vdots \\ \delta_{k,i} &= 0, \quad i = 1, 2, \dots, n_k-1. \end{aligned}$$

Hence

$$\sum_{i=1}^k \delta_{i, n_i} p_{i, n_i}^T = 0,$$

and the lemma is proved.

The corollaries below follow readily from Lemma 3.7.

Corollary 3.8 If the input is a scalar quantity, i.e. $r = 1$, then a necessary condition for the system to be c.s.c. (c.s.o.) is that no two Jordan blocks contain the same mode.

Corollary 3.9 The system with distinct modes is c.s.c. (c.s.o.) if and only if each row of P^T (column of W) is different from zero.

Kalman [18] shows that the number of controllable state variables in a time-invariant linear dynamical system S is the same for every representation of S . The effect of feedback on this number is expressed in the lemma below.

Lemma 3.10 The number of controllable state variables in a time-invariant linear dynamical system S remains invariant with respect to the application of linear feedback to S .

Proof Let S be an n^{th} order system with n_c controllable state variables (s.v.). A valid representation of this system is given by (2.4). Apply control in the form of

$$m = m_f + m_o \quad (3.11)$$

$$\text{where } m_f = [F_{11} : F_{12}] \bar{x}$$

$$m_o = \text{arbitrary}$$

It is obvious that the number of controllable s.v. of the resulting system

$$\dot{\bar{x}} = \left[\begin{array}{c|c} \bar{A}_{11} + \bar{C}_{11} F_{11} & \bar{A}_{12} + \bar{C}_{11} F_{12} \\ \hline 0 & \bar{A}_{22} \end{array} \right] \bar{x} + \left[\begin{array}{c} \bar{C}_{11} \\ 0 \end{array} \right] m_0 \quad (3.12)$$

does not increase. Assume that the number of controllable s.v. decreases to $n_c - k$, ($1 \leq k \leq n_c$). A nonsingular transformation $\bar{x} = Nx$ then exists which represents the system characterized in (3.12) as

$$\dot{x} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array} \right] x + \left[\begin{array}{c} C_{11} \\ 0 \end{array} \right] m_0 \quad (3.13)$$

where A_{11} and C_{11} are $(n_c - k \times n_c - k)$ and $(n_c - k \times r)$ respectively. Now let $m_0 = -F\bar{x}$ in (3.12) and its equivalent $m_0 = -FNx$ in (3.13). The representation resulting from (3.12) is the original representation (2.4) having n_c controllable s.v., while the representation resulting from (3.13) has at most $n_c - k$ controllable s.v. But this contradicts the fact that all representations of a system have the same number of controllable s.v. Therefore, the assumption that a linear feedback law exists which alters the number of controllable s.v. is false, and the lemma is proved.

The particular case of a completely controllable system, i.e. $n_c = n$, provides the following corollary to Lemma 3.10.

Corollary 3.14 If the pair (A, C) is c.s.c. and F is any (rxn) matrix, then the pair $(A + CF, C)$ is c.s.c.

Proof of Theorem 3.4: (\implies) The proof is by contradiction.

Assume that the system is c.m.c., but not c.s.c. Then as a consequence of Lemma 3.5 there exists a representation of the system which contains a state equation of the form $\dot{x}_\ell = \lambda x_\ell$, where λ is a mode of the system. Since this mode is uncoupled from the control it can not be altered. Therefore the system is not c.m.c. contradicting the initial assumption.

(\impliedby) This part of the proof proceeds in two steps. The approach taken is to first apply feedback to create a system with distinct modes, and then determine the additional feedback required to shift the distinct modes to the desired locations.

(i) The modes of the system are first moved to distinct locations. Since it is postulated that the given system S is c.s.c. there exists a representation for it in the canonical form of (2.6). The modes of S are the roots of the equation

$$\prod_{i=1}^{\xi} \det[sI - \hat{A}_{ii}] = 0 .$$

Therefore the modes of S can be changed by altering the elements of the \hat{A}_{ii} . Consider the matrix

$$\hat{A}_{kk} = \begin{bmatrix} 0 & & 1 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & 1 \\ -a_{k,n_k} & & -a_{k,n_k-1} & & & -a_{k,1} \end{bmatrix},$$

with its associated characteristic equation

$$s^{n_k} + a_{k,1} s^{n_k-1} + \dots + a_{k,n_k} = 0. \quad (3.15)$$

Partition the state vector to correspond to the diagonal blocks of \hat{A} as follows

$$\hat{x}^T = [\hat{x}_{\xi,n_\xi}, \dots, \hat{x}_{\xi,1}, \dots, \hat{x}_{1,n_1}, \dots, \hat{x}_{1,1}] = [\hat{x}^\xi \dots \hat{x}^1].$$

Let m_k , the k^{th} component, of m be given by

$$m_k = \sum_{i=1}^{n_k} (a_{k,i} - a_{k,i}^0) \hat{x}_{k,i},$$

and all of the other components of m be zero. Then the only diagonal block of \hat{A} changed is \hat{A}_{kk} , and its new characteristic equation is given by (3.15) with the $a_{k,i}$ replaced by $a_{k,i}^0$ ($i = 1, 2, \dots, n_k$). In a similar manner the characteristic equation of each diagonal block of \hat{A} could be changed to produce a system S^d with distinct modes. The only restriction on the mode locations of S^d obtained by employing real valued feedback is that each diagonal block, \hat{A}_{ii} , of odd order contain at least one real mode.

(ii) It is now shown that additional feedback may be determined to move the distinct modes of S^d to arbitrary locations. Since the initial system S is c.s.c. it follows from Corollary 3.14 that S^d is also c.s.c. System S^d has a representation of the form (2.3). As a consequence of Corollary 3.9 each row of P^T is a nonzero (r -dimensional) vector. The required feedback may be determined from (4.19) where the δ_k ($k = 1, 2, \dots, n$) are found from equation (4.27) with n_m replaced by n . It only remains to be shown that a vector g_0 can be chosen so that $\alpha_{k0} = \langle p_k, g_0 \rangle \neq 0$ for $k = 1, 2, \dots, n$.

The existence of such a vector, g_0 , is shown by the following construction. As an initial guess let the r -dimensional vector be $g_0 = (1, 1, \dots, 1)^T$. Calculate the values of the corresponding α_{k0} . Let Π be the set of all p_k such that $\alpha_{k0} = \langle p_k, g_0 \rangle = 0$. If Π is empty, then the initial g_0 is acceptable, otherwise proceed as follows. Let p_β be an element of Π . Since $p_\beta \neq 0$ it has a non-zero element, call it $p_{i\beta}$. Increment the i^{th} element of g_0 by $\epsilon > 0$ to form $g_0^{(1)}$. Then $\alpha_{\beta 0}^{(1)} = \langle p_\beta, g_0^{(1)} \rangle = \epsilon p_{i\beta} \neq 0$. Choose the value of ϵ so that the nonzero values of α_{k0} remain nonzero. This can always be done. First try an arbitrary $\epsilon > 0$, if this fails try 2ϵ , and so on. Since the number of vectors is finite an acceptable increment must be found before $(n-1)$ steps. Once an acceptable ϵ is selected the number of elements in Π is decreased by at least one to form $\Pi_1 \subset \Pi$. Continue the process in the same manner

by selecting an element of Π_1 . Since Π_1 has a finite number of elements a g_0 is eventually found. This proves the theorem.*

Although c.s.c. is both a necessary and a sufficient condition for c.m.c. it is possible to control a subset of the modes of a system without requiring that it be c.s.c.

Definition 3.16 A set Λ_v of $n_v \leq n$ modes is said to be controllable if there exists a real control m which allows the modes of Λ_v to be moved to an arbitrary set of complex numbers $\Gamma_v = \{\gamma_1, \gamma_2, \dots, \gamma_v\}$ such that any γ_i with $\text{Im}(\gamma_i) \neq 0$ appears in Γ_v in a conjugate pair, and furthermore every complex mode in the resulting system appears in a complex conjugate pair.

Comment: It follows from Definition 3.16 that every controllable set of modes must contain complex modes in complex conjugate pairs. A set of modes Λ_1 may be called controllable when they form a subset of a larger set of controllable modes Λ_2 . When used in this context the expression ' Λ_1 is a controllable set of modes' indicates that the modes in Λ_1 along with their complex conjugates can be moved to achieve an arbitrary specified distribution of modes (satisfying the complex conjugate pair

* It has been brought to the author's attention that the results of Theorem 3.4 have also recently been obtained by Wonham [40].

condition). This terminology is consistent with the fact that for real systems if $\lambda_i = \lambda_k^*$ is a member of a controllable set of modes, then so is λ_k .

Assume that a time-invariant system S has a representation given by (2.4), i.e.

$$\begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_u \end{bmatrix} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_u \end{bmatrix} + \begin{bmatrix} \bar{C}_{11} \\ 0 \end{bmatrix} m .$$

Since no control can exert any influence on $\dot{\bar{x}}_u$ there exist states that can not be taken to the origin of the state space in a finite amount of time. Therefore the controllable states of the system take the form $(\bar{x}_c : 0)$ in this representation. It can also be shown that the modes of the system are the roots of the equation.

$$\det[sI - \bar{A}_{11}] \det[sI - \bar{A}_{22}] = 0 .$$

By isolating the controllable part of the system S

$$\dot{\bar{x}}_c = \bar{A}_{11} \bar{x}_c + \bar{C}_{11} m$$

the following corollary to Theorem 3.4 can be formed.

Corollary 3.17 Given a time-invariant linear dynamical system S

(i) The modes associated with the controllable part of S are controllable.

(ii) The number of controllable modes of S is equal to the number of states in the controllable part of S , which is invariant under feedback.

(iii) The set of modes not associated with the controllable part of S remains invariant under feedback.

For systems with distinct modes the following theorem proves to be useful.

Theorem 3.18 A mode λ_i of a system with distinct eigenvalues represented by (2.3) is a member of a controllable set of modes if and only if $p_i \neq 0$.

Proof. The necessity of $p_i \neq 0$ is obvious from the representation (2.3) where $\dot{z}_i = \lambda_i z_i + \sum_{k=1}^r p_{ki} m_{ki}$. If $p_i = 0$ then the control vector can not influence the mode λ_i . Sufficiency follows from the mode shifting algorithms of the next two chapters that actually show how to move the mode when $p_i \neq 0$.

The question of the observability of the modes will be considered in detail only for the case of a system with distinct modes. For such systems the zero-input response can be represented as

$$y(t) = H \sum_{i=1}^n \langle v_i, x(0) \rangle e^{\lambda_i t} = W e^{At} z(0), \quad (3.19)$$

or more generally as

$$y(t) = \sum_{i=1}^n a_i e^{\lambda_i t} . \quad (3.20)$$

Based on the form of (3.20) modal observability is introduced by the restricted definition.

Definition 3.21 A mode λ_i of a system with distinct modes is said to be observable if a zero-input response exists which includes a term of the form $a_i e^{\lambda_i t}$ with $a_i \neq 0$. If every mode is observable the system is said to be completely (mode) observable with distinct modes or c.m.o.d.

Theorem 3.22 A mode λ_i of a system with distinct modes is observable if and only if $w_i \neq 0$.

Proof: Follows by inspection of (3.19).

Comment: Gilbert [11] has obtained results similar to those of Theorem (3.18) and Theorem (3.22).

It can be seen from Corollary (3.9) and Theorem (3.22) that for systems with distinct modes the property of c.s.o. is equivalent to that of c.m.o.d.

Theorems (3.18) and (3.22) have interesting interpretations when related to the conditions that they imply on the matrices C and H . Consider the expansions of the columns of C and the

rows of H .

$$k^{\text{th}} \text{ column of } C : c_k = \sum_{i=1}^n p_{ki} u_i , \quad u = 1, 2, \dots, r$$

$$\ell^{\text{th}} \text{ row of } H : h^\ell = \sum_{i=1}^n w_{\ell i} v_i , \quad \ell = 1, 2, \dots, e .$$

Corollary 3.23 A mode λ_i of a system with distinct modes is controllable (observable) if and only if there exists a nonzero $p_{ki}(w_{\ell i})$ for some $k(\ell)$.

Although the number of controllable modes remains invariant with respect to feedback the same can not be said about the number of observable modes. Gilbert [11] has shown that for time-invariant linear dynamical systems with distinct modes that

"If the composite system S is a feedback configuration with S_1 in the forward path and a static, i.e. algebraic, system S_2 in the return path then S is c.c. and c.o. if and only if S_1 is c.c. and c.o." .

The following example illustrates the loss in observability that may result when the feedback can not be expressed as a linear combination of the output variables.

Example 3.24 Consider the c.c. and c.o. system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m$$

$$y = [-3 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with corresponding transfer function $T_o(s) = \frac{s-3}{(s+1)(s+2)}$.

Let $m = m_o + m_f$ where $m_f = 8x_1 + 4x_2$ is derived from state variable feedback. Note that m_f violates the condition of being obtainable algebraically from the system output. Then the closed-loop system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} m_o$$

$$y = [-3 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with corresponding transfer function $T_c(s) = \frac{s-3}{(s-3)(s+2)}$.

After cancellation the effective closed-loop transfer function $T_c(s) = \frac{1}{s+2}$ does not indicate the presence of the unobservable unstable mode. This example illustrates the power of using state

variable feedback. As is well known from root locus theory, without dynamic compensation, a pole can only cancel a zero in the limit as the gain tends to infinity.

$$\text{Note also that with } U_c \doteq \begin{bmatrix} 1 & 1 \\ -2 & 3 \end{bmatrix}, \text{ then } V_c^T = \frac{1}{5} \begin{bmatrix} 3 & -1 \\ 2 & 1 \end{bmatrix},$$

$$C = -\frac{1}{5} u_1 + \frac{1}{5} u_2, \text{ and } H = -5 V_1^T$$

which checks with Corollary (3.23).

B. Existence and Uniqueness of Modal Controllers

The controllability of a set of modes guarantees the existence of a linear state variable feedback law which enables the modes to be arbitrarily changed. Armed with the characterization of controllability developed in the preceding section it is a simple matter to determine whether or not a given set of modes is controllable. For control purposes it is also important to know if the feedback law required to achieve a desired distribution of modes is unique, or if many feedback laws exist which allows for a greater flexibility in design. Results along this direction are presented below.

Theorem 3.25 If S is a c.c. single-input system, then any distribution of modes for S can be realized in a unique manner using linear state variable feedback.

Proof: Every c.c. single-input system has a unique representation in companion matrix form (2.5). Let the representation for an n^{th} order system S be described by (2.5). Assume that the desired modes are the roots of the equation

$$s^n + d_1 s^{n-1} + \dots + d_n = 0. \quad (3.26)$$

Then the only linear state variable feedback control law which yields the desired modes is

$$m = [a_n - d_n, \dots, a_1 - d_1] \hat{x} \hat{F} \hat{x}. \quad (3.27)$$

The assumption of c.c. in Theorem (3.25) is not only required to allow for control over all of the mode locations, but is a necessary condition for the uniqueness of the control law.

Consider the single-input system

$$\dot{\bar{x}} = \bar{A} \bar{x} + \bar{c}m \quad (3.28)$$

$$\text{where } \bar{A} = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, \quad \bar{c} = \begin{bmatrix} \bar{c}_{11} \\ 0 \end{bmatrix}$$

\bar{A}_{11} , \bar{A}_{12} , \bar{A}_{22} , \bar{c}_{11} are $(n_c \times n_c)$, $(n_c \times n - n_c)$, $(n - n_c \times n - n_c)$, and $(n_c \times 1)$ respectively, and the pair $[\bar{A}_{11}, \bar{c}_{11}]$ is c.c. This representation is perfectly general and may be obtained from any system with n_c controllable modes by a nonsingular linear transformation of state. Perform a nonsingular transformation $\bar{x} = Tx$ such that

$$T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}$$

$$\text{where } T_{11}^{-1} \bar{A}_{11} T_{11} = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_{n_c} & \dots & \dots & -a_1 \end{bmatrix}, \quad T_{11}^{-1} \bar{c}_{11} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

and T_{22} is an arbitrary nonsingular matrix. Performing this change of coordinates on system (3.28) yields the representation

$$\begin{bmatrix} \dot{x}_c \\ \dot{x}_u \end{bmatrix} = \begin{bmatrix} T_{11}^{-1} \bar{A}_{11} T_{11} & T_{11}^{-1} \bar{A}_{12} T_{22} \\ 0 & T_{22}^{-1} \bar{A}_{22} T_{22} \end{bmatrix} \begin{bmatrix} x_c \\ x_u \end{bmatrix} + \begin{bmatrix} T_{11}^{-1} \bar{c}_{11} \\ 0 \end{bmatrix} m \quad (3.29)$$

From this form it can be observed by inspection that the modes of the system are composed of an invariant set corresponding to the modes of \bar{A}_{22} and the controllable modes found from the equation $s^{n_c} + a_1 s^{n_c-1} + \dots + a_{n_c} = 0$. In order to move the controllable modes to their desired locations a linear state variable feedback control law of the form

$$m = [F_{11} : F_{12}]x = [F_{11} T_{11}^{-1} : F_{12} T_{22}^{-1}] \bar{x} \quad (3.30)$$

can be used where

$$F_{11} = [a_{n_c} - d_{n_c}, \dots, a_1 - d_1] \text{ is } (1 \times n_c), \text{ and}$$

$$F_{12} \text{ is an arbitrary } (n - n_c)\text{-dimensional row vector.}$$

Because of the arbitrariness of both F_{12} and T_{22} obviously many control laws exist which do the job. The results of the above analysis are summarized in the

Theorem 3.31 Given a single-input system that is not c.c. any distribution of the controllable modes can be realized in many ways using linear state variable feedback.

$$\dot{x} = A_1 x + c m_0, \quad (A_1 = A + cF) .$$

Only the elements in row n_c can differ in the A and A_1 matrices. Therefore the closed-loop transfer matrix takes the form

$$T_1(s) = \frac{H[\text{adj}(sI-A_1)]c}{\det[sI-A_1]} = \frac{H[A_{n_c 1}(s), A_{n_c 2}(s), \dots, A_{n_c n}(s)]^T}{\det[sI-A_1]} \quad (3.34)$$

Comparison of (3.33) to (3.34) clearly indicates that the zeros remain invariant while only the modes may change.

To summarize: Controllable modes of a single-input system may be moved to arbitrary locations while the zeros remain invariant. The control necessary to achieve this relocation is unique if and only if the system is c.c.

For multi-input, ($r \geq 2$), systems c.c. is no longer a sufficient condition for the uniqueness of control or invariance of zeros as illustrated in the example below. However, a class of systems does exist in which arbitrary modes can be created while the zeros remain invariant. Such a class of systems is considered in Section (3.C).

Example 3.35

Consider the c.c. and c.o. system

$$\dot{x} = Ax + Cm$$

$$y = Hx$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad H = [1 \quad 1 \quad 1]$$

$$\text{with } T(s) = \frac{1}{s(s+0.5)(s+2)} [0.125(s+1.256)(s+12.744) \quad (s+2)].$$

The effects of two different control laws which yield the same distribution of modes will be illustrated. Let $m = m_o + m_k$ (where $m_k = F_k x$) be used to produce a system

$$\dot{x} = A_k x + C m_o$$

$$y = Hx$$

with corresponding transfer function matrix $y(s) = T_k(s)m_o(s)$

$$\text{control 1 : } F_1 = \begin{bmatrix} -6 & -3 & -7 \\ 6 & 3 & 6.5 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 1 & 0 \\ -6 & -5 & -7 \\ 0 & 0 & -1 \end{bmatrix}$$

$$T_1(s) = \frac{1}{(s+1)(s+2)(s+3)} [2s^2 \quad (s+.414)(s-2.414)]$$

$$\text{control 2 : } F_2 = \begin{bmatrix} -2 & -1 & 0 \\ 2 & 1 & -2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 \\ -2 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$T_2(s) = \frac{1}{(s+1)(s+2)(s+3)} [(s+1)(2s+5) \quad (s+1)(s+2)]$$

Note that the factor $(s+1)$ cancels in all terms making the mode "-1" unobservable. Thus the effective transfer function is

$$T_2(s) = \frac{1}{(s+2)(s+3)} [(2s+5) \quad (s+2)] .$$

C. Special Case: A Class of Systems with Invariant Zeros

A class of systems is examined in which the zeros of the transfer function matrix elements remain invariant with respect to a form of state feedback. In addition to providing an interesting application of modal theory this class of systems yields insight into the design of transfer function matrices of large systems.

The transfer function corresponding to (2.1) is given in (2.7) as

$$T_x(s) = H[sI-A]^{-1}C . \quad (3.36)$$

Let $x = N \hat{x}$ be any nonsingular transformation of state.

Then the system representation (2.1) becomes

$$\dot{\hat{x}} = \hat{A} \hat{x} + \hat{C}m \quad (\hat{A} = N^{-1}AN , \quad \hat{C} = N^{-1}C) \quad (3.37a)$$

$$y = \hat{H} \hat{x} \quad (\hat{H} = HN) . \quad (3.37b)$$

The transfer function corresponding to (3.37) is

$$T_{\hat{x}}(s) = \hat{H}[sI-\hat{A}]^{-1}\hat{C} . \quad (3.38)$$

Expressing the new matrices in terms of the old, and simplifying yields

$$\begin{aligned}
 T_x(s) &= HN[sI - N^{-1}AN]^{-1} N^{-1}C \\
 &= H[sI - A]^{-1}C \\
 &= T_x(s) \quad .
 \end{aligned}
 \tag{3.39}$$

This well known result just verifies the fact that the transfer function, which relates the input-output pairs of the system, is independent of the state parameterization used to define a particular representation. Therefore in order to arrive at results related to transfer functions any convenient system representation may be employed.

Let the free part of (2.1), $\dot{x} = Ax$, describe the physical model of a c.c. system having distinct modes. Usually the matrix H , which defines the outputs of the system is fixed, but in some applications it may not be. More freedom is available in the selection of the matrix C . Each column of C may be considered as an actuating vector for a different component of control.

As a result of the assumption on the modes of A it is possible to expand each column of C as a linear combination of the eigenvectors of A . Let

$$c_k = \sum_{i=1}^n p_{ki} u_i, \quad k = 1, 2, \dots, r. \tag{3.40}$$

The p -parameters can be thought of as influence coefficients. A nonzero value of p_{ki} indicates that the k^{th} actuating vector c_k exerts influence on the i^{th} mode of A . A measure of the effectiveness of this influence can be obtained by proper normalization, but this is not relevant to the present discussion.

Define the set of modes capable of being influenced by c_k as $\{\lambda^{(k)}\}$. With the aid of this notation the class of systems under consideration may be described as those systems satisfying the conditions

$$\bigcup_{k=1}^r \{\lambda^{(k)}\} = \{\lambda_1, \lambda_2, \dots, \lambda_n\}, \text{ and} \quad (3.41a)$$

$$\{\lambda^{(i)}\} \cap \{\lambda^{(k)}\} = \emptyset, \text{ for } i \neq k. \quad (3.41b)$$

The first condition guarantees that the system is completely controllable, while the second indicates that each actuating vector influences a different set of modes. This class of systems will be referred to as 'systems with disjoint control.' Note that all c.c. single-input systems are members of this class.

The class of systems with disjoint control has a very simple representation in terms of the canonical form of Anderson and Luenberger [1] given by

$$\dot{\hat{x}} = \hat{A} \hat{x} + \hat{C}m \quad (3.42)$$

$$\hat{A} = \begin{bmatrix} \hat{A}_r & & & \\ & \hat{A}_{r-1} & & \\ & & \ddots & \\ & & & \hat{A}_1 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} | & | & | & | & | \\ 0 & \dots & 0 & & 0 \\ \hline & & & & \\ 0 & \dots & 1 & & \\ \hline & & & & \\ 0 & & & & \\ \vdots & & & & \\ 0 & \dots & 0 & & 0 \\ \hline 1 & & & & \end{bmatrix}.$$

Each \hat{A}_k ($k = 1, 2, \dots, r$) is a companion matrix corresponding to the modes influenced by c_k . This class of systems has a unique representation except for the ordering of the \hat{A}_k .

Analogous results could be obtained using the representation of (2.3) which for these systems is

$$\dot{z} = \Lambda z + P^T m \quad (3.43)$$

where

$$\Lambda = \begin{bmatrix} \Lambda_r & & & \\ & \Lambda_{r-1} & & \\ & & \ddots & \\ & & & \Lambda_1 \end{bmatrix}, \quad P^T = \begin{bmatrix} | & | & | & | & | \\ & & & & 1 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & & 1 \\ \hline & & & & \\ 0 & \dots & 1 & & 0 \\ \hline & & & & \\ 1 & & & & \\ \vdots & & & & \\ 1 & \dots & 0 & & 0 \end{bmatrix}.$$

Each Λ_k ($k = 1, 2, \dots, r$) is a diagonal matrix containing the modes influenced by c_k . Note that Rosenbrock's ideal case of Section (7.A) is a special case of a system with disjoint control.

Every actuating vector of system (7.1) influences only one mode. Therefore, the system is uncontrollable if $r < n$.

For ease of presentation the representation of (3.42) is used throughout the rest of this section. A typical submatrix, \hat{A}_k , of \hat{A} is described by

$$\hat{A}_k = \begin{bmatrix} 0 & 1 & 0 \\ \vdots & \vdots & \vdots \\ -a_{k,n_k} & -a_{k,n_k-1} & \dots & -a_{k,1} \end{bmatrix}, \quad (3.44)$$

with corresponding characteristic equation

$$s^{n_k} + a_{k,1} s^{n_k-1} + \dots + a_{k,n_k} = 0. \quad (3.45)$$

Since the system is completely controllable it follows that

$\sum_{k=1}^r n_k = n$. The characteristic equation of \hat{A} is given by

$$\det[sI - \hat{A}] = \prod_{k=1}^r \det[sI - \hat{A}_k] = 0. \quad (3.46)$$

Every mode can be altered by changing the characteristic equation of the companion matrix it is associated with. For example to change the dynamics corresponding to \hat{A}_k let the k^{th} component of control be

$$m_k = [m_{k,1} \quad \vdots \quad m_{k,2} \quad \vdots \quad \dots \quad \vdots \quad m_{k,r}] \hat{x} \quad (3.47)$$

$$\text{where } m_{k,i} = \begin{cases} 0 & (i \neq k) \\ [a_{k,n_k} - a_{k,n_k}^{\circ}, \dots, a_{k,1} - a_{k,1}^{\circ}] & (i = k) . \end{cases}$$

If all of the modes are changed in this manner, the resulting system, \hat{A}° , is still of the form (3.42) except that each \hat{A}_k is replaced by \hat{A}_k° . That is, only the elements in the last row of each A_k are changed by the shifting of the modes.

In order to find the transfer function matrix of the system it is necessary to find the inverse of $[sI - \hat{A}]$. Since this matrix has a block diagonal structure its inverse takes a simple form, which is also block diagonal.

$$[sI - \hat{A}]^{-1} = \begin{bmatrix} [sI - \hat{A}_r]^{-1} & & & \\ & [sI - \hat{A}_{r-1}]^{-1} & & \\ & & \ddots & \\ & & & [sI - \hat{A}_1]^{-1} \end{bmatrix} . \quad (3.48)$$

A typical block of $[sI - \hat{A}_k]^{-1}$, of $[sI - \hat{A}]^{-1}$ is of the form

$$[sI - \hat{A}_k]^{-1} = \frac{1}{\det[sI - \hat{A}_k]} \begin{bmatrix} \diagdown & & & & 1 \\ & \diagup & & & s \\ & & \ddots & & \vdots \\ & & & \diagdown & s \\ & & & & s^{n_k-1} \end{bmatrix} . \quad (3.49)$$

Only the last column proves to be of importance, so the rest of the matrix is not described here.

The transfer function matrix may now be expressed as

$$T(s) = \hat{H}[sI - \hat{A}]^{-1}\hat{C} = \hat{H} \begin{bmatrix} & & & 1 \\ & 0 & & \vdots \\ & & \dots & s^{n_r-1} \\ \hline \vdots & & & \\ & 1 & & \\ & \vdots & & \\ 0 & & \dots & 0 \\ & s^{n_2-1} & & \\ \hline 1 & & & \\ \vdots & & & \\ & 0 & \dots & 0 \\ s^{n_1-1} & & & \end{bmatrix} \quad (3.50)$$

Let the i^{th} row of \hat{H} be given as

$$\hat{h}^i = [\hat{h}_{r,0}^i, \dots, \hat{h}_{r,n_r-1}^i \vdots \vdots \hat{h}_{2,0}^i, \dots, \hat{h}_{2,n_2-1}^i \vdots \hat{h}_{1,0}^i, \dots, \hat{h}_{1,n_1-1}^i] \quad (3.51)$$

where the partitions are introduced to correspond to the horizontal partitions of $[sI - \hat{A}]^{-1}\hat{C}$. Then a typical element of the transfer function matrix relating the k^{th} input to the i^{th} output is

$$\frac{y_i(s)}{m_k(s)} = \frac{\hat{h}_{k,0}^i + s\hat{h}_{k,1}^i + \dots + s^{n_k-1}\hat{h}_{k,n_k-1}^i}{\det[sI - \hat{A}_k]} \quad (3.52)$$

Note that the zeros of this typical element of the transfer function matrix depend only on \hat{H} which is fixed by the system. Recall that $\det[sI - \hat{A}_k]$ is only a function of the last row of elements of \hat{A}_k (see (3.45)). Therefore, feedback of the type described in (3.47) may be used to create arbitrary modes while leaving the zeros of the system invariant.

To summarize: A typical element of the transfer function of a system with disjoint control is described by (3.52). The order of the denominator is fixed by the number of modes the k^{th} actuator vector can influence. These modes can be arbitrarily placed using the proper state feedback, while the zeros of the system remain invariant. It is possible to cancel zeros with modes, but they can not be moved by feedback of the type described in (3.47). If a different form of feedback is employed the zeros will change because the system matrix will lose its block diagonal structure.

In systems of this type a great deal of freedom exists in the transfer function matrices that can be synthesized. It should be noted that although the results of this section are derived through the use of the canonical form of Anderson and Luenberger the actual design may be carried out more easily by utilizing the mode shifting technique developed in the next chapter.

CHAPTER IV

A TECHNIQUE FOR MODAL CONTROL

A. Introduction

An important factor in determining the response characteristics of system (2.1) is the distribution of its modes. The complete response, of course, is a function of A , C , and H . If C is non-singular then any arbitrary system matrix can be obtained with linear state variable feedback. However, in the general case, ($r < n$), there are less degrees of freedom available for shaping the response. In many applications it is not necessary to achieve some prespecified response as for example in the stabilization of a system, or in speeding up a sluggish response. When this is the case a satisfactory design may be obtained by simply moving a small number of modes.

Along with an idea of what constitutes an acceptable response the design objectives sometimes include constraints on the required feedback gains. These constrained gain problems arise from power constraints, the desire to eliminate the need for certain measurements, or the fact that when certain values of gain are exceeded the linear model breaks down. Very often the response characteristics are flexible, and the real usefulness of a design

depends on how well the implementation constraints are satisfied. Yet in most of the control literature much has been done in developing optimal control theory with very little attention paid to the problems mentioned.

Actually the optimum regulator problem is concerned with limiting large excursions of error and control effort. There is no known algorithm available for altering the weighting matrices to adjust the resulting feedback gains. In practice the designer must either approximate the optimal design with hardware that meets the given specifications or redesign the system by performing a tedious numerical search in gain space. It is not obvious that for either procedure the resulting system is "optimal". Therefore, in effect, these systems are designed by trial and error with optimal control theory merely supplying the initial guess.

In the sequel a technique is developed which provides the designer with a powerful new tool to help overcome some of the problems associated with the design of large multivariable systems. The technique is based on controlling the system's modes. It is shown that the amount of design effort required is in some sense proportional to the number of modes moved. Therefore very little computational effort is expended for designing systems with simple objectives. Another attractive feature of the technique is that it gives some choice over the feedback gains required to achieve the desired distribution of modes so that other objectives can be

met.

It is assumed that an accurate model of the system is available, derived either from first principles or actual testing of the system. This physical representation of the system is taken to be characterized by (2.1). Therefore, any implementation constraints are imposed in the state space associated with this representation. Except for the development in Chapter 6 it is also assumed that the system has distinct modes, which implies that (2.3) is another valid representation for the system. All results obtained through the simpler canonical form of (2.3) are related to the physical model via the unique nonsingular transformation of state relating the two representations.

B. The Modal Decomposition Property

Before proceeding with the actual derivation of the mode shifting algorithms an important property of the system matrix is discussed. This property plays a vital role in simplifying both the conceptual and computational aspects of modal control. In order to streamline the development of the modal decomposition property several powerful results from the theory of determinants have to be utilized. These and other useful properties of determinants are listed in Appendix B.

Having assumed distinct modes, the description in (2.3) is a valid representation of the system. Expressed in terms of the

canonical state variables $z \doteq V^T x$ this representation is

$$\dot{z} = \Lambda z + P^T m, \quad \Lambda = \text{diagonal} [\lambda_1, \lambda_2, \dots, \lambda_n]. \quad (4.1)$$

The modes of the system are the roots of the equation

$$\det[sI - \Lambda] = 0. \quad (4.2)$$

For this representation it is a trivial operation to find the modes because (4.2) can be simplified to the decomposed form

$$\prod_{i=1}^n (s - \lambda_i) = 0. \quad (4.3)$$

Since the decomposition is complete, i.e. the equation factors into n linear terms, it can be seen by inspection that the modes are just the diagonal entries of Λ .

If feedback is introduced into (4.1) in the form of

$$m = Gz, \quad (G \text{ is an } rxn \text{ constant matrix}) \quad (4.4)$$

then the modes of the closed-loop system must be determined from the equation

$$\det[sI - \bar{\Lambda}] = 0, \quad \text{where } \bar{\Lambda} \doteq \Lambda + P^T G. \quad (4.5)$$

In general the modes of the closed-loop system are not the diagonal entries of the matrix $\bar{\Lambda}$. However, by judicious choice of the matrix G the expression (4.5) can be considerably simplified.

The modal decomposition property which enables (4.5) to be simplified is best introduced through a specific example. Consider the representation (4.1) with $n = 5$. Let the feedback law be given by

$$m = g_1 z_1 + g_3 z_3 + g_4 z_4 \quad (4.6)$$

where $g_1, g_2,$ and g_4 are r -dimensional vectors. Then $\bar{\Lambda}$ is

$$\bar{\Lambda} = \begin{bmatrix} \lambda_1 + \alpha_{11} & 0 & \alpha_{13} & \alpha_{14} & 0 \\ \alpha_{21} & \lambda_2 & \alpha_{23} & \alpha_{24} & 0 \\ \alpha_{31} & 0 & \lambda_3 + \alpha_{33} & \alpha_{34} & 0 \\ \alpha_{41} & 0 & \alpha_{43} & \lambda_4 + \alpha_{44} & 0 \\ \alpha_{51} & 0 & \alpha_{53} & \alpha_{54} & \lambda_5 \end{bmatrix} \quad (4.7)$$

where $\alpha_{ik} = \langle p_i, g_k \rangle$.

Careful use of (B.2) enables the characteristic equation of $\bar{\Lambda}$ to be written in the form

$$\det[sI - \bar{A}] = \det \begin{bmatrix} s - \lambda_1 - \alpha_{11} & -\alpha_{13} & -\alpha_{14} & 0 & 0 \\ -\alpha_{31} & s - \lambda_3 - \alpha_{33} & -\alpha_{34} & 0 & 0 \\ -\alpha_{41} & -\alpha_{43} & s - \lambda_4 - \alpha_{44} & 0 & 0 \\ -\alpha_{21} & -\alpha_{23} & -\alpha_{24} & s - \lambda_2 & 0 \\ -\alpha_{51} & -\alpha_{53} & -\alpha_{54} & 0 & s - \lambda_5 \end{bmatrix} = 0. \quad (4.8)$$

A further simplification is obtained by utilizing properties (B.1) and (B.6) to give the decomposed form of (4.8)

$$\det[sI - \bar{A}] = (s - \lambda_2)(s - \lambda_5) \det \begin{bmatrix} s - \lambda_1 - \alpha_{11} & -\alpha_{13} & -\alpha_{14} \\ -\alpha_{31} & s - \lambda_3 - \alpha_{33} & -\alpha_{34} \\ -\alpha_{41} & -\alpha_{43} & s - \lambda_4 - \alpha_{44} \end{bmatrix} = 0. \quad (4.9)$$

The features of the modal decomposition property can easily be extrapolated from this example. For the general case in which $n_f \leq n$ canonical state variables are used in the feedback law it is found that

- (i) The characteristic equation of the closed-loop system factors into $(n - n_f)$ linear terms and the determinant of an n_f -th order system.

- (ii) The linear terms are composed of modes unchanged by feedback because their corresponding canonical state variables are not feedback.
- (iii) The n_f -th order determinant is the determinant of the submatrix formed by crossing out the rows and columns in $[sI-\bar{A}]$ corresponding to the canonical state variables omitted in the feedback law.

If any of the canonical state variables used in the feedback law correspond to uncontrollable modes, then a further simplification is possible. For example in the illustration above if λ_1 , is not controllable, then $p_1 = 0$ which implies that $\alpha_{1k} = 0$, $k = 1, 2, \dots, 5$. In this case (4.9) reduces to

$$\det[sI-\bar{A}] = (s-\lambda_1)(s-\lambda_2)(s-\lambda_5) \det \begin{bmatrix} s-\lambda_3-\alpha_{33} & -\alpha_{34} \\ -\alpha_{43} & s-\lambda_4-\alpha_{44} \end{bmatrix} = 0 \quad (4.10)$$

C. The Basic Technique

Assume that the given physical system representation (2.1) has distinct modes. For ease of derivation the representation (2.3) is used. The general results expressed in terms of the canonical state variables are related to the original state variables x through the nonsingular linear transformation $x = Uz$, where the columns of U are the eigenvectors of A . Only (2.3a) need be

considered to arrive at the mode shifting algorithm. This part of the representation is given explicitly by

$$\dot{z} = \Lambda z + P^T m, \quad \Lambda = \text{diagonal} [\lambda_1, \lambda_2, \dots, \lambda_n] \quad (4.11)$$

where for notational convenience it is assumed that the first $n_m \leq n$ modes are the controllable modes to be shifted.

As a consequence of the modal decomposition property it is known that the first n_m canonical state variables must go into making up the feedback law if their corresponding modes are to be changed. Therefore, let

$$m = \sum_{k=1}^{n_m} g_k z_k + m_o = m_f + m_o \quad (4.12)$$

where the g_k , $k = 1, 2, \dots, n_m$, are r -dimensional vectors to be determined.

When control (4.12) is substituted into (4.11) the result is the closed-loop system representation

$$\dot{z} = \bar{\Lambda} z + P^T m_o \quad (4.13)$$

where

$$\bar{\Lambda} = \left[\begin{array}{c|c} \bar{\Lambda}_{11} & 0 \\ \hline \bar{\Lambda}_{21} & \bar{\Lambda}_{22} \end{array} \right] = \left[\begin{array}{cccc|ccc} \lambda_1 + \alpha_{11} & \alpha_{12} & \dots & \alpha_{1n_m} & & & \\ \alpha_{21} & \lambda_2 + \alpha_{22} & \dots & \alpha_{2n_m} & & & \\ \vdots & \vdots & & \vdots & & & \\ \vdots & \vdots & & \vdots & & & \\ \alpha_{n_m 1} & \alpha_{n_m 2} & \dots & \lambda_{n_m} + \alpha_{n_m n_m} & & & \\ \hline \alpha_{n_m+1,1} & \alpha_{n_m+1,2} & \dots & \alpha_{n_m+1,n_m} & \lambda_{n_m+1} & 0 & \\ \vdots & \vdots & & \vdots & 0 & \ddots & \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{n,n_m} & & & \lambda_n \end{array} \right]$$

It follows from the modal decomposition property that the modes of the closed-loop system are the roots of the characteristic equation

$$\det[sI - \bar{\Lambda}] = (s - \lambda_{n_m+1}) \dots (s - \lambda_n) \det[sI - \bar{\Lambda}_{11}] = 0. \quad (4.14)$$

Thus the modes of the closed-loop system are composed of the modes of $\bar{\Lambda}_{11}$, and the last $(n - n_m)$ modes of the open-loop system. Because of the decomposition property it is only necessary to consider the $n_m \times n_m$ matrix $\bar{\Lambda}_{11}$ to determine the effect of the feedback on the modes. Recall that in general the submatrix containing all of the required information is obtained by crossing out all of the rows and columns of the closed-loop system matrix associated with the fixed modes.

A set of r -dimensional vectors g_k , ($k=1,2,\dots,n_m$), causing the matrix $\bar{\Lambda}_{11}$ of (4.13) to have the required distribution of modes must now be determined. Usually many such sets exist. A general procedure for obtaining the g_k is to first find the characteristic equation of $\bar{\Lambda}_{11}$

$$\det[sI - \bar{\Lambda}_{11}] = s^{n_m} + f_1 s^{n_m-1} + \dots + f_{n_m-1} s + f_{n_m} = 0 \quad (4.15)$$

where the f_i , $i=1,2,\dots,n_m$, are nonlinear (linear if $r=1$) functions of the components of the g_k , and then compare it to the characteristic equation whose roots are the desired modes $\gamma_1, \gamma_2, \dots, \gamma_{n_m}$

$$(s-\gamma_1)(s-\gamma_2)\dots(s-\gamma_{n_m}) = s^{n_m} + d_1 s^{n_m-1} + \dots + d_{n_m-1} s + d_{n_m} = 0 \quad (4.16)$$

The condition that the characteristic equations of (4.15) and (4.16) have identical roots is that coefficients of like powers of s be equal, i.e.

$$f_i = d_i, \quad i=1,2,\dots,n_m \quad (4.17)$$

The simplicity of expression (4.17) is deceiving because for $r > 1$ it usually represents a set of n_m nonlinear equations in the $n_m \cdot r$ unknown components of the g_k . Every solution to (4.17)

represents a different linear state variable feedback control law which yields the desired distribution of modes. If the number of modes being moved, n_m , is equal to the number of controllable modes, n_c , then every system matrix obtainable by linear feedback corresponds to a solution of (4.17) with $n_m = n_c$.

Canonical state variables associated with uncontrollable modes of the system may be feedback without altering the system matrix. For example, if the n^{th} mode is uncontrollable, then (4.12) could be changed to

$$m = \sum_{k=1}^{n_m} g_k z_k + g_n z_n + m_0 \quad (4.18)$$

where g_n is arbitrary.

This implies that the control could be changed by an amount $g_n z_n$ (i.e., $g_n < v_n, x >$ in the state space of (2.1)) without affecting the modes of the system.

Although it is possible to develop algorithms to obtain solutions to (4.17) for the general case of arbitrary values of r and n_m only a special case in which r is effectively equal to unity is treated in this chapter. A powerful recursive technique is developed in the next chapter which enables a small number of modes to be changed at each step, thus eliminating the need to work with a large underdetermined set of equations.

D. Special Case - Fixed Ratio of Control Elements

A special case develops when the elements of the control vector m are restricted to be multiples of one another. For this class of controllers the expression (4.17) represents a set of n_m linear simultaneous equations in n_m unknowns, which can easily be solved. Analytically this class of feedback controllers takes the form

$$m_F = g_o \sum_{k=1}^{n_m} \delta_k z_k = g_o \sum_{k=1}^{n_m} \delta_k v_k, x > . \quad (4.19)$$

where the elements of the r -dimensional vector g_o , $(g_{10}, g_{20}, \dots, g_{r0})$, fix the ratio of control elements, and the scalar weights δ_k are determined to achieve the desired changes in the modes. Scalar controllers for single-input systems form a subset of this class with g_o defined to be unity.

Actually this class of controllers reduce an r -input system to an effective single-input system. With the feedback control restricted as in (4.19) the r -input representation of (2.1) can be replaced by

$$\dot{x} = Ax + \tilde{c} \tilde{m}_F, \quad (4.20)$$

where the column vector $\tilde{c} = \sum_{i=1}^r g_{i0} c_i$, and the effective scalar control $\tilde{m}_F = \sum_{k=1}^{n_m} \delta_k \langle v_k, x \rangle$.

The only restriction on the elements of g_0 is that they be chosen to preserve the controllability of the modes to be changed.

Proposition 4.21 The distinct modes $\lambda_1, \lambda_2, \dots, \lambda_{n_m}$ are controllable in the effective system (4.20) if and only if $\alpha_{i0} = \langle p_i, g_0 \rangle \neq 0$, $i = 1, 2, \dots, n_m$.

Proof: Consider the diagonal canonical representation of (4.20).

$$\dot{z} = \Lambda z + \tilde{P}^T m_f \quad (4.22)$$

where $\Lambda = V^T A U$, and $\tilde{P}^T = V^T c$.

The vector \tilde{P}^T can be expanded to give

$$\tilde{P}^T = \sum_{i=1}^r g_{i0} V^T c_i = \sum_{i=1}^r g_{i0} \begin{bmatrix} p_{i1} \\ p_{i2} \\ \vdots \\ p_{in} \end{bmatrix} = \begin{bmatrix} \langle p_1, g_0 \rangle \\ \langle p_2, g_0 \rangle \\ \vdots \\ \langle p_n, g_0 \rangle \end{bmatrix} = \begin{bmatrix} \alpha_{10} \\ \alpha_{20} \\ \vdots \\ \alpha_{n0} \end{bmatrix} \quad (4.23)$$

Application of Corollary (3.9) then clearly shows that a mode, λ_i , is controllable in (4.20) if and only if $\alpha_{i0} \neq 0$. Note that for a mode to be controllable in system (4.20) it is necessary that it be controllable in system (2.1).

It is shown in Appendix C that for feedback controllers of the form (4.19) the f_i in (4.15) are given by

$$f_i = (-1)^i [P_i(\lambda) + \sum_{k=1}^{n_m} \delta_k \alpha_{k0} P_{k-1}(\lambda | \lambda_k)] , \quad i = 1, 2, \dots, n_m . \quad (4.24)$$

The characteristic polynomial of the desired roots is

$$s^{n_m} - P_1(\gamma)s^{n_m-1} + \dots + (-1)^{n_m} P_{n_m}(\gamma) , \quad (4.25)$$

thus the d_i in (4.16) are

$$d_i = (-1)^i P_i(\gamma) , \quad i = 1, 2, \dots, n_m . \quad (4.26)$$

Equating the corresponding f_i and d_i , as in (4.17), yields

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ P_1(\lambda | \lambda_1) & P_1(\lambda | \lambda_2) & \dots & P_1(\lambda | \lambda_{n_m}) \\ \vdots & \vdots & & \vdots \\ P_{n_m-1}(\lambda | \lambda_1) & P_{n_m-1}(\lambda | \lambda_2) & \dots & P_{n_m-1}(\lambda | \lambda_{n_m}) \end{bmatrix} \begin{bmatrix} \delta_1 \alpha_{10} \\ \delta_2 \alpha_{20} \\ \vdots \\ \delta_{n_m} \alpha_{n_m 0} \end{bmatrix} = \begin{bmatrix} P_1(\gamma) - P_1(\lambda) \\ P_2(\gamma) - P_2(\lambda) \\ \vdots \\ P_{n_m}(\gamma) - P_{n_m}(\lambda) \end{bmatrix} . \quad (4.27)$$

which is written symbolically as $RD = Q$.

Proposition 4.28 A unique solution exists for the δ_k , $k = 1, 2, \dots, n_m$, if and only if the modes $\lambda_1, \lambda_2, \dots, \lambda_{n_m}$ are

distinct and $\alpha_{k0} \neq 0$, $k = 1, 2, \dots, n_m$.

Proof: See Appendix D.

It should be noted that a vector g_0 of ratios can always be found to satisfy the condition $\alpha_{k0} \neq 0$, $k = 1, 2, \dots, n_m$, if $\lambda_1, \lambda_2, \dots, \lambda_{n_m}$ are distinct and controllable in system (2.1).

Computational Aspects of Solving Equation (4.27)

Since only real systems are considered all complex quantities must occur in complex conjugate pairs. It follows that $P_i(\lambda)$ and $P_i(\gamma)$ are always real, but $P_i(\lambda|\lambda_k)$ is real if and only if λ_k is real. Thus the matrix R is real if and only if all of the modes to be changed are real. It turns out, however, that the matrix equation (4.27) can always be reduced to an equivalent equation in real quantities as is shown below.

(1) All Real Modes

From property (A.6) it follows that the p_i , $i = 1, 2, \dots, n_m$, are all real. Thus once R is inverted it is trivial to obtain the scalar δ_k 's from the solutions $\delta_k \alpha_{k0}$, $k = 1, 2, \dots, n_m$.

For large values of n_m a more convenient method for determining the $P_i(\lambda|\lambda_k)$ than that given in (C.11) is now derived. Recall that by definition

$$\begin{aligned}
& (s-\lambda_k)^{n_m-1} [s^{n_m-1} - P_1(\lambda|\lambda_k)s^{n_m-2} + \dots + (-1)^{n_m-1} P_{n_m-1}(\lambda|\lambda_k)] \\
& = s^{n_m} - P_1(\lambda)s^{n_m-1} + \dots + (-1)^{n_m} P_{n_m}(\lambda) . \quad (4.29)
\end{aligned}$$

Equating coefficients of like powers in s yields the relations

$$P_i(\lambda|\lambda_k) = P_i(\lambda) - \lambda_k P_{i-1}(\lambda|\lambda_k) , \quad \begin{array}{l} i = 1, 2, \dots, n_m-1 \\ k = 1, 2, \dots, n_m \end{array} \quad (4.30)$$

As a check on the numerical accuracy of the $P_i(\lambda|\lambda_k)$ calculated recursively from (4.30) the relation

$$P_{n_m}(\lambda) = \lambda_k P_{n_m-1}(\lambda|\lambda_k) \quad (4.31)$$

obtained from equating the zeroth order coefficients can be used.

(2) Complex Modes

This case will be treated via an illustration in which $n_m = 3$. The extension to arbitrary n_m follows readily.

$$\begin{aligned}
\text{Assume } \lambda_1 &= \lambda_1' + j\lambda_1'' \\
\lambda_2 &= \lambda_1^* = \lambda_1' - j\lambda_1'' \\
\lambda_3 &\text{ real}
\end{aligned}$$

In general it is true that $\lambda_\ell = \lambda_k^*$ implies that

$P_i(\lambda|\lambda_\ell) = P_i^*(\lambda|\lambda_k)$ and that $P_i(\lambda|\lambda_\ell, \lambda_k)$, i.e. $P_i(\lambda)$ with λ_ℓ and λ_k set equal to zero, is real. It also follows from property (A.6) that $\alpha_{\ell 0} = \alpha_{k 0}^*$. Thus in order to satisfy (4.27) it must be true that $\delta_\ell = \delta_k^*$, which in addition guarantees that m is real.

Defining $\delta_1 = \delta_1' + j\delta_1''$ and $\alpha_{10} = \alpha_{10}' + j\alpha_{10}''$ (4.27) may be rewritten in terms of real quantities only as

$$\begin{aligned}
 & \begin{bmatrix} 1 & 0 & 1 \\ P_1'(\lambda|\lambda_1) & P_1''(\lambda|\lambda_1) & P_1(\lambda|\lambda_3) \\ P_2'(\lambda|\lambda_1) & P_2''(\lambda|\lambda_1) & P_2(\lambda|\lambda_3) \end{bmatrix} \begin{bmatrix} 2 \operatorname{Re}(\delta_1 \alpha_{10}') \\ -2 \operatorname{Im}(\delta_1 \alpha_{10}'') \\ \delta_3 \alpha_{30} \end{bmatrix} \\
 &= \begin{bmatrix} P_1(\gamma) - P_1(\lambda) \\ P_2(\gamma) - P_2(\lambda) \\ P_3(\gamma) - P_3(\lambda) \end{bmatrix} \tag{4.32}
 \end{aligned}$$

If n_m is a large number it may be easier to compute $P_i'(\lambda|\lambda_1)$ and $P_i''(\lambda|\lambda_1)$ from the real quantities $P_i(\lambda|\lambda_1, \lambda_2)$. The general result for n_m modes containing a complex conjugate pair λ_ℓ, λ_k is

$$P_i(\lambda|\lambda_\ell) = P_i(\lambda|\lambda_\ell, \lambda_k) + \lambda_k P_{i-1}(\lambda|\lambda_\ell, \lambda_k), \quad i = 1, 2, \dots, n_m - 1 \tag{4.33}$$

where

$$P_0(\lambda | \lambda_\ell, \lambda_k) \doteq 1$$

$$P_{n_m-1}(\lambda | \lambda_\ell, \lambda_k) \doteq 0 .$$

It then follows that the real and imaginary components of the $P_i(\lambda | \lambda_\ell)$ are

$$P_i'(\lambda | \lambda_\ell) = P_i(\lambda | \lambda_\ell, \lambda_k) + \lambda_\ell' P_{i-1}(\lambda | \lambda_\ell, \lambda_k) \quad (4.34a)$$

$$P_i''(\lambda | \lambda_\ell) = -\lambda_\ell'' P_i(\lambda | \lambda_\ell, \lambda_k) \quad (4.34b)$$

The actual feedback control law for the illustrative example is then

$$m_f = g_0(\delta_1 z_1 + \delta_2 z_2 + \delta_3 z_3)$$

$$= g_0 \langle 2\delta_1' v_1' - 2\delta_1'' v_1'' + \delta_3 v_3, x \rangle \quad (4.35)$$

E. State Decomposition

The modal decomposition property of Section (4.B) shows that knowledge of a mode and its associated reciprocal basis vector is sufficient to arbitrarily change it. Since this information can be directly obtained from the system without first transforming it into a canonical representation or computing the characteristic equation, this property affords a tremendous computational savings. It would be nice if an analogous state decomposition property existed which would allow the mode and reciprocal basis vector to

be calculated from a system whose state space representation is smaller than that of the original. In general such a property does not exist. However, special results are obtained in the cases discussed below.

1. Systems Connected in Parallel

Consider a set of systems, S_i , described by

$$\dot{x}_i = A_{ii}x_i + C_i m_i \quad (i = 1, 2, \dots, \ell) \quad (4.36)$$

where A_{ii} is $n_i \times n_i$ with modes $\{\lambda_{i1}, \dots, \lambda_{in_i}\}$

C_i is $n_i \times r_i$

$$\sum_{i=1}^{\ell} n_i = n$$

$$\sum_{i=1}^{\ell} r_i = r .$$

Let a composite system, S , be formed by connecting the S_i in parallel to give

$$\dot{x} = Ax + Cm \quad (4.37a)$$

$$y = Hx \quad (4.37b)$$

where

$$A = \begin{bmatrix} A_{11} & & & 0 \\ & A_{22} & & \\ & & \ddots & \\ 0 & & & A_{\ell\ell} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_\ell \end{bmatrix}$$

The modes of A are the roots of the factorable characteristic equation

$$\det[sI - A] = \prod_{i=1}^{\ell} \det[sI - A_{ii}] = 0. \quad (4.38)$$

Furthermore, it follows from the block diagonal structure of A that the eigenvectors and reciprocal basis vectors take a form similar to the structure of C . Therefore it is sufficient to consider only the much smaller subsystems independently when changing the modes of A .

In order to change λ_{kl} , a mode of A_{kk} , only the reciprocal basis vector v_{kl} , of A_{kk} , associated with λ_{kl} need be calculated. The required control will be of the form

$$m^T = [m_{(1)} \vdots m_{(2)} \vdots \dots \vdots m_{(\ell)}]$$

where $m_{(i)} = \begin{cases} 0 & (i \neq k) \\ \xi_k^T \langle v_{kl}, x_k \rangle & (i = k) \end{cases}$

and g_k is an r_k - dimensional vector which achieves the desired shift in λ_{kl} .

2. Systems Connected in Cascade

Let the S_i of (4.37) be connected in cascade to form a composite system, S , described by (4.37) with

$$A = \begin{bmatrix} A_{11} & 0 & \dots & 0 \\ A_{21} & A_{22} & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{\ell 1} & A_{\ell 2} & \dots & A_{\ell \ell} \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & & & 0 \\ & C_2 & & \\ & & \ddots & \\ 0 & & & C_\ell \end{bmatrix} \quad (4.39)$$

The submatrices A_{ik} ($k < i$) represent interaction matrices of state coupling.

Again the characteristic equation of A is factorable as in (4.38). Since the coupling between the systems is only in one direction the subsystems may be treated independently to achieve arbitrary mode locations. Thus in order to change the modes of A occurring in A_{kk} it is sufficient to determine a feedback law $m_k = F_k x_k$ from (4.36) with $i = k$. The states of x_k which go into all S_i , $i > k$, are treated effectively as inputs to those systems, and do not effect their dynamics.

3. Cascade-Parallel Connections of Systems

The arguments of the previous section also apply to composite

systems formed by arbitrary cascade-parallel connections of the subsystems. In this case some of the submatrices A_{ik} , $i > k$, in (4.39) will be zero. This does not effect any of the previous results.

4. Weakly Coupled Systems

In an effort to extend the concept of state decomposition Milne[25] treats a class of systems that he defines as "weakly coupled dynamical systems." His results, unlike the previous results, are approximations.

Consider the following system with distinct modes

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} m_1(t) \\ m_2(t) \end{bmatrix} \quad (4.40)$$

where

$$A_{11} \text{ is } n_1 \times n_1$$

$$A_{22} \text{ is } n_2 \times n_2$$

$$n_1 + n_2 = n .$$

The solution of (4.40) is given by

$$x(t) = \sum_{i=1}^n e^{\lambda_i t} [\langle v_i, x(0) \rangle + \int_0^t \langle v_i, m(\tau) \rangle e^{-\lambda_i \tau} d\tau] u_i \quad (4.41)$$

where the λ_i , u_i and v_i are respectively the eigenvalues, eigenvectors and reciprocal basis vectors of the $n \times n$ matrix A . Each mode of A is a root of the n^{th} degree characteristic equation

$$\det[sI-A] = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n = 0. \quad (4.42)$$

The results of the procedure can be stated in the following proposition.

Proposition 4.43 When the subsystems A_{11} and A_{22} are "weakly coupled" (defined in (4.49))-

- (i) An approximation to the set of eigenvalues of A is given by the union of the sets of eigenvalues of two derived matrices \tilde{A}_{11} and \tilde{A}_{22} .
- (ii) Furthermore an approximation to the eigenvectors of A is simply expressible in terms of the eigenvectors of \tilde{A}_{11} and \tilde{A}_{22} .

The basis for the approximation resides in the fact that if the eigenvalues of A form two sets having n_1 and n_2 elements which are widely separated in modulus, then the members of each set can be approximated by solving the n_1 -th and n_2 -th order polynomial equations obtained from the first (last) n_1+1 and last (first) n_2+1 terms of (4.42). This approximation is not directly applicable to decomposition for two reasons. First it requires

a priori knowledge of the modes of A , and secondly it requires that the characteristic equation of A be determined.

To circumvent these difficulties it is postulated that the modes of A_{11} and A_{22} form disjoint sets which are widely separated in modulus. Then conditions are found on the elements of the coupling matrices A_{12} and A_{21} such that the modes of A also form two sets widely separated in modulus.

Let the matrices A_{11} and A_{22} be diagonalized

$$\text{diagonal}[\lambda_1^{(1)}, \lambda_2^{(2)}, \dots, \lambda_{n_1}^{(1)}] \doteq \Lambda_1 = U_1^{-1} A_{11} U_1 \quad (4.44a)$$

$$\text{diagonal}[\lambda_1^{(2)}, \lambda_2^{(2)}, \dots, \lambda_{n_2}^{(2)}] \doteq \Lambda_2 = U_2^{-1} A_{22} U_2 \quad (4.44b)$$

Assuming that the sets of modes are disjoint, with those of A_{11} smaller, define

$$d = \max_i |\lambda_i^{(1)}| \quad (i = 1, 2, \dots, n_1) \quad (4.45a)$$

$$D = \min_i |\lambda_i^{(2)}| \quad (i = 1, 2, \dots, n_2) \quad (4.45b)$$

In addition define two matrices as follows

$$\Gamma = U_1^{-1} A_{12} U_2 \quad (\gamma = \max_{i,k} |\gamma_{i,k}|) \quad (4.46a)$$

$$\Delta = U_2^{-1} A_{21} U_1 \quad (\delta = \max_{i,k} |\delta_{i,k}|) \quad (4.46b)$$

Note that if the transformation of state

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (4.47)$$

is performed then $\dot{x} = Ax$ becomes

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \Lambda_1 & \Gamma \\ \Delta & \Lambda_2 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \quad (4.48)$$

The conditions for "weak coupling" that give a priori justification for the procedure are shown by Milne to be

$$\begin{aligned} \text{(i)} \quad & d/D \ll 1 \quad \text{and} \quad (4.49) \\ \text{(ii)} \quad & \gamma n_1 / D^2 \ll 1 \end{aligned}$$

If these conditions are satisfied then (4.50)

$$\text{(i)} \quad \det[sI - \Lambda_1 + \Gamma \Lambda_2^{-1} \Delta]$$

approximates the last n_1+1 terms of $[\det(sI-A)/\alpha_{n_2}]$ which yield an approximation to the n_1 smaller modes of A ,

$$\text{(ii)} \quad \det[sI - \Lambda_2] = 0$$

approximates the first n_2+1 terms of $[\det(sI-A)/s^{n_1}]$ which yield an approximation to the n_2 larger modes of A ,

(iii) the eigenvectors of A can be approximated by

$$\begin{bmatrix} \tilde{u}_i^{(1)} \\ -\Lambda_2^{-1} \Delta \tilde{u}_i^{(1)} \end{bmatrix} \quad (i=1,2, \dots, n_1) \quad \text{and} \quad \begin{bmatrix} \frac{1}{\lambda_k^{(2)}} \Gamma u_k^{(2)} \\ u_k^{(2)} \end{bmatrix} \quad (k=1,2, \dots, n_2)$$

where

$$\begin{aligned} \tilde{u}_i^{(1)} & \text{ are eigenvectors of } [\Lambda_1 - \Gamma \Lambda_2^{-1} \Delta] \\ u_k^{(2)} \text{ and } \lambda_k^{(2)} & \text{ are eigenvectors and eigenvalues of } [\Lambda_2]. \end{aligned}$$

Milne's decomposition technique appears to be quite limited because of the restriction it imposes on the matrix A . Even if the modes of A do separate into two widely separated sets there is no guarantee that these sets correspond to the modes of A_{11} and A_{22} .

The importance of Milne's technique stems from its ability to analyze the special class of large systems which are weakly coupled. For these systems an estimate of $x(t)$ can readily be obtained via (4.41) by performing a spectral analysis of only A_{11} and A_{22} , and not of the entire $n \times n$ matrix A . If the approximations to the λ_i and u_i prove to check reasonably well then they may be utilized as a guide in defining a feedback law for modal control.

CHAPTER V

RECURSIVE DESIGN

A. Introduction

The previous chapter derives a general set of equations which must be solved to find an r -dimensional control vector producing a specified distribution of n_m ($n_m \leq n$) modes. A class of feedback controls is introduced which essentially reduces the problem to one of inverting an $n_m \times n_m$ matrix. However, if other controls outside this class are to be investigated it is necessary to consider methods for solving (4.17). In order to circumvent the difficulties of working with a large number of undetermined equations a recursive technique is developed. The technique is recursive in the sense that it allows a small number of modes to be moved to their desired locations at each iteration.

Consider again a system with distinct modes characterized by (2.1). For clarity it is necessary to use superscripts to distinguish between certain quantities at each stage of the design. The open-loop system, $S^{(0)}$, is denoted by

$$\dot{x} = A^{(0)}x + Cm \quad (5.1)$$

with the corresponding canonical representation

$$\dot{z} = \Lambda^{(0)} z + P^{(0)T} m \quad (5.2)$$

where $\Lambda^{(0)} = V^{(0)T} A^{(0)} U^{(0)} = \text{diagonal} [\lambda_1, \lambda_2, \dots, \lambda_n]$,
and $P^{(0)T} = V^{(0)T} C$.

The first step in the recursive design procedure consists of finding a linear state variable feedback law

$$m^{(1)} = G^{(1)} z = F^{(1)} x, \quad (F^{(1)} \doteq G^{(1)} V^{(0)T}), \quad (5.3)$$

which moves a selected number of modes to specified locations while keeping the others fixed. For illustrative purposes assume that $m^{(1)}$ is chosen to change λ_1 and λ_2 to γ_1 and γ_2 respectively. Incorporating this feedback law into $S^{(0)}$ yields a new system, $S^{(1)}$, described by

$$\dot{x} = A^{(1)} x + C m \quad \text{where} \quad A^{(1)} \doteq A^{(0)} + C F^{(1)}, \quad (5.4a)$$

$$\text{or} \quad \dot{z} = \bar{\Lambda}^{(0)} z + P^{(0)T} m \quad \text{where} \quad \bar{\Lambda}^{(0)} \doteq \Lambda^{(0)} + P^{(0)T} G. \quad (5.4b)$$

If the modes of $S^{(1)}$ are distinct, then it may be represented in canonical form by

$$\dot{z} = \Lambda^{(1)} z + P^{(1)T} m \quad (5.5)$$

where $\Lambda^{(1)} = V^{(1)T} A^{(1)} U^{(1)} = \text{diagonal} [\gamma_1, \gamma_2, \lambda_3, \dots, \lambda_n]$, and

$$P^{(1)T} = V^{(1)T}C .$$

Note that $\Lambda^{(1)}$ is the canonical form of $A^{(1)}$, and is thus a diagonal matrix having the modes of $S^{(1)}$ as its entries. Compare this with $\bar{\Lambda}^{(0)}$ defined in (5.4b).

The system denoted by $S^{(1)}$ is nothing more than the closed-loop system obtained by employing the feedback law $m^{(1)}$ in system $S^{(0)}$. For the purpose of recursive design it may also be viewed effectively as a new open-loop system with its system matrix given by $A^{(1)}$. If $S^{(1)}$ is found to be satisfactory the design is complete. Otherwise an additional feedback controller

$$m^{(2)} = G^{(2)}z = F^{(2)}x, \quad (F^{(2)} \doteq G^{(2)}V^{(1)T}), \quad (5.6)$$

can be derived to change the modes of $S^{(1)}$.

The recursive procedure can be continued indefinitely with any number of modes altered at each iteration. After σ steps the system is described by

$$\begin{aligned} \dot{x} &= A^{(0)}x + C(m^{(1)} + m^{(2)} + \dots + m^{(\sigma)}) \\ &= A^{(\sigma)}x + Cm, \quad \text{where } A^{(\sigma)} = A^{(0)} + C \sum_{i=1}^{\sigma} F^{(i)}. \end{aligned} \quad (5.7)$$

If the modes of the resulting system, $S^{(\sigma)}$, are distinct, then its corresponding canonical representation is

$$\dot{z} = \Lambda^{(\sigma)} z + P^{(\sigma)T} m \quad (5.8)$$

where $\Lambda^{(\sigma)} = V^{(\sigma)T} A^{(\sigma)} U^{(\sigma)}$

$$P^{(\sigma)T} = V^{(\sigma)T} C .$$

A pictorial interpretation of the recursive design procedure may be found in Figures 1 through 3.

Obviously if the recursive algorithm is to alleviate the overall computational effort each step in the design must be easier to perform than the single step required for the simultaneous movement of all of the modes. A tremendous savings in effort is afforded by the modal decomposition property when only a small number of modes are moved at one time because the amount of effort required to determine a control increases exponentially with the number of modes moved. It is necessary in the recursive algorithm to compute the $v_i^{(k)}$ (reciprocal basis vectors) corresponding to the modes moved in the k^{th} step, but an efficient algorithm is developed to determine these quantities. As a byproduct, if desired, the eigenvectors, $u_i^{(k)}$, can also be determined with very little additional computational effort.

The following two sections develop the algorithms for recursively shifting one and two modes respectively. Algorithms for recursively moving an arbitrary number of modes may readily be generalized from these. In the last section a special application

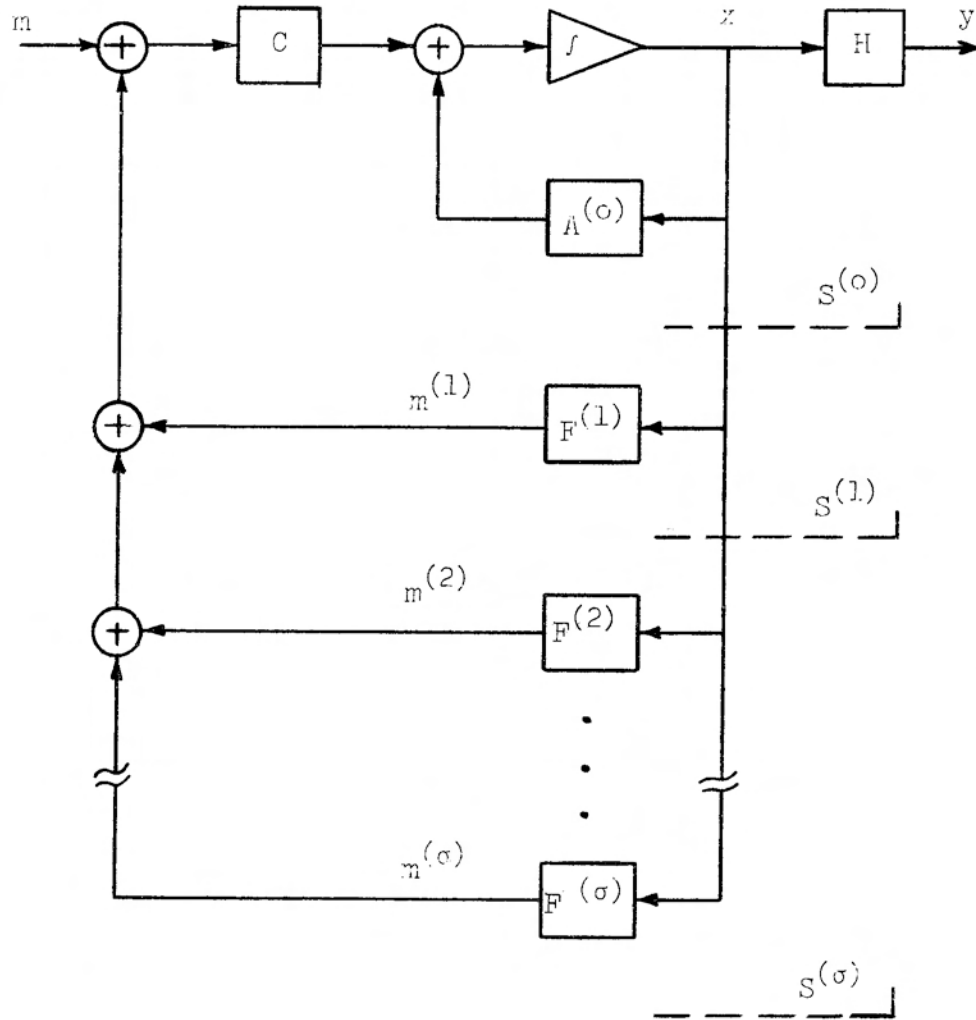


Figure 1. Recursive Design (σ Stages)

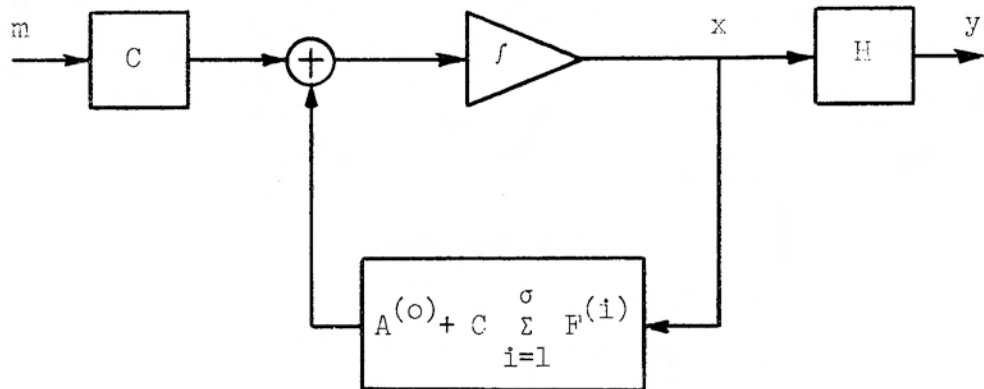


Figure 2. Effective System (σ Stages)

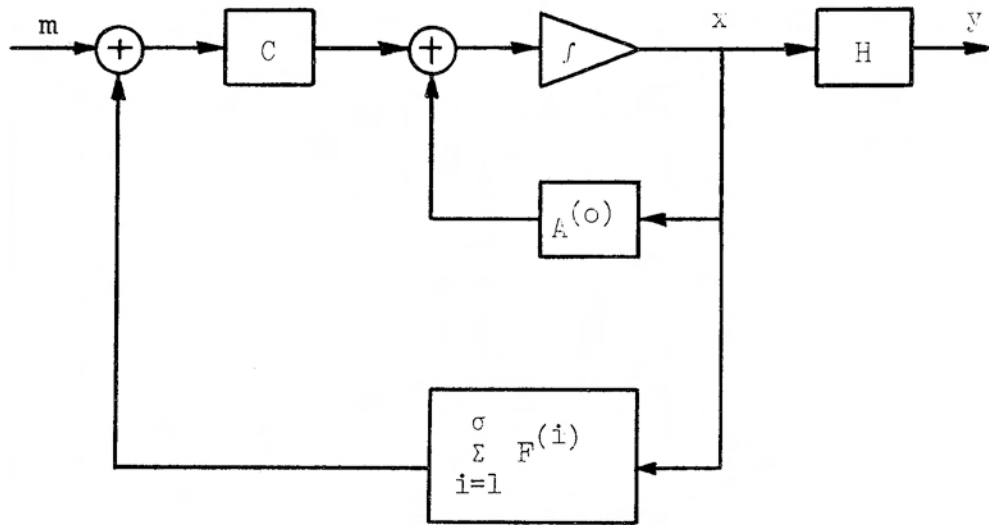


Figure 3. Final Design (σ Stages)

of the recursive technique is employed to overcome state measurement problems. It is shown that the measurement of $n_m - 1$ ($n_m \leq r$) states can be eliminated if at least n_m modes take part in the shifting algorithms at each step.

B. Algorithms for Single Mode Shift

1. Derivation of Control

For notational convenience it is assumed that the first (real) mode, λ_1 , is to be changed to the (real) mode, γ_1 . Only a real to real change is treated to preserve the realness of the system at each stage in the design.

As a consequence of the modal decomposition property the form of $m^{(1)}$ is chosen as

$$m^{(1)} = g_1 z_1 = \begin{bmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{r1} \end{bmatrix} \langle v_1^{(0)}, x \rangle . \quad (5.9)$$

Substitution of (5.9) into (5.2) yields

$$\bar{\Lambda}^{(0)} = \begin{bmatrix} \lambda_1 + a_{11}^{(0)} & | & 0 \\ \hline a_{21}^{(0)} & | & \lambda_2 & 0 \\ \vdots & | & \ddots & \ddots \\ a_{n1}^{(0)} & | & 0 & \ddots & \lambda_n \end{bmatrix} \quad (5.10)$$

where $\alpha_{k1}^{(0)} \doteq \langle p_k^{(0)}, g_1 \rangle$.

As expected $m^{(1)}$ only effects the first mode. The characteristic equation of the closed-loop system is

$$\det[sI - \bar{A}^{(0)}] = (s - \lambda_1 - \alpha_{11}^{(0)}) (s - \lambda_2) \dots (s - \lambda_n) = 0. \quad (5.11)$$

Thus any g_1 satisfying

$$\lambda_1 + \langle p_1^{(0)}, g_1 \rangle = \gamma_1 \quad (5.12)$$

will achieve the desired shift. Note that the result (5.12) is in perfect agreement with Theorem (3.18)

Observations

(i) For the case of scalar control g_1 and $p_1^{(0)} \doteq \langle v_1^{(0)}, c \rangle$ are also scalars. Therefore, $m^{(1)}$ is uniquely determined by

$$m^{(1)} = \frac{(\gamma_1 - \lambda_1)}{p_1^{(0)}} z_1 = \frac{(\gamma_1 - \lambda_1)}{\langle v_1^{(0)}, c \rangle} \langle v_1^{(0)}, x \rangle \quad (5.13)$$

A measure of the controllability of the first mode is given by the absolute value of the normalized $p_1^{(0)}$. That is,

$$\overline{p_1^{(0)}} \doteq \left| \frac{\langle v_1^{(0)}, c \rangle}{\langle v_1^{(0)}, v_1^{(0)} \rangle^{\frac{1}{2}}} \right| \quad (5.14)$$

varies inversly with the absolute value of the gains required to yield a given displacement of the first mode. This agrees

intuitively with the fact that when $\overline{p_1^{(0)}} = 0$ the mode λ_1 is uncontrollable.

(ii) In the general case, $r > 1$, the components of the control vector $m^{(1)}$ are proportional to the corresponding elements of the r -dimensional vector g_1 . A simple procedure for obtaining a unique solution to (5.12) is to specify a desired ratio for the components of $m^{(1)}$. Such a ratio may be based on the reliability, sensitivity, cost of the corresponding control variables, or be chosen to reduce the absolute value of the required feedback gains.

Assume that

$$g_1 = \delta g_0 \quad (5.15)$$

where δ is a scalar to be determined, and the vector g_0 specifies the ratio of control variables. Then

$$m^{(1)} = \delta g_0 \langle v_1^{(0)}, x \rangle \quad (5.16)$$

where $\delta = \frac{\gamma_1 - \lambda_1}{\langle p_1^{(0)}, g_0 \rangle}$ is obtained from (5.12).

Recall that this procedure reduces the given system with r inputs to an effective single-input system. Proposition (4.21) states that the first mode of this effective system is controllable if and only if g_0 is chosen such that $\langle p_1^{(0)}, g_0 \rangle \neq 0$. A measure of controllability for the first mode of the effective

single-input system may be defined as

$$\left| \frac{\langle p_1^{(0)}, g_0 \rangle}{\langle v_1^{(0)}, v_1^{(0)} \rangle^{\frac{1}{2}} \|g_0\|} \right|, \text{ where } \|g_0\| = \max_{i=1, \dots, r} |g_{i0}|. \quad (5.17)$$

Appendix E shows that the selection of the elements of g_0 by the rule

$$g_{i0} = (\text{sign } p_{i1}^{(0)}) , \quad i = 1, 2, \dots, r \quad (5.18)$$

maximizes the measure of controllability. Hence the ratios defined by (5.18) require the least absolute value of feedback gains.

2. Derivation of Updated Eigenvectors

After determining an appropriate feedback law $m^{(1)}$, defined in (5.9), the open-loop system $S^{(0)}$ is transformed into $S^{(1)}$ which is represented by

$$\dot{x} = A^{(1)}x + Cm. \quad (5.19)$$

The new system matrix satisfies the relations

$$\begin{aligned} A^{(1)} &= A^{(0)} + Cg_1 v_1^{(0)T} \\ &= A^{(0)} + \sum_{k=1}^r c_k g_{k1} v_1^{(0)T} \end{aligned} \quad (5.20)$$

It is readily verified that

$$u_i^{(1)} = u_i^{(0)}, \quad i = 2, 3, \dots, n \quad (5.21)$$

because $A^{(1)} u_i^{(0)} = A^{(0)} u_i^{(0)} = \lambda_i u_i^{(0)}$, $i = 2, 3, \dots, n$.

The revised first eigenvector $u_1^{(1)}$ must satisfy

$$[A^{(0)} + \sum_{k=1}^r c_k g_{k1} v_1^{(0)T}] u_1^{(1)} = \gamma_1 u_1^{(1)} \quad (5.22)$$

Since the $u_i^{(0)}$ form an n -dimensional basis $u_1^{(1)}$ can be represented by

$$u_1^{(1)} = \sum_{i=1}^n q_i u_i^{(0)} \quad (5.23)$$

where the q_i are scalars to be determined. Expanding (5.22) using (5.23) yields

$$\sum_{i=1}^n q_i \lambda_i u_i^{(0)} + q_1 \sum_{k=1}^r c_k g_{k1} = \gamma_1 \sum_{i=1}^n q_i u_i^{(0)} \quad (5.24)$$

but by definition $c_k = \sum_{i=1}^n p_{ki}^{(0)} u_i^{(0)}$, therefore

$$\sum_{i=1}^n [q_1 \alpha_{i1}^{(0)} - q_i (\gamma_1 - \lambda_i)] u_i^{(0)} = 0. \quad (5.25)$$

The fact that the $u_i^{(0)}$ are linearly independent implies that

$$q_1 \alpha_{i1}^{(0)} = q_i (\gamma_1 - \lambda_i) \quad i = 1, 2, \dots, n \quad (5.26)$$

If a resulting set of eigenvalues, $\{\gamma_1, \lambda_2, \dots, \lambda_n\}$, remains distinct then $q_1 \neq 0$, and may be set equal to unity. Thus

$$q_i = \frac{\alpha_{i1}^{(0)}}{\gamma_1 - \lambda_i} \quad i = 2, 3, \dots, n \quad \text{with } q_1 = 1. \quad (5.27)$$

Note that if γ_1 is set equal to another eigenvalue it is possible for $u_1^{(1)}$ not to exist. This agrees with the well known fact that if an n^{th} order matrix has repeated eigenvalues then it may have less than n linearly independent eigenvectors.

3. Derivation of Updated Reciprocal Basis

The updated set of eigenvectors is now known, and given by (5.21) and (5.23). By definition $V^{(1)T} = U^{(1)-1}$, however, it is more convenient to compute the reciprocal basis vectors using the relations $\langle u_i^{(1)}, v_k^{(1)} \rangle = \delta_{ik}$. Making use of the fact that $\langle u_i^{(0)}, v_k^{(0)} \rangle = \delta_{ik}$ it is found that

$$v_1^{(1)} = v_1^{(0)} \quad (5.28a)$$

$$v_k^{(1)} = v_k^{(0)} - q_k v_1^{(0)}, \quad k = 2, 3, \dots, n \quad (5.28b)$$

It can also be verified that the method used to define the updated variables preserves properties (A.2) and (A.6).

C. Algorithms for Shifting a Pair of Modes

The algorithms in this section are developed in a manner

analogous to those of the single mode shift. This derivation illustrates how to handle the additional difficulty encountered when complex modes are changed. Since the system is real it is known that the complex modes occur in complex conjugate pairs (c. c. p.) . In order to preserve the realness of the system it is assumed that the pair of modes to be changed is a c.c.p. or a real pair (r. p.) , and likewise the resulting pair is a c.c.p. or r. p . of modes. Thus four types of transformations are considered (r.p. \rightarrow r.p. , r.p. \rightarrow c.c.p. , c.c.p. \rightarrow r.p. , c.c.p. \rightarrow c.c.p.) .

1. Derivation of Control

The starting point of the derivation is again the canonical system representation (5.2) of the system. For notational simplicity it is assumed that the first two modes (λ_1, λ_2) are to be changed to γ_1 and γ_2 . Consider the control

$$m^{(1)} = g_1 z_1 + g_2 z_2 = \begin{bmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{r1} \end{bmatrix} \langle v_1^{(0)}, x \rangle + \begin{bmatrix} g_{12} \\ g_{22} \\ \vdots \\ g_{r2} \end{bmatrix} \langle v_2^{(0)}, x \rangle \quad (5.29)$$

In the development conditions will be imposed on the vectors g_1 and g_2 to insure that m is real.

Substitution of (5.29) into (5.2) yields

$$\bar{\Lambda}^{(o)} = \left[\begin{array}{cc|ccc} \lambda_1 + \alpha_{11}^{(o)} & \alpha_{12}^{(o)} & & & \\ \alpha_{21}^{(o)} & \lambda_2 + \alpha_{22}^{(o)} & & & \\ \alpha_{31}^{(o)} & \alpha_{32}^{(o)} & \lambda_3 & & \\ \vdots & \vdots & & \ddots & \\ \alpha_{n1}^{(o)} & \alpha_{n2}^{(o)} & 0 & & \lambda_n \end{array} \right] \quad \text{where } \alpha_{ik}^{(o)} = \langle p_i^{(o)}, g_k \rangle \quad (5.30)$$

It is evident that only the first two modes can be changed. In order to determine the two revised modes it follows from the modal decomposition property that it is sufficient to examine the effect of g_1 and g_2 on the eigenvalues of

$$\bar{\Lambda}_{11}^{(o)} = \left[\begin{array}{cc} \lambda_1 + \alpha_{11}^{(o)} & \alpha_{12}^{(o)} \\ \alpha_{21}^{(o)} & \lambda_2 + \alpha_{22}^{(o)} \end{array} \right] \quad (5.31)$$

The problem is now to determine g_1 and g_2 such that the eigenvalues of $\bar{\Lambda}_{11}^{(o)}$ are γ_1 and γ_2 . Although the problem is conceptually simple to solve it is convenient to break it up into separate cases in order to investigate some interesting special results. The major distinction depends on what type of pair (r.p. or c.c.p.) the original modes are.

The characteristic equation of $\bar{\Lambda}_{11}^{(o)}$ in (5.31) is

$$s^2 - s(\lambda_1 + \lambda_2 + \alpha_{11}^{(o)} + \alpha_{22}^{(o)}) + \lambda_1 \lambda_2 + \lambda_1 \alpha_{22}^{(o)} + \lambda_2 \alpha_{11}^{(o)} + \alpha_{11}^{(o)} \alpha_{22}^{(o)} - \alpha_{12}^{(o)} \alpha_{21}^{(o)} = 0 \quad (5.32)$$

If the roots of (5.32) are to be γ_1 and γ_2 it must be equivalent to

$$s^2 - s(\gamma_1 + \gamma_2) + \gamma_1\gamma_2 = 0 \quad (5.33)$$

Equating coefficients of like powers of s yields the equations that g_1 and g_2 must satisfy.

$$\alpha_{11}^{(o)} + \alpha_{22}^{(o)} = \epsilon_1 \quad (5.34a)$$

$$\lambda_2 \alpha_{11}^{(o)} + \lambda_1 \alpha_{22}^{(o)} + \alpha_{11}^{(o)} \alpha_{22}^{(o)} - \alpha_{12}^{(o)} \alpha_{21}^{(o)} = \epsilon_2 \quad (5.34b)$$

where $\epsilon_1 \doteq \gamma_1 + \gamma_2 - \lambda_1 - \lambda_2 = P_1(\gamma_1, \gamma_2) - P_1(\lambda_1, \lambda_2)$

$$\epsilon_2 \doteq \gamma_1\gamma_2 - \lambda_1\lambda_2 = P_2(\gamma_1, \gamma_2) - P_2(\lambda_1, \lambda_2)$$

Note that ϵ_1 and ϵ_2 are real regardless of whether the modes are real or form complex conjugate pairs.

Some special cases which simplify the solution of (5.34) are now considered.

a. Real λ_1 and λ_2 ($\lambda_1 \neq \lambda_2$)

As a result of property (A.6) $v_1^{(o)}$ and $v_2^{(o)}$ are real. Therefore by restricting g_1 and g_2 to be real the feedback control law (5.29) will also be real.

(1) Single-Input ($r = 1$)

The control (5.29) reduces to the scalar

$$m^{(1)} = g_1 z_1 + g_2 z_2 = \langle g_1 v_1^{(o)} + g_2 v_2^{(o)}, x \rangle \quad (5.35)$$

If λ_1 and λ_2 are controllable, i.e. $p_1 \neq 0$ and $p_2 \neq 0$, then a unique solution exists. This solution is given by

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} p_1^{(o)} & p_2^{(o)} \\ \lambda_2 p_1^{(o)} & \lambda_1 p_2^{(o)} \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \quad (5.36)$$

(2) Fixed Ratio of Feedback Gains

Replace g_1 and g_2 by $\delta_1 g_o$ and $\delta_2 g_o$ respectively.

Then

$$m^{(1)} = \delta_1 g_o z_1 + \delta_2 g_o z_2 = g_o \langle \delta_1 v_1^{(o)} + \delta_2 v_2^{(o)}, x \rangle \quad (5.37)$$

where δ_1 and δ_2 are real scalar constants fixing the ratio of feedback from each mode. One basis for choosing the δ 's is to fix the feedback gains from one state at zero. For example if x_i is not to be feedback then let $\delta_1 v_{i1}^{(o)} + \delta_2 v_{i2}^{(o)} = 0$. Equations (5.34) can be represented as a pair of linear simultaneous equations in the r unknown components of g_o .

$$\langle g_o, \delta_1 p_1^{(o)} + \delta_2 p_2^{(o)} \rangle = \epsilon_1 \quad (5.38a)$$

$$\langle g_o, \lambda_2 \delta_1 p_1^{(o)} + \lambda_1 \delta_2 p_2^{(o)} \rangle = \epsilon_2 \quad (5.38b)$$

In general for a solution to exist both δ_1 and δ_2 must be chosen to be different from zero.

(3) Fixed Ratio of Control Components

This case is similar to (2) except that now the elements in g_0 are prespecified to define ratios between the elements of the control vector, and the δ 's are to be determined. The δ 's are found from

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \alpha_{10}^{(o)} & \alpha_{20}^{(o)} \\ \lambda_2 \alpha_{10}^{(o)} & \lambda_1 \alpha_{20}^{(o)} \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \end{bmatrix} \quad (5.39)$$

where $\alpha_{ko}^{(o)} = \langle p_k^{(o)}, g_0 \rangle$.

A unique solution is obtained for any specification of ratios in which $\alpha_{ko}^{(o)} \neq 0$, $k = 1, 2$. Then

$$m^{(1)} = g_0 \langle \delta_1 v_1^{(o)} + \delta_2 v_2^{(o)}, x \rangle \quad (5.40)$$

(4) Movement of Only One Mode ($\gamma_2 = \lambda_2$)

For the single input case the equations defining the g_1 are

$$p_1^{(o)} g_1 + p_2^{(o)} g_2 = \gamma_1 - \lambda_1 \quad (5.41a)$$

$$\lambda_2 p_1^{(o)} g_1 + \lambda_1 p_2^{(o)} g_2 = \lambda_2 (\gamma_1 - \lambda_1) \quad (5.41b)$$

The only solution is $g_1 = (\gamma_1 - \lambda_1)/p_1^{(o)}$, $g_2 = 0$ which is identical to the result of the single mode shift algorithm.

In the multiple input case, however, the defining relations can be expressed by the nonlinear equations

$$\alpha_{11}^{(o)} + \alpha_{22}^{(o)} = \gamma_1 - \lambda_1 \quad (5.42a)$$

$$\alpha_{22}^{(o)}(\lambda_1 - \lambda_2 + \alpha_{11}^{(o)}) - \alpha_{12}^{(o)}\alpha_{21}^{(o)} = 0 \quad (5.42b)$$

which in general has many solutions for g_1 and g_2 . For example $g_1 = g_2 = g_0$ where $\alpha_{10}^{(o)} = \gamma_1 - \lambda_1$ and $\alpha_{20}^{(o)} = 0$. Thus modes may be included in the shifting algorithm even though they are not moved in order to change the value of gains required in the feedback law.

b. Case of Complex Conjugate Modes

Define

$$\lambda_1 = \lambda_1' + j\lambda_1'' \quad \text{and} \quad \lambda_2 = \lambda_1^* = \lambda_1' - j\lambda_1'' \quad , \quad (5.43)$$

then by property (A.6) it follows that

$$v_1^{(o)} = v_1^{(o)'} + jv_1^{(o)''} \quad , \quad v_2^{(o)} = v_1^{(o)*} \quad (5.44a)$$

and

$$p_1^{(o)} = p_1^{(o)'} + jp_1^{(o)''} \quad , \quad p_2^{(o)} = p_1^{(o)*} \quad . \quad (5.44b)$$

In order to insure that m is real let

$$g_2 = g_1^* = g_1' - jg_1'' \quad (5.45)$$

Thus

$$\begin{aligned} m^{(1)} &= g_1 z_1 + g_2 z_2 \quad (5.46) \\ &= 2g_1' \langle v_1^{(o)'} \rangle_{,x} - 2g_1'' \langle v_1^{(o)''} \rangle_{,x} . \end{aligned}$$

It is convenient to incorporate (5.43 - 5.46) into the development and rewrite $\bar{\Lambda}_{11}^{(o)}$ of (5.31) as

$$\bar{\Lambda}_{11}^{(o)} = \begin{bmatrix} (\lambda_1' + \beta_{11} - \beta_{22}) + j(\lambda_1'' + \beta_{12} + \beta_{21}) & (\beta_{11} + \beta_{22}) - j(\beta_{12} - \beta_{21}) \\ (\beta_{11} + \beta_{22}) + j(\beta_{12} - \beta_{21}) & (\lambda_1' + \beta_{11} - \beta_{22}) - j(\lambda_1'' + \beta_{12} + \beta_{21}) \end{bmatrix} \quad (5.47)$$

$$\begin{aligned} \text{where } \beta_{11} &\doteq \langle p_1^{(o)'} \rangle_{,g_1'} & \beta_{12} &\doteq \langle p_1^{(o)'} \rangle_{,g_1''} \\ \beta_{21} &\doteq \langle p_1^{(o)''} \rangle_{,g_1'} & \beta_{22} &\doteq \langle p_1^{(o)''} \rangle_{,g_1''} \end{aligned}$$

The α 's used previously are related to the β 's by the relations

$$\alpha_{11}^{(o)} = \beta_{11} - \beta_{22} + j(\beta_{12} + \beta_{21}) \quad (5.48a)$$

$$\alpha_{22}^{(o)} = \beta_{11} - \beta_{22} - j(\beta_{12} + \beta_{21}) \quad (5.48b)$$

$$\alpha_{12}^{(o)} = \beta_{11} + \beta_{22} - j(\beta_{12} - \beta_{21}) \quad (5.48c)$$

$$\alpha_{21}^{(0)} = \beta_{11} + \beta_{22} + j(\beta_{12} - \beta_{21}) \quad (5.48d)$$

The characteristic equation of $\overline{\Lambda}_{11}^{(0)}$ in (5.47) may be expressed in terms of the real β 's as

$$\begin{aligned} s^2 - 2s(\lambda_1' + \beta_{11} - \beta_{22}) + (\lambda_1')^2 + (\lambda_1'')^2 + 2\lambda_1'(\beta_{11} - \beta_{22}) + 2\lambda_1''(\beta_{12} + \beta_{21}) - \\ - 4(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) = 0 \end{aligned} \quad (5.49)$$

Equating coefficients of like powers of s in (5.33) and (5.49) yields the equations that g_1 and g_2 must satisfy

$$\beta_{11} - \beta_{22} = \epsilon_1/2 \quad (5.50a)$$

$$\lambda_1'(\beta_{11} - \beta_{22}) + \lambda_1''(\beta_{12} + \beta_{21}) - 2(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) = \epsilon_2/2 \quad (5.50b)$$

where ϵ_1 and ϵ_2 are defined in (5.34).

A simplification of (5.50b) is obtained by substitution of (5.50a) into it to give

$$\lambda_1''(\beta_{12} + \beta_{21}) - 2(\beta_{11}\beta_{22} - \beta_{12}\beta_{21}) = (\epsilon_2 - \lambda_1'\epsilon_1)/2 \quad (5.50b')$$

The equations of (5.50) are equivalent to those of (5.34) for complex elements rewritten in terms of the real quantities β . Some special solutions to (5.50) are now considered.

(1) Single-Input (r = 1)

The control (5.46) reduces to the scalar

$$m^{(1)} = g_{11}z_1 + g_{12}z_2 = 2 \langle g_{11}'v_1^{(o)'} - g_{11}''v_1^{(o)''}, x \rangle \quad (5.51)$$

If $\lambda_2 = \lambda_1^*$ is controllable, i.e. $p_2^{(o)} = p_1^{(o)*} \neq 0$, then a unique solution exists. This solution is given by

$$\begin{bmatrix} g_{11}' \\ g_{11}'' \end{bmatrix} = \begin{bmatrix} p_1^{(o)'} & -p_1^{(o)''} \\ p_1^{(o)''} & p_1^{(o)'} \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_1/2 \\ (\epsilon_2 - \lambda_1' \epsilon_1)/2\epsilon_1'' \end{bmatrix} \quad (5.52)$$

(2) Fixed Ratio of Feedback Gains

Simplify (5.46) by replacing g_1' and g_1'' by $\delta_1 g_0$ and $\delta_2 g_0$ respectively. Then

$$m^{(1)} = 2g_0 \langle \delta_1 v_1^{(o)'} - \delta_2 v_1^{(o)''}, x \rangle \quad (5.53)$$

where δ_1 and δ_2 are real scalar constants chosen to adjust the ratio of feedback gains. The real vector g_0 can be determined from (5.50) which reduces to

$$\langle g_0, \delta_1 p_1^{(o)'} - \delta_2 p_1^{(o)''} \rangle = \epsilon_1/2 \quad (5.54a)$$

$$\langle g_0, \delta_2 p_1^{(o)'} + \delta_1 p_1^{(o)''} \rangle = (\epsilon_2 - \lambda_1' \epsilon_1)/2\lambda_1'' \quad (5.54b)$$

When the real and imaginary components of the vector $p_1^{(o)}$ are

not proportional to each other two special cases arise.

$$(i) \quad \delta_1 = 0, \quad \delta_2 = 1 : m^{(1)} = 2g_0 \langle v_1^{(o)'} , x \rangle$$

$$(ii) \quad \delta_1 = 1, \quad \delta_2 = 0 : m^{(1)} = -2g_0 \langle v_1^{(o)''} , x \rangle$$

(3) Fixed Ratio of Control Components

Choose a real vector g_0 the components of which will be proportional to the desired ratio of control elements. Then let $g_1 = \delta g_0$ and $g_2 = \delta^* g_0$ where δ is a complex scalar given by the solution of (5.50) as

$$\begin{bmatrix} \delta' \\ \delta'' \end{bmatrix} = \begin{bmatrix} \langle p_1^{(o)'} , g_0 \rangle & - \langle p_1^{(o)''} , g_0 \rangle \\ \langle p_1^{(o)''} , g_0 \rangle & \langle p_1^{(o)'} , g_0 \rangle \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_1/2 \\ (\epsilon_2 - \lambda_1' \epsilon_1)/2\lambda_1'' \end{bmatrix} \quad (5.55)$$

A unique solution is obtained for any specification of ratios in which $\langle p_1^{(o)'} , g_0 \rangle$ or $\langle p_1^{(o)''} , g_0 \rangle$ is different from zero. The control is

$$m^{(1)} = 2g_0 \langle \delta' v_1^{(o)'} - \delta'' v_1^{(o)''} , x \rangle \quad (5.56)$$

2. Derivation of Updated Eigenvectors

The system matrix resulting from the use of (5.29) in (5.1) is

$$A^{(1)} = [A^{(0)} + \sum_{k=1}^r c_k (g_{k1} v_1^{(0)} + g_{k2} v_2^{(0)})^T] \quad (5.57)$$

It is readily verified that

$$u_i^{(1)} = u_i^{(0)}, \quad i = 3, 4, \dots, n \quad (5.58)$$

because $A^{(1)} u_i^{(0)} = A^{(0)} u_i^{(0)} = \lambda_i u_i^{(0)}$, $i = 3, 4, \dots, n$

The revised first eigenvector $u_1^{(1)}$ must satisfy

$$A^{(1)} u_1^{(1)} = \gamma_1 u_1^{(1)} \quad (5.59)$$

Since the $u_i^{(0)}$ form an n -dimensional basis $u_1^{(1)}$ can be represented by

$$u_1^{(1)} = \sum_{i=1}^n q_{1i} u_i^{(0)} \quad (5.60)$$

where the q_{1i} are scalars to be determined. Expanding (5.59) using (5.57) and (5.60) yields

$$\sum_{i=1}^n q_{1i} \lambda_i u_i^{(0)} + \sum_{k=1}^r c_k (q_{11} g_{k1} + q_{12} g_{k2}) u_i^{(0)} = \gamma_1 \sum_{i=1}^n q_{1i} u_i^{(0)} \quad (5.61)$$

but by definition $c_k = \sum_{i=1}^n p_{ki}^{(0)} u_i^{(0)}$, therefore

$$\sum_{i=1}^n \{q_{11} \alpha_{i1}^{(0)} + q_{12} \alpha_{i2}^{(0)} - q_{1i} (\gamma_1 - \lambda_i)\} u_i^{(0)} = 0$$

The fact that the $u_i^{(0)}$ are linearly independent implies that

$$q_{11}\alpha_{i1}^{(0)} + q_{12}\alpha_{i2}^{(0)} = q_{1i}(\gamma_1 - \lambda_i) \quad , \quad i = 1, 2, \dots, n \quad (5.62)$$

If (5.62) is examined for i set equal to one and two the results are

$$\frac{q_{12}}{q_{11}} = \frac{\gamma_1 - \lambda_1 - \alpha_{11}^{(0)}}{\alpha_{12}^{(0)}} \quad , \quad (i = 1) \quad (5.63)$$

$$\frac{q_{12}}{q_{11}} = \frac{\alpha_{21}^{(0)}}{\gamma_1 - \lambda_2 - \alpha_{22}^{(0)}} \quad , \quad (i = 2) \quad (5.64)$$

It can be shown that the right hand sides in (5.63) and (5.64) are equivalent, hence either one may be used to define the ratio of q_{12} to q_{11} . This result is not surprising since equation (5.59) only fixes the eigenvector $u_1^{(1)}$ to within a scalar (real or complex) multiplier. The freedom available in the normalization of the eigenvectors will be used to preserve property (A.6).

If the shift is not of the type c.c.p. to r.p. then a set of q_{1i} ($i = 1, 2, \dots, n$) can be determined directly from (5.62) by setting q_{11} equal to unity. The q_{1i} are then defined by

$$q_{11} = 1 \quad (5.65a)$$

$$q_{12} = \frac{\alpha_{21}^{(o)}}{\gamma_1 - \lambda_2 - \alpha_{22}^{(o)}} \quad \text{from (5.64), and} \quad (5.65b)$$

$$q_{1i} = \frac{q_{11}\alpha_{i1}^{(o)} + q_{12}\alpha_{i2}^{(o)}}{\gamma_1 - \lambda_i} \quad , \quad i = 3, 4, \dots, n. \quad (5.65c)$$

In order to determine the q coefficients for the case of complex conjugate λ_1, λ_2 and real γ_1, γ_2 a different normalization factor is chosen to preserve property (A.6). Under restriction (5.45) this requirement is satisfied if $q_{12} = q_{11}^*$. After some algebraic manipulation (see Appendix F) a set of satisfactory q coefficients is found to be

$$q_{11} = 1 + j \left[\frac{\lambda_1' - \gamma_1 + 2 \langle p_1^{(o)'}, g_1' \rangle}{\lambda_1'' + 2 \langle p_1^{(o)'}, g_1'' \rangle} \right] \quad (5.66a)$$

$$q_{12} = q_{11}^* \quad (5.66b)$$

$$q_{1i} = \frac{\langle p_i^{(o)}, q_{11}g_1 + q_{12}g_2 \rangle}{(\gamma_1 - \lambda_i)} = \frac{2 \langle p_i^{(o)}, g_1 - q_{11}''g_1'' \rangle}{(\gamma_1 - \lambda_i)} \quad , \quad (5.66c)$$

$i = 3, 4, \dots, n$

The second revised eigenvector can be represented as

$$u_2^{(1)} = \sum_{i=1}^n q_{2i} u_i^{(o)}. \quad (5.67)$$

Expressions for the q_{2i} , ($i = 1, 2, \dots, n$), can easily be obtained by simply replacing γ_1 with γ_2 everywhere in the preceding development.

Some relations which follow from the property (A.6) are cited below to indicate how computational labor can be reduced.

(i) r.p. or c.c.p. to r.p.

$$\lambda_\ell \text{ real implies } q_{k\ell} \text{ real, } \lambda_\ell = \lambda_i^* \text{ implies}$$

$$q_{k\ell} = q_{ki}^*, \quad (k = 1 \text{ or } 2).$$

(ii) r.p. or c.c.p. to c.c.p.

$$u_2^{(1)} = u_1^{(1)*}.$$

3. Derivation of Updated Reciprocal Basis

If the updated set of eigenvectors are calculated, then the updated set of reciprocal basis vectors may be obtained from the basic definition

$$[V^{(1)}]^T = \begin{bmatrix} v_1^{(1)T} \\ \hline v_2^{(1)T} \\ \vdots \\ \hline v_n^{(1)T} \end{bmatrix} = \begin{bmatrix} u_1^{(1)} & | & u_2^{(1)} & | & \dots & | & u_n^{(1)} \end{bmatrix}^{-1} = [U^{(1)}]^{-1}.$$

(5.68)

Properties (A.7) and (A.8) can be used to avoid the difficulty of

working with complex quantities. This allows the complex conjugate pairs of eigenvectors to be replaced by scalar multiples of their real and imaginary parts. For example, in a fifth order system where $\lambda_2 = \lambda_1^*$, $\lambda_4 = \lambda_3^*$ and λ_5 is real, the reciprocal basis can be obtained by inverting a real matrix as shown below

$$\begin{bmatrix} v_1^T \\ \vdots \\ v_1''^T \\ \vdots \\ v_3^T \\ \vdots \\ v_3''^T \\ \vdots \\ v_5^T \end{bmatrix} = \begin{bmatrix} 2u_1' & -2u_1'' & 2u_3' & -2u_3'' & u_5 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}^{-1} \quad (5.69)$$

where $v_k = v_k' + jv_k''$, $v_{k+1} = v_k^*$, ($k = 1$ and 3).

Furthermore additional simplifications are possible since only two columns of U are changed at each stage in the design procedure.

Actually the eigenvectors of the system matrix are not required in the mode shifting algorithms. Therefore, a method is presented to update the reciprocal basis vectors which is only a function of the q -coefficients.

Recall that $u_k^{(1)} = \sum_{i=1}^n q_{ki} u_i^{(0)}$, $k = 1, 2$.

Let

$$v_k^{(1)} = r_{k1} v_1^{(0)} + r_{k2} v_2^{(0)}, \quad k = 1, 2 \quad (5.70a)$$

$$v_k^{(1)} = r_{k1}v_1^{(0)} + r_{k2}v_2^{(0)} + v_k^{(0)}, \quad k = 3, 4, \dots, n \quad (5.70b)$$

where the r -coefficients are to be determined.

Application of the relations $\langle u_i^{(1)}, v_k^{(1)} \rangle = \delta_{ik}$,
($i, k = 1, 2, \dots, n$) immediately yields the sought after
expressions. For example to determine r_{11} and r_{12} consider

$$\langle v_1^{(1)}, u_1^{(1)} \rangle = r_{11}q_{11} + r_{12}q_{12} = 1 \quad (5.71a)$$

$$\langle v_1^{(1)}, u_2^{(1)} \rangle = r_{11}q_{21} + r_{12}q_{22} = 0, \quad \text{then} \quad (5.71b)$$

$$\begin{bmatrix} r_{11} \\ r_{12} \end{bmatrix} = Q^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \text{where } Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}. \quad (5.71c)$$

In a similar manner the other r -coefficients are found to be

$$\begin{bmatrix} r_{21} \\ r_{22} \end{bmatrix} = Q^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \text{and} \quad (5.72)$$

$$\begin{bmatrix} r_{k1} \\ r_{k2} \end{bmatrix} = -Q^{-1} \begin{bmatrix} q_{1k} \\ q_{2k} \end{bmatrix}, \quad k = 3, 4, \dots, n. \quad (5.73)$$

For completeness the following proposition is proven.

Proposition 5.74 Q^{-1} exists if the set of eigenvalues remains distinct.

Proof: Assume that the eigenvalues remain distinct. Then there exist a set of n linearly independent updated eigenvectors. Consider an arbitrary linear combination of them, viz.,

$$L = \sum_{i=1}^n a_i u_i^{(1)} .$$

Expand this sum in terms of the previous set of linearly independent eigenvectors to get

$$L = (a_1 q_{11} + a_2 q_{21}) u_1^{(0)} + (a_1 q_{12} + a_2 q_{22}) u_2^{(0)} + \sum_{i=3}^n (a_1 q_{1i} + a_2 q_{2i} + a_i) u_i^{(0)} .$$

If Q^{-1} does not exist, i.e. $q_{11}q_{22} - q_{12}q_{21} = 0$, then there exists a non-trivial set of a_i such that $L = 0$. This is impossible because the eigenvectors are linearly independent, therefore Q^{-1} must exist.

D. Reduction in the Number of State Variable Measurements

There are a variety of methods for assigning state variables to a system with each state space description being an equally valid representation of the system. For design purposes it is very convenient to select a set of states which corresponds to conveniently measured attributes of the system. Unfortunately this

is not always possible. Some states are too expensive or take too much time to measure, others may be physically unobtainable or severely corrupted by noise. Therefore when using a feedback law generated by optimal control theory some form of state estimation technique must be employed to account for these states.

The modal approach to control does not require the measurement of each state. Actually the need to measure up to $r-1$ states can be eliminated without effecting the realizable mode locations. This is accomplished by an extension of the "fixed ratio of feedback gains" approach discussed in Section (5.C).

Assume for notational simplicity, that the first n_m , ($n_m \leq r$) modes of the system are to be changed* in the first stage of the recursive design. Then $m^{(1)}$ must be of the form.

$$g_1 z_1 + g_2 z_2 + \dots + g_{n_m} z_{n_m} \quad (5.75)$$

In order to fix the ratio of feedback gains this form is specialized to

$$\begin{aligned} m^{(1)} &= g_o (\delta_1 z_1 + \delta_2 z_2 + \dots + \delta_{n_m} z_{n_m}) \\ &= g_o \langle v^{(o)}, x \rangle, \text{ where } v^{(o)} = (\delta_1 v_1^{(o)} + \dots + \delta_{n_m} v_{n_m}^{(o)}) \end{aligned} \quad (5.76)$$

* In reality it is not necessary that each mode change its value. It is merely required that m have the form required for such a change.

where g_0 is a real r -dimensional vector to be determined, after the scalar weights δ are chosen to make $m^{(1)}$ real and independent of a set of states. Solution for the unknown parameters can be described by the following two step procedure.

Step 1. Determination of the δ_i , $i = 1, 2, \dots, n_m$.

As a consequence of property (A.6) the reciprocal basis vectors corresponding to complex conjugate pairs of modes are also complex conjugates. Therefore, to insure that $m^{(1)}$ is real the weights δ corresponding to such pairs must also be complex conjugates. The strategy for eliminating the feedback gains for a number of selected states is best illustrated by a specific example.

Let $n_m = r = 3$ and $\lambda_2 = \lambda_1^*$. Then

$$m^{(1)} = g_0 \langle v, x \rangle \quad (5.77)$$

$$= \begin{bmatrix} g_{10} \\ g_{20} \\ g_{30} \end{bmatrix} \langle 2\delta_1' \begin{bmatrix} v_{11}' \\ v_{21}' \\ \vdots \\ v_{n1}' \end{bmatrix} - 2\delta_1'' \begin{bmatrix} v_{11}'' \\ v_{21}'' \\ \vdots \\ v_{n1}'' \end{bmatrix} + \delta_3 \begin{bmatrix} v_{13} \\ v_{23} \\ \vdots \\ v_{n3} \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \rangle .$$

The weights δ_1' , δ_1'' and δ_3 are now chosen to make the components of v vanish which correspond to the states that are not to be measured. Since v_1 , v_2 , and v_3 are linearly

independent the rank of the matrix

$$\begin{bmatrix} v'_{11} & v''_{11} & v_{13} \\ v'_{21} & v''_{21} & v_{23} \\ \vdots & \vdots & \vdots \\ v'_{n1} & v''_{n1} & v_{n3} \end{bmatrix} \quad (5.78)$$

is three which implies that at least two components of v can be set equal to zero. To eliminate the measurement of the i^{th} and k^{th} states the following equations must be satisfied.

$$2\delta'_1 v'_{i1} - 2\delta''_1 v''_{i1} + \delta_3 v_{i3} = 0 \quad (5.79a)$$

$$2\delta'_1 v'_{k1} - 2\delta''_1 v''_{k1} + \delta_3 v_{k3} = 0. \quad (5.79b)$$

Step 2. Determination of the g_{i0} , $i = 1, 2, \dots, r$.

When control (5.76) is substituted into (5.2) application of the modal decomposition property makes it sufficient to consider the resulting submatrix

$$\bar{\Lambda}_{11}^{(o)} = \begin{bmatrix} \lambda_1 + \delta_1 \alpha_{10}^{(o)} & \delta_2 \alpha_{10}^{(o)} & \dots & \delta_{n_m} \alpha_{10}^{(o)} \\ \delta_1 \alpha_{20}^{(o)} & \lambda_2 + \delta_2 \alpha_{20}^{(o)} & \dots & \delta_{n_m} \alpha_{20}^{(o)} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_1 \alpha_{n_m 0}^{(o)} & \delta_2 \alpha_{n_m 0}^{(o)} & \dots & \lambda_{n_m} + \delta_{n_m} \alpha_{n_m 0}^{(o)} \end{bmatrix} \quad (5.80)$$

where $\alpha_{i0}^{(o)} = \langle p_i^{(o)}, g_o \rangle$.

It is shown in Appendix C that the characteristic equation of (5.80) is given by

$$s^{n_m} + f_1 s^{n_m-1} + f_2 s^{n_m-2} + \dots + f_{n_m} = 0 \quad (5.81)$$

$$\text{where } f_i = (-1)^i [P_i(\lambda) + \sum_{k=1}^{n_m} \delta_k \alpha_{ko} P_{k-1}(\lambda | \lambda_k)] .$$

The characteristic equation of the desired modes is

$$s^{n_m} - P_1(\gamma) s^{n_m-1} + P_2(\gamma) s^{n_m-2} + \dots + (-1)^{n_m} P_{n_m}(\gamma) = 0 . \quad (5.82)$$

Equating like powers of s in (5.81) and (5.82) yields the equations that the elements of g_o must satisfy.

$$RD^{(o)} = Q \quad (5.83)$$

where R and Q are defined in (4.27),

$$D^{(o)} = \begin{bmatrix} \delta_1 \alpha_{10}^{(o)} \\ \delta_2 \alpha_{20}^{(o)} \\ \vdots \\ \delta_{n_m} \alpha_{n_m}^{(o)} \end{bmatrix} = M^{(o)} g_o , \text{ and} \quad (5.84)$$

$$M^{(o)} = \begin{bmatrix} \delta_{1p_{11}}^{(o)} & \cdots & \delta_{1p_{1r}}^{(o)} \\ \delta_{2p_{21}}^{(o)} & \cdots & \delta_{2p_{2r}}^{(o)} \\ \vdots & & \vdots \\ \delta_{n_m p_{n_m 1}}^{(o)} & \cdots & \delta_{n_m p_{n_m r}}^{(o)} \end{bmatrix}$$

Proposition 5.85 If the first n_m modes are distinct and the rank of $M^{(o)}$ is n_m , then a vector g_o can be found such that the feedback control law (5.76) is independent of (n_m-1) states.

Proof: Recall (Appendix D) that the distinctness of the modes means that R has an inverse. The fact that $M^{(o)}$ is of full rank implies that by an appropriate interchange of the elements of $M^{(o)}$ and g_o the right hand side of (5.84) can be written as

$$M^{(o)} g_o = [M_{(1)} \mid M_{(2)}] \begin{bmatrix} g_{(1)} \\ - - - \\ g_{(2)} \end{bmatrix} \quad (5.86)$$

where the $(n_m \times n_m)$ submatrix $M_{(1)}$ is nonsingular.

Then the components of g_o which satisfy the required conditions are found from the relation

$$g_{(1)} = M_{(1)}^{-1} (R^{-1}Q - M_{(2)}g_{(2)}) \quad (5.87)$$

where the components of g_o in $g_{(2)}$ can be arbitrarily specified.

Here again, as shown in the previous chapter, it is not necessary to work with complex quantities. For example if $\lambda_2 = \lambda_1^*$ then (5.83) may be replaced by

$$\begin{bmatrix} 2\delta_1' & -2\delta_1'' & \dots & \delta_{n_m} \\ 2\text{Re}[\delta_1 P_1(\lambda|\lambda_1)] & -2\text{Im}[\delta_1 P_1(\lambda|\lambda_2)] & \dots & \delta_{n_m} P_1(\lambda|\lambda_{n_m}) \\ \vdots & & & \\ 2\text{Re}[\delta_1 P_{n_m-1}(\lambda|\lambda_1)] & -2\text{Im}[\delta_1 P_{n_m-1}(\lambda|\lambda_2)] & \dots & \delta_{n_m} P_{n_m-1}(\lambda|\lambda_{n_m}) \end{bmatrix} \cdot$$

$$\begin{bmatrix} \langle p_1^{(o)'} , g_o \rangle \\ \langle p_1^{(o)''} , g_o \rangle \\ \vdots \\ \langle p_{n_m}^{(o)} , g_o \rangle \end{bmatrix} = Q \quad (5.88)$$

After the first stage of the design is completed a feedback control law is obtained which moves the first n_m modes to desired locations while the others remain unchanged. In addition this control law is independent of n_m-1 states. The form of this control is given by

$$m^{(1)} = F^{(1)} x, \text{ where } F^{(1)} = g_o (\delta_1 v_1^{(o)} + \dots + \delta_{n_m} v_{n_m}^{(o)})^T \quad (5.89)$$

$F^{(1)}$ has a zero column corresponding to each component of x that does not have to be measured. If the final design takes σ stages, then

$$m_F = \sum_{i=1}^{\sigma} F^{(i)} x . \quad (5.90)$$

In this case it is only necessary that $F = \sum_{i=1}^{\sigma} F^{(i)}$ contains a zero column corresponding to each component of x that does not have to be measured.

CHAPTER VI

SYSTEMS WITH REPEATED MODES

A. Introduction

Up to now there has been no restriction on the systems under consideration with respect to controllability. For both the derivation of the shifting algorithms and the characterization of controllability the Jordan canonical form proves to be a useful system representation. In the case of a system with distinct modes, with the aid of the modal decomposition property, only those modes being changed actually need to be considered. The other modes, controllable or not, do not influence the procedure.

When repeated modes exist, some of which are not controllable, the analysis is greatly simplified if the system is first transformed into a reduced form isolating the controllable modes. Kalman[18] shows that any system with n_c controllable modes may be put into the form shown in (2.4).

The controllable modes of the system correspond to the eigenvalues of A_{11} . For the mode shifting algorithms only a portion of the system given by (2.4), i.e.

$$\dot{\bar{x}}_{11} = \bar{A}_{11} \bar{x}_{11} + \bar{C}_{11} m \quad (6.1)$$

is of value. This portion of the system which is c.c. will be called the controllable form of the system.

It is assumed that throughout the rest of this chapter that the discussion pertains to the controllable form of the system. For notational convenience \bar{x}_{11} , \bar{A}_{11} , and \bar{C}_{11} are replaced by x , A , and C , respectively.

B. Shifting of Repeated Modes

1. Phase I Approach

If it is suspected that the system matrix A has repeated modes, first precondition the matrix. This preconditioning, or Phase I, of the design is easily accomplished by choosing an arbitrary initial control

$$m^{(1)} = F^{(1)}x \quad . \quad (6.2)$$

Generally a sparse $F^{(1)}$ with small elements is sufficient to cause the resulting matrix

$$A^{(1)} = A^{(0)} + CF^{(1)} \quad (6.3)$$

to have distinct roots. Therefore, the influence of $m^{(1)}$ on the total control law

$$m_F = \sum_{i=1}^{\sigma} F^{(i)}x \quad (6.4)$$

is insignificant. In the case of a system with a single input variable the control used in Phase I does not effect the final control at all. This is a consequence of Theorem (4.3) which states that the total control is only a function of the initial and final distribution of modes.

The Phase I approach is perfectly general, and may always be utilized. Except for very unusual systems, however, it is felt that the special case discussed in the next section complements the algorithms already presented for practical applications.

2. Special Case: One (Real) Double Mode

Let the Jordan canonical form of the system be given by

$$\dot{z} = J^{(o)} z + P^{(o)T} m \quad (6.5)$$

$$\text{where } J^{(o)} = V^{(o)T} A^{(o)} U^{(o)} = \begin{bmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & & & \\ & & \lambda_3 & & \\ & & & \ddots & \\ & & & & \lambda_n \end{bmatrix} .$$

For this system the $u_i^{(o)}$, $i = 1, 2, \dots, n$, of $U^{(o)}$ are not all true eigenvectors. These vectors satisfy the relations

$$A^{(o)} u_i^{(o)} = \lambda_i u_i^{(o)}, \quad i = 1, 3, 4, \dots, n \quad (6.6a)$$

$$A^{(o)} u_2^{(o)} = \lambda_1 u_2^{(o)} + u_1^{(o)}. \quad (6.6b)$$

The vectors $v_i^{(0)}$, $i = 1, 2, \dots, n$, are still defined by the relation

$$V^{(0)T} = U^{(0)-1} \quad (6.7)$$

A procedure for moving all of the modes simultaneously could be developed in the same manner as is presented in Section (4.C). This will not be derived here. Instead a recursive procedure will be employed in which the first iteration results in a system having distinct modes on which any of the previously described algorithms may be used. The derivation is broken up into the three familiar parts (control, resulting eigenvectors, and resulting reciprocal basis vectors).

a. Derivation of Control

The purpose of the control used in the first stage of the design is to alter the repeated modes so that the resulting system has the set of distinct modes $\{\gamma_1, \gamma_2, \lambda_3, \dots, \lambda_n\}$. To accomplish this let

$$m^{(1)} = g_1 z_1 + g_2 z_2 = g_1 \langle v_1^{(0)}, x \rangle + g_2 \langle v_2^{(0)}, x \rangle \quad (6.8)$$

Substitute (6.8) into (6.3) to get

$$\bar{J}^{(0)} = \left[\begin{array}{cc|c} \lambda_1 + \alpha_{11}^{(0)} & 1 + \alpha_{12}^{(0)} & 0 \\ \alpha_{21}^{(0)} & \lambda_1 + \alpha_{22}^{(0)} & \\ \hline \alpha_{31}^{(0)} & \alpha_{32}^{(0)} & \lambda_3 \\ \vdots & \vdots & \ddots \\ \alpha_{n1}^{(0)} & \alpha_{n2}^{(0)} & \lambda_n \end{array} \right] \quad (6.9)$$

where $\alpha_{ik}^{(0)} = \langle p_i^{(0)}, g_k \rangle$.

Again, as in previous derivations, the modal decomposition property makes it sufficient to consider the leading 2×2 submatrix to determine the required values of g_1 and g_2 . The characteristic equation of this leading submatrix is

$$s^2 - s(2\lambda_1 + \alpha_{11}^{(0)} + \alpha_{22}^{(0)}) + \lambda_1^2 + \lambda_1(\alpha_{11}^{(0)} + \alpha_{22}^{(0)}) + \alpha_{11}^{(0)}\alpha_{22}^{(0)} - \alpha_{21}^{(0)}(1 + \alpha_{12}^{(0)}) = 0 \quad (6.10)$$

Comparison with the desired equation

$$(s - \gamma_1)(s - \gamma_2) = s^2 - s(\gamma_1 + \gamma_2) + \gamma_1\gamma_2 = 0 \quad (6.11)$$

yields the relations that g_1 and g_2 must satisfy.

$$\alpha_{11}^{(0)} + \alpha_{22}^{(0)} = \epsilon_1 \quad (6.12a)$$

$$\lambda_1(\alpha_{11}^{(0)} + \alpha_{22}^{(0)}) + \alpha_{11}^{(0)}\alpha_{22}^{(0)} - \alpha_{21}^{(0)}(1 + \alpha_{12}^{(0)}) = \epsilon_2 \quad (6.12b)$$

where

$$\begin{aligned}\epsilon_1 &= \gamma_1 + \gamma_2 - 2\lambda_1 \\ \epsilon_2 &= \gamma_1 \gamma_2 - \lambda_1^2.\end{aligned}$$

The form of the equations (6.12) can be simplified by inserting (6.12a) into (6.12b) forming

$$\alpha_{11}^{(o)} \alpha_{22}^{(o)} - \alpha_{21}^{(o)} (1 + \alpha_{12}^{(o)}) = \epsilon_2 - \lambda_1 \epsilon_1. \quad (6.12b')$$

Special solutions of (6.12) can be obtained for the categories illustrated in Section (5.C). For example, let g_0 be a real vector fixing the ratios between the elements of the control vector. Replace g_1 and g_2 by $\delta_1 g_0$ and $\delta_2 g_0$. Then

$$m^{(1)} = g_0 \langle \delta_1 v_1^{(o)} + \delta_2 v_2^{(o)}, x \rangle \quad (6.13)$$

where the weights δ_1 and δ_2 are defined by

$$\begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} \alpha_{10}^{(o)} & \alpha_{20}^{(o)} \\ -\alpha_{20}^{(o)} & 0 \end{bmatrix}^{-1} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 - \lambda_1 \epsilon_1 \end{bmatrix}. \quad (6.14)$$

Note that for a single input system

$$m^{(1)} = \langle g_1 v_1^{(o)} + g_2 v_2^{(o)}, x \rangle \quad (6.15)$$

where

$$\begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \frac{1}{(p_2^{(o)})^2} \begin{bmatrix} 0 & -p_2^{(o)} \\ p_2^{(o)} & p_1^{(o)} \end{bmatrix} \begin{bmatrix} \epsilon_1 \\ \epsilon_2 - \lambda_1 \epsilon_1 \end{bmatrix}. \quad (6.16)$$

This checks with Lemma (3.7) which states that for complete controllability of the first two modes it is necessary and sufficient that $p_2^{(o)} \neq 0$.

b. Derivation of Updated Eigenvectors ($\gamma_1 \neq \gamma_2$)

It is easily verified that

$$u_i^{(1)} = u_i^{(o)}, \quad i = 3, 4, \dots, n. \quad (6.17)$$

The revised first eigenvector $u_1^{(1)}$ must satisfy

$$A^{(1)} u_1^{(1)} = \gamma_1 u_1^{(1)} \quad (6.18)$$

where

$$A^{(1)} = \left[A^{(o)} + \sum_{k=1}^r c_k (g_{k1} v_1^{(o)} + g_{k2} v_2^{(o)})^T \right] \quad (6.19)$$

Although $u_2^{(o)}$ is not a true eigenvector the columns of $U^{(o)}$ are linearly independent which allows $u_1^{(1)}$ and c_k to be represented as

$$u_1^{(1)} = \sum_{i=1}^n q_{1i} u_i^{(o)} \quad (6.20a)$$

$$c_k = \sum_{i=1}^n p_{ki} u_i^{(0)} \quad (6.20b)$$

Expanding (6.18) using (6.19) and (6.20) yields

$$\begin{aligned} q_{11} \lambda_1 u_1^{(0)} + q_{12} (u_1^{(0)} + \lambda_1 u_2^{(0)}) + \sum_{i=3}^n q_{1i} \lambda_i u_i^{(0)} + \sum_{i=1}^n \langle p_i, q_{11} g_1 + q_{12} g_2 \rangle u_i^{(0)} \\ = \gamma_1 \sum_{i=1}^n q_{1i} u_i^{(0)} \end{aligned} \quad (6.21)$$

From the independence of the $u_i^{(0)}$ it follows that

$$q_{11} \lambda_1 + q_{12} \langle p_1, q_{11} g_1 + q_{12} g_2 \rangle = \gamma_1 q_{11}, \quad i = 1 \quad (6.22a)$$

$$q_{12} \lambda_1 + \langle p_2, q_{11} g_1 + q_{12} g_2 \rangle = \gamma_1 q_{12}, \quad i = 2 \quad (6.22b)$$

$$q_{1i} \lambda_1 + \langle p_i, q_{11} g_1 + q_{12} g_2 \rangle = \gamma_1 q_{1i}, \quad i=3,4,\dots,n \quad (6.22c)$$

It can be shown that equations (6.22a) and (6.22b) are not independent, thus the ratio of q_{11} and q_{12} can be determined from either one. After selecting values for q_{11} and q_{12} the other q_{1i} are found from (6.22c).

The second revised eigenvector is represented as

$$u_2^{(1)} = \sum_{i=1}^n q_{2i} u_i^{(0)} \quad (6.23)$$

where the q_{2i} are obtained from equations (6.22) with γ_1 replaced by γ_2 .

c. Derivation of Updated Reciprocal Basis

Since $V^{(1)}$ is determined directly from $U^{(1)}$ the results of this section are identical to those already obtained in Section (5.C).

C. Numerical Example

For the purpose of illustration the two procedures described in this chapter are used to design a simple feedback control law.

Consider the system

$$\dot{x} = A^{(0)}x + cm \quad (6.24)$$

where $A^{(0)} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix}$, $c = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and

$$\det[sI - A^{(0)}] = (s + 2)^2 = 0.$$

A state variable feedback control law is to be found that yields a closed-loop system with modes at (-3) and (-4). The complete controllability of the system guarantees that such a control law exists. Furthermore Theorem (3.19) states that the law is unique.

Phase I Approach

An arbitrary first stage control law, $m^{(1)}$, is selected to yield a system with distinct modes. Let

$$m^{(1)} = F^{(1)}x \quad \text{where } F^{(1)} = [1 \ 1] . \quad (6.25)$$

Then

$$A^{(1)} = A^{(0)} + cF^{(1)} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} . \quad (6.26)$$

It also follows that

$$A^{(1)} = U^{(1)}\Lambda^{(1)}V^{(1)T} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} , \quad (6.27)$$

and

$$P^{(1)T} = \begin{bmatrix} p_1^{(1)} \\ p_2^{(1)} \end{bmatrix} = V^{(1)T}c = \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \quad (6.28)$$

From equation (5.36) it is found that $g_1 = -6$ and $g_2 = -2$.

Therefore

$$\begin{aligned} m^{(2)} &= F^{(2)}x & (6.29) \\ &= \langle -6v_1^{(1)} - 2v_2^{(1)}, x \rangle \\ &= [-6 \ -4] x \end{aligned}$$

The final design is given by

$$\begin{aligned} m_f &= m^{(1)} + m^{(2)} = (F^{(1)} + F^{(2)})x & (6.30) \\ &= [-5 \ -3] x . \end{aligned}$$

This control yields a closed-loop system matrix

$$A^{(f)} = \begin{bmatrix} -1 & 1 \\ -6 & -6 \end{bmatrix} \quad (6.31)$$

with a corresponding characteristic equation

$$\det[sI - A^{(f)}] = (s + 3)(s + 4) = 0. \quad (6.32)$$

The eigenvectors and reciprocal basis vectors corresponding to $A^{(f)}$ are just those obtained by updating $U^{(1)}$ and $V^{(1)}$.

From equations (5.65) it is found that $q_{11} = 1$, $q_{12} = -2$, $q_{21} = 1$, and $q_{22} = -\frac{3}{2}$. Therefore,

$$u_1^{(f)} = u_1^{(1)} - 2u_2^{(1)} = [-1 \quad 2]^T \quad (6.33a)$$

$$u_2^{(f)} = u_1^{(1)} - \frac{3}{2}u_2^{(1)} = \frac{1}{2}[-1 \quad 3]^T \quad (6.33b)$$

Equations (5.71) and (5.72) yield $r_{11} = -3$, $r_{12} = -2$, $r_{21} = 4$, and $r_{22} = 2$. Therefore,

$$v_1^{(f)} = -3v_1^{(1)} - 2v_2^{(1)} = [-3 \quad -1]^T \quad (6.34a)$$

$$v_2^{(f)} = 4v_1^{(1)} + 2v_2^{(1)} = [4 \quad 2]^T \quad (6.34b)$$

Direct Approach

Let the Jordan representation of (6.24) be given by

$$\dot{z} = J^{(1)}z + P^{(o)T}m \quad (6.35)$$

$$\text{where } J^{(o)} = V^{(o)T}A^{(o)}U^{(o)} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$$

$$P^{(o)T} = \begin{bmatrix} p_1^{(o)} \\ p_2^{(o)} \end{bmatrix} \quad V^{(o)T}c = \begin{bmatrix} 0 \\ 1 \end{bmatrix} .$$

Note that the vectors u_1 and u_2 satisfy (6.6). Hence only u_1 is a true eigenvector of $A^{(o)}$.

The solution of (6.16), ($g_1 = -2$, $g_2 = -3$), substituted into (6.15) yields the feedback control law

$$m^{(1)} = [-5 \quad -3] x \quad (6.36)$$

which checks with the result obtained in (6.30) by the Phase I Approach.

Eigenvectors and reciprocal basis vectors corresponding to the final system matrix derived by the methods in Section (6.B) are

$$u_1^{(1)} = u_1^{(0)} - u_2^{(0)} = [1 \quad -2]^T \quad (6.37a)$$

$$u_2^{(1)} = u_1^{(0)} - 2u_2^{(0)} = [1 \quad -3]^T \quad (6.37b)$$

$$v_1^{(1)} = 2v_1^{(0)} + v_2^{(0)} = [3 \quad 1]^T \quad (6.38a)$$

$$v_2^{(1)} = -v_1^{(0)} - v_2^{(0)} = [-2 \quad -1]^T \quad (6.38b)$$

Note that as a consequence of not requiring that the eigenvectors be normalized in a standard manner these results differ from those of (6.33) and (6.34) by a scale factor.

CHAPTER VII

A COMPARISON OF MODE SHIFTING TECHNIQUES

The idea of moving the modes of a system to improve its dynamical response is not new. Unfortunately, direct methods for achieving arbitrary distributions of the modes have only appeared quite recently in the literature. This chapter is devoted to a comparison of these methods to the technique developed in the body of this report. Only a brief description of each method is presented here to illustrate its salient features. For convenience the mode shifting technique developed in the preceding chapters is referred to as M.S.T.

A. Rosenbrock's Modal Control

Perhaps the first serious consideration of the use of mode shifting to aid in process control can be attributed to Rosenbrock [32]. The results of his work along with extensions to distributed parameter systems can also be found in [12]. An introduction to modal analysis with attention focussed on the theory and application of shifting a single real mode of a single-input system is given in [9]. The resulting control algorithm obtained in [9] is identical to the special case of M.S.T. contained in (5.13).

It is interesting to develop Rosenbrock's method in an evolutionary manner starting with an ideal system to explore its fundamental limitations. Imposing realistic constraints on the system complicates the analysis, and also severely limits the applicability of the method in some cases.

1. Ideal Case

Consider the system

$$\dot{x} = Ax + Cm . \quad (7.1)$$

Assume A is an $n \times n$ real matrix with distinct real eigenvalues

$$\lambda_i[A] , \quad i = 1, 2, \dots , n . \quad (7.2a)$$

$$\text{Let } \lambda_n[A] < \lambda_{n-1}[A] < \dots < \lambda_1[A] \leq 0 . \quad (7.2b)$$

This assumption is made for convenience, it is not essential.

$$\text{Choose } C = U_r = [u_1 : u_2 : \dots : u_r] , \quad (7.2c)$$

i.e. an $n \times r$ real matrix whose columns are the eigenvectors of A corresponding to the first r modes of the system.

The matrix C is chosen (assumption (7.2c)) to enable the first r real modes of the system to be made more negative. This has the effect of speeding up the transient response of the process. Let

$$m = K V_r^T x \quad (7.3)$$

where $K = \text{diagonal } [k_1, k_2, \dots, k_r]$, and

$$V_r = [v_1 : v_2 : \dots : v_r] \quad (\text{recall } \langle u_i, v_k \rangle = \delta_{ik})$$

Utilization of the feedback control law (7.3) yields the closed-loop system

$$\dot{x} = \bar{A} x, \quad \bar{A} = [A + U_r K V_r^T]. \quad (7.4)$$

It is easily verified that the modes of the closed-loop system are

$$\lambda_i[\bar{A}] = \lambda_i[A] + k_i, \quad i = 1, 2, \dots, r \quad (7.5a)$$

$$\lambda_i[\bar{A}] = \lambda_i[A], \quad i = r+1, \dots, n \quad (7.5b)$$

Hence the first (real) r modes can be moved to any real locations by appropriately specifying the elements of the diagonal matrix K .

An interesting feature occurring in the ideal case is that the open-loop and closed-loop system matrices have the same set of eigenvectors. This implies that the zeros of the elements in the corresponding transfer function matrices are identical. Actually the behavior of the system is similar to one consisting of n first order (for real λ) non-interacting loops. Unfortunately this property is lost when the matrix C can not be chosen arbitrarily.

Even in the ideal case, where the actuating matrix C can

arbitrarily be specified, several inherent drawbacks of this approach are evident.

(1) The eigenvalue, eigenvector and reciprocal basis vector corresponding to each mode moved must be computed. Compare this with M.S.T. in which it is not necessary to calculate the eigenvectors.

(2) Only r modes can be changed, where r is the number of control elements. In M.S.T. every mode can be arbitrarily changed by just one control element.

(3) An extension is possible to include systems with complex modes, however, only the real parts of the modes may be changed. Thus the time constant and damping ratio associated with these conjugate pairs are not independent. No such restriction occurs in M.S.T.

(4) The extension to general systems with repeated modes is difficult. Complications arise in the use of the Phase I approach of Section (6.B) because this approach randomly relocates the modes. If more than r modes are made dominant, then more than r control elements must be employed to shift them.

2. Fixed Actuating Matrix

Consider again the system of (7.1), but assume that the actuating matrix C is fixed by the physics of the plant. The

procedure consists of defining a pseudo actuating matrix D whose columns are composed of linear combinations of the physically available actuating vectors c_i , $i = 1, 2, \dots, r$. Let D be defined in the following manner

$$D = C(V_r^T C)^{-1} = C(P_r^T)^{-1}, \quad P_r = [p_1 : p_2 : \dots : p_r] \quad (7.6)$$

Note that if $C = U_r$, as in the ideal case, then $D = C$. The columns of D defined by (7.6) can be represented as

$$d_i = u_i + \sum_{k=r+1}^n \ell_{ik} u_k, \quad i = 1, 2, \dots, r \quad (7.7)$$

With m chosen as in (7.3), and connected so as to achieve the pseudo actuating matrix D , the effective closed-loop system is given by

$$\dot{x} = \bar{A} x, \quad \bar{A} = [A + DK V_r^T] \quad (7.8)$$

In order to display the effect of feedback on the modes \bar{A} is transformed to a simpler form via a similarity transformation. It follows readily that

$$V \bar{A} U = \left[\begin{array}{ccc|ccc} \lambda_1 + k_1 & & 0 & & & \\ & \ddots & & & & \\ 0 & & \lambda_r + k_r & & & \\ \hline & & & \lambda_{r+1} & & \\ X & & & & \ddots & \\ & & & & & \lambda_n \end{array} \right] \quad (7.9)$$

Therefore once again only the first r modes are changed by an amount determined by the matrix K . Inspection of (7.9) clearly indicates that the eigenvectors associated with the first r modes of the closed-loop system differ from those of the open-loop system. Thus the simple analogy with a system of n first order loops non-interacting loops is destroyed.

A serious difficulty with this approach is that a suitable pseudo actuating matrix D may fail to exist. Independence of the columns of C , i.e. the given actuating vectors, is required by the method. However, this condition is only necessary, not sufficient, for the existence of $(P_r)^{-1}$. Practically, one is really not concerned with the existence of a matrix inverse, but rather with the complications that arise from an ill conditioned matrix. In other words when the determinant of the matrix P_r is close to zero it implies that there will be great difficulty in physically implementing the control law because of the high gains required to synthesize the pseudo actuating matrix.

M.S.T. experiences no difficulty of the type described above. It is not even necessary to require that the columns of C be independent. Two examples are presented below to illustrate the difficulties of Rosenbrock's approach as compared to M.S.T.

Example 7.10

Consider a simple system of form (7.1) where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -10 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} . \quad (7.11)$$

A control law is derived which moves the first two modes to (-5) and leaves the mode at (-10) unchanged.

The first step of Rosenbrock's method is to determine the pseudo matrix D .

$$D = C(V_r^T C)^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^{-1} \quad (7.12)$$

Although the columns of C are independent the matrix D fails to exist. Hence this approach fails.

Application of algorithm (5.34) from M.S.T. generates a class of feedback controllers of the form

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad (7.13)$$

$$\text{where} \quad \begin{cases} g_{11} + g_{21} = -16 \\ g_{12} + g_{22} = 9 \end{cases} .$$

Any controller from this class yields the desired distribution of modes. In particular, the controller requiring the minimum largest absolute value of gain is given by a control vector with the equal elements

$$m_1 = m_2 = -8x_1 + 4.5x_2 . \quad (7.14)$$

Example 7.15

Let the matrix C of the previous example be changed to

$$C = \begin{bmatrix} 1.0 & 0.9 \\ 1.1 & 1.0 \\ 2.0 & 0.0 \end{bmatrix} . \quad (7.16)$$

The pseudo actuating matrix is defined by

$$D = C(V_r^T C)^{-1} = C \begin{bmatrix} 100 & -90 \\ -110 & 100 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 200 & -180 \end{bmatrix} \quad (7.17)$$

In order to move the first two modes of the pseudo system

$$\dot{x} = Ax + Dm \quad (7.18)$$

to(-5) the controller required is

$$\begin{bmatrix} m_1 \\ m_2 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4x_1 \\ -3x_2 \end{bmatrix}. \quad (7.19)$$

That is, the controller (7.19) accomplishes the desired task if m is fed into D . Therefore the effective d_i terminals must be synthesized by the rule of (7.17). Since the d_i and c_i are related by the equations

$$d_1 = 100c_1 - 110c_2 \quad (7.20a)$$

$$d_2 = -90c_1 + 100c_2 \quad (7.20b)$$

this requires large values of gain.

If M.S.T. is used to design the system the class of required controllers is again given by (7.13), but now the gains satisfy

$$\alpha_{11} = g_{11} + 0.9g_{12} = -16 \quad (7.21a)$$

$$\alpha_{22} = 1.1g_{12} + g_{22} = 9. \quad (7.21b)$$

Note that the solution given by M.S.T. is much less sensitive to changes in the actuating matrix than that of Rosenbrock's method. In addition the gains obtained for this particular example by M.S.T. are almost two orders of magnitude lower than those required

in the other method.

B. Methods Derived From Phase-Variable Canonical Form

A variety of mode shifting methods for single-input systems are derived either directly or indirectly from the phase-variable canonical representation of a system. Recall that if the system represented by

$$\dot{x} = Ax + Cm \quad (A_{n \times n}, C_{n \times 1}) \quad (7.22)$$

is completely controllable then a unique nonsingular transformation of state $x = T \hat{x}$ exists which transforms (7.22) to

$$\dot{\hat{x}} = \hat{A}\hat{x} + \hat{c}m \quad (7.23)$$

$$\text{where } \hat{A} = T^{-1}AT = \begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \\ -a_n & -a_{n-1} & & -a_1 \end{bmatrix}, \text{ and } \hat{c} = T^{-1}c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The characteristic polynomial of the system is

$$s^n + a_1 s^{n-1} + \dots + a_n. \quad (7.24)$$

Assume that the desired dynamics of the system are described by the characteristic polynomial

$$s^n + d_1 s^{n-1} + \dots + d_n. \quad (7.25)$$

Then the unique (in each representation) linear state variable feedback control law that realizes the desired dynamics is

$$m = Fx \quad \text{where} \quad F = [a_n - d_n, \dots, a_1 - d_1]T^{-1}. \quad (7.26)$$

Thus the information required to design a mode shifting feedback controller consists of the characteristic equations of the given and desired system, and the inverse of the transformation that takes the given system to the phase-variable canonical representation.

Some of the algorithms used to compute the feedback matrix F are described below to give a flavor of the type of operations involved. The algorithms appear to fall naturally into two groups: those that require a computation of the eigenvalues, and those that do not.

Group 1. Johnson and Wonham [14] show that for a completely controllable system with distinct eigenvalues the required inverse is given by

$$T^{-1} = MKV^T \quad (7.27)$$

where

$$M = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} \quad (\text{Vandermonde Matrix}) \quad (7.28)$$

$$V = [v_1 : v_2 : \dots : v_n] \quad (\text{reciprocal basis vectors}) \quad (7.29)$$

$$K = \begin{bmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & \ddots & \\ 0 & & & k_n \end{bmatrix} \quad \text{where} \quad (7.30)$$

$$\begin{bmatrix} k_1 \\ k_2 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} [M] \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_n \end{bmatrix} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad p_i = \langle v_i, c \rangle.$$

A procedure for deriving the transformation when repeated modes exist is presented by Mufti [27]. Chidambra [5] proposes methods which he believes are simpler than those of Johnson and Worham, or Mufti. Actually if the objective is just to alter the modes, then the additional complexities associated with repeated modes can be circumvented by the Phase I approach of Section (6.B).

This approach uses an initial control to create distinct modes, and does not effect the final design.

Group 2. Since the eigenvalues and eigenvectors of the system matrix are not explicitly related to the problem of constructing the transformation to phase-variable form many algorithms have been developed which avoid their computation. Johnson and Wonham [15] propose a method which generates both the transformation and its inverse. The coefficients of the characteristic equation are obtained directly from the results of the controllability test that they employ. Silverman [35] treats the time-varying case in a similar manner, and arrives at the identical result for time-invariant systems.

More efficient approaches for time-invariant systems have been developed by Bass and Gura [3], and Morgan [26]. These papers suggest that the Leverrier algorithm [10] be employed to compute the coefficients of the characteristic equation. (This algorithm is presented in Appendix G). Tuel [37] and Rane [31] independently discovered procedures that yield the same algorithm as that of Morgan.

Morgan generates the matrix T in the form

$$T = [t_1 : t_2 : \dots : t_n] \quad (7.31)$$

where $t_n = c$

$$t_{n-i} = R_i c, \quad i = 1, 2, \dots, n-1$$

$$R_i = A^i + a_1 A^{i-1} + \dots + a_i I = A R_{i-1} + a_i I.$$

Both the a_i and R_i can be obtained from the Leverrier algorithm.

Bass and Gura derive the feedback (row) vector F in (7.26) directly. Their result yields

$$F^T = \sum_{i=1}^n (a_{n-i+1} - d_{n-i+1}) (A^T)^{i-1} b \quad (7.32)$$

where $b^T = [0 \ 0 \ \dots \ 0 \ 1] [c \ A c \ \dots \ A^{n-1} c]^{-1}$.

It is impossible in comparing these techniques with M.S.T. to conclude that one of them is best for every application. The effectiveness of a technique in an actual design depends on many factors such as computing algorithms and facilities available, design objectives, and the type of information known and desired about the system. Certain features of the various approaches are described below to aid in characterizing their usefulness.

(1) One fundamental difference between the techniques presented above and M.S.T. is that M.S.T. does not require that the system be completely controllable. In using M.S.T. the controllable modes are readily discernible. A set of controllable

modes can be arbitrarily changed regardless of whether or not the remaining modes are controllable. Also the technique is not influenced by the existence of uncontrollable modes, except when they correspond to repeated modes whose Jordan blocks are not single elements. Equation (4.18) and its associated comments indicate the effect of uncontrollable modes on the system.

Before any of the methods above can be utilized it is necessary to first isolate the controllable modes of the system. This requires the additional computational burden of transforming the system representation to the controllable form of (2.4). In this case the control is not unique, and depends on the particular controllable form chosen.

It should be stressed that the problem is not only with the uncontrollable modes, but also with the difficulties caused by nearly uncontrollable modes. Complications arise in the phase-variable transformation from the fact that certain matrices closely related to a measure of controllability must be inverted.

(2) A disadvantage of the class of techniques discussed in this section is that the canonical transformation must be determined even if only one mode is to be changed. In M.S.T. the only information required to change one mode is its value and associated reciprocal basis vector. This information can be determined directly without having to calculate the characteristic equation

or any other eigenvectors. A tremendous savings in computation can thus be achieved when only a small number of modes must be moved, as for example in the stabilization of a large system.

(3) By their very nature the phase-variable techniques are limited to systems with only a single input, while M.S.T. is not. A pseudo single-input system may be formed by fixing the ratio of control elements as discussed in Section (4.D), but care must be taken to insure preservation of complete controllability. If the reciprocal basis vectors are available, then the simple test provided by Proposition (4.21) can be employed. Note that if only one mode, λ_i , is to be altered knowledge of v_i is sufficient to determine whether or not it is controllable in the pseudo system. However, in order to use the phase-variable transformation every mode of the pseudo system must be tested to insure complete controllability.

Another factor to consider in the design of a multi-input system by the 'fixed ratio of control elements' approach is the flexibility of the techniques. Before arriving at the final design it may be necessary to explore different ratios. In the methods of group one and M.S.T. changing the ratio of control elements is much easier to perform than in the methods of group two.

(4) In the design of constrained gain type systems, i.e. systems with limits set on the absolute values of the feedback

gains, M.S.T. offers a conceptual advantage over the phase-variable form techniques. One reason for this advantage is that the system representation used to design the required controller is intuitively appealing. In addition the transformation relating the original state space, in which the constraints are formulated, and the canonical state space is fixed by the system matrix A alone.

Recall that in representation (2.3) each component of m is a linear combination of the z_i , ($z_i = \langle v_i, x \rangle$), corresponding to the modes to be moved. Therefore to insure that no constraints are violated one merely has to monitor the linear combinations of the v_i under consideration. This can give valuable information in guiding the selection of the modes to move along with an idea of how far they can be moved while maintaining acceptable gains.

The phase-variable canonical form is really not a natural representation for the system if constraints are imposed upon the gains. Each component of m is composed of linear combinations of the rows of T^{-1} . Unfortunately, T is not only a function of A , but of the actuating vector also. Thus in exploring the effects of different actuating vectors the building blocks used to construct the control change. Another difficulty arises from the fact that a change in one mode alters every coefficient of the characteristic equation. Hence a linear combination of every row of T^{-1} must be incorporated into m making it very difficult to predict the effect of changing a mode on the values of required gains.

(5) Computationally the methods of group one require more labor than M.S.T. Actually these methods first transform the system to Jordan form. It can be seen from (7.27) that $\hat{x} = MKV^T x$, but since $z = V^T x$ it follows that

$$\hat{x} = \hat{T}z \quad (7.33)$$

$$\hat{T} = MK \quad (7.34)$$

Thus if the system is completely controllable the Jordan (distinct eigenvalue) representation can be transformed to phase-variable canonic form via the nonsingular state transformation (7.33).

Bass and Morgan recommend that the Leverrier algorithm be employed to compute the characteristic equation directly. If there is no desire to calculate the modes of the open-loop system, and all of the modes are to be changed, then this approach may yield the fastest design. However, if the modes are obtained, the eigenvectors and reciprocal basis vectors may be conveniently determined from the results of the Leverrier algorithm (see Appendix G), so that the more flexible M.S.T. can be employed. If only a small number of modes are to be changed there is no need to find the characteristic equation. Therefore, a more feasible approach would be to use M.S.T. along with an efficient algorithm which just computes the required information.

C. Method of Anderson and Luenberger

The method considered in this section, developed by Anderson

and Luenberger [1], is based on a generalization of the phase-variable canonical form. It is shown that for each completely controllable multiple-input system,

$$\dot{x} = Ax + Cm \quad (A_{n \times n} \text{ and } C_{n \times r}), \quad (7.35)$$

a nonsingular transformation of state,

$$x = T \hat{x}, \quad (7.36)$$

exists which allows it to be represented in a form similar to that described by (2.6). Once the system representation takes this form a feedback law to achieve a specified distribution of modes can be designed by inspection.

In general a multi-input system has many representations similar in structure to that of (2.6). Thus the design obtained for a desired distribution of modes depends not only on the initial and final mode locations, but also very strongly on the particular representation chosen.

Before discussing the relative merits of this method it is necessary to understand the mechanism by which the different transformations are generated. Therefore a brief outline of the procedure developed by Anderson and Luenberger is first presented. An interpretation of the method based on the spectral properties of the system is also discussed. It is felt that this alternate characterization yields much valuable insight into the problem of

selecting representations.

Anderson and Luenberger Approach

Consider the following array of vectors

$$\begin{array}{cccc}
 c_1 & Ac_1 & A^2c_1 & \dots & A^{n-1}c_1 \\
 c_2 & Ac_2 & A^2c_2 & \dots & A^{n-1}c_2 \\
 \vdots & \vdots & \vdots & & \vdots \\
 c_r & Ac_r & A^2c_r & \dots & A^{n-1}c_r \quad .
 \end{array} \tag{7.37}$$

Selecting elements from the first row, in order, let n_1 be the least integer i such that $A^i c_1$ is a linear combination of the earlier members of the sequence. Then there exist a set of α 's such that

$$A^{n_1} c_1 + \alpha_1 A^{n_1-1} c_1 + \dots + \alpha_{n_1} c_1 = 0 \tag{7.38}$$

Choose the last n_1 columns of the matrix T as

$$\begin{aligned}
 t_n &= c_1 \\
 t_{n-1} &= (A + \alpha_1 I) c_1 \\
 t_{n-2} &= (A^2 + \alpha_1 A + \alpha_2 I) c_1 \\
 \vdots & \\
 t_{n-n_1+1} &= (A^{n_1-1} + \alpha_1 A^{n_1-2} + \dots + \alpha_{n_1-1} I) c_1.
 \end{aligned} \tag{7.39}$$

Note the similarity between (7.39) and (7.31). If $n_1 = n$, then

the entire matrix T is determined. This indicates that the system is completely controllable with just the first component of control, m_1 , operating. In this case the canonical representation obtained is identical to that of (7.23) with $m = m_1$.

The selection above, which partially specifies a transformation T , imposes the following constraints on the canonical matrices \hat{A} and \hat{C} .

$$\hat{A} = \left[\begin{array}{c|cccc} & & & & \circ \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ \hline & 0 & 1 & \dots & 0 \\ & \vdots & \vdots & \ddots & \vdots \\ & -\alpha_{n_1} & -\alpha_{n_1-1} & \dots & -\alpha_1 \end{array} \right] \quad (7.40)$$

$$\hat{C} = \left[\begin{array}{c|c} 0 & \diagup \quad \diagdown \\ \vdots & \vdots \\ 0 & \diagdown \quad \diagup \\ \hline 1 & \end{array} \right]$$

More columns of T can be found by examining the second row of (7.37). Starting with c_2 proceed along the row until $A^{n_2} c_2$, which is the first vector of the second row dependent on the earlier sequence members and the vectors t_n, \dots, t_{n-n_1+1} . Therefore a linear combination of the vectors may be found such that

$$\hat{C} = \left[\begin{array}{cc|c} \vdots & \vdots & \vdots \\ \hline 0 & 0 & \vdots \\ \vdots & \vdots & \vdots \\ 0 & 0 & 1 \\ \hline \vdots & \vdots & \vdots \\ 0 & 0 & \vdots \\ 1 & 0 & \vdots \end{array} \right]$$

Note: the horizontal partitions in the matrix \hat{C} correspond to those in the matrix \hat{A} .

The complete controllability assumption allows the continuation of this process until the unknown part of \hat{A} disappears. Then \hat{A} is almost the direct sum of a number (say $\xi < r$) of companion matrices. The first ξ columns of \hat{C} are almost null except for an entry of unity in each column corresponding to the last row of a companion matrix.

The number of unity columns in \hat{C} , which determines the number of effective control components, is equal to the number of companion matrices in \hat{A} . This number is also equal to the number of rows of the array (7.37) whose elements go into the construction of T .

Assume that the completed construction of T yields

$$\hat{A} = T^{-1}AT = \begin{bmatrix} \hat{A}_\gamma & 0 & 0 \\ x & \hat{A}_\beta & 0 \\ x & x & \hat{A}_\alpha \end{bmatrix}, \quad \hat{C} = T^{-1}C = \begin{bmatrix} 0 & 0 & \vdots & 1 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 1 & \vdots & 0 \\ \vdots & 0 & 0 & 0 \\ 1 & \vdots & \vdots & \vdots \end{bmatrix} \quad (7.43)$$

where

$$\hat{A}_\alpha = \begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & 1 \\ -\alpha_{n_1} & -\alpha_{n_1-1} & \dots & -\alpha_1 \end{bmatrix}, \text{ etc. } (n = n_1 + n_2 + n_3)$$

The modes of \hat{A} are the roots of

$$\det[sI - \hat{A}] = 0, \quad (7.44)$$

but

$$\begin{aligned} \det[sI - \hat{A}] &= \det[sI - \hat{A}_\alpha] \det[sI - \hat{A}_\beta] \det[sI - \hat{A}_\gamma] \\ &= (s^{n_1 + \alpha_1} s^{n_1 - 1} + \dots + \alpha_{n_1}) (s^{n_2 + \beta_1} s^{n_2 - 1} + \dots + \beta_{n_2}) (s^{n_3 + \gamma_1} s^{n_3 - 1} + \dots \\ &\quad + \gamma_{n_3}) \end{aligned} \quad (7.45)$$

Therefore, the modes of the system can be controlled by using feedback to change the α , β , and γ .

Let the modes of the desired system be given by the roots of

$$\det[sI - \hat{A}^{\circ}] = (s^{n_1 + \alpha_1^{\circ}} s^{n_1 - 1} + \dots + \alpha_{n_1}^{\circ}) (s^{n_2 + \beta_1^{\circ}} s^{n_2 - 1} + \dots + \beta_{n_2}^{\circ}) (s^{n_3 + \gamma_1^{\circ}} s^{n_3 - 1} + \dots + \gamma_{n_3}^{\circ}) . \quad (7.46)$$

The feedback law which realizes these modes is

$$m = \begin{bmatrix} 0 & | & 0 & | & m_{\alpha} \\ 0 & | & m_{\beta} & | & 0 \\ m_{\gamma} & | & 0 & | & 0 \\ 0 & | & 0 & | & 0 \\ \vdots & | & \vdots & | & \vdots \\ 0 & | & 0 & | & 0 \end{bmatrix} T^{-1} x \quad (7.47)$$

where

$$m_{\alpha} = [\alpha_{n_1 - \alpha_1^{\circ}}, \alpha_{n_1 - 1 - \alpha_1^{\circ}}, \dots, \alpha_1 - \alpha_1^{\circ}] , \text{ etc.}$$

Spectral Interpretation

Assume that the eigenvalues of A are distinct. This implies that the eigenvectors of A are linearly independent and form a basis. Thus each column vector of C may be represented as

$$c_k = \sum_{i=1}^n p_{ki} u_i , \quad k = 1, 2, \dots, r \quad (7.48)$$

where $p_{ki} = \langle c_k, v_i \rangle$.

Paralleling the previous development first examine the

expansion of c_1 . Assume that only n_1 , for convenience let it be the first n_1 , of the parameters $p_{11}, p_{12}, \dots, p_{1r}$ differ from zero. This implies that the first actuating vector, c_1 , can only influence the first n_1 modes of A . Equation (7.38) may be written down immediately with the α_i determined by the relations.

$$\alpha_i = (-1)^i P_i(\lambda_1, \lambda_2, \dots, \lambda_{n_1}), \quad i = 1, 2, \dots, n_1, \quad (7.49)$$

where the P_i are defined in (C.11).

It follows that if $n_1 = n$, then c_1 can influence every mode of the system. The system is then completely controllable under the action of just the first component of control. In this case the α_i are the coefficients of the characteristic equation of A , and the canonical representation is identical to that of (7.23) with $m = m_1$.

If $n_1 < n$ the transformation T is only partially specified. The second column of C is then investigated. Three general cases arise depending on how the set of modes influenced by c_1 is related to the set of modes influenced by c_2 . These sets of modes are denoted by $\{\lambda^{(1)}\}$ and $\{\lambda^{(2)}\}$ respectively.

(i) $\{\lambda^{(2)}\} \subseteq \{\lambda^{(1)}\}$: In this case c_2 can be expressed as a linear combination of vectors defined in (7.39). Hence no additional columns of T are defined. In effect this means that

the method discards the second component of m . Note that if $c_2 = \sum_{i=1}^{n_0} p_{2i} u_i$ where $n_0 < n_1$, then by reversing the order of c_2 and c_1 , the form of the resulting canonical matrix would change. Instead of having one companion matrix containing the set of modes $\{\lambda_1, \lambda_2, \dots, \lambda_{n_1}\}$ it would have two smaller companion matrices corresponding to the sets $\{\lambda_1, \dots, \lambda_{n_0}\}$ and $\{\lambda_{n_0+1}, \dots, \lambda_{n_1}\}$.

(ii) $\{\lambda^{(1)}\} \cap \{\lambda^{(2)}\} = \emptyset$: When the sets of modes are disjoint the ordering of c_1 and c_2 only effects the ordering of their corresponding companion matrices in \hat{A} . Equation (7.41) can be written down immediately as

$$A^{n_2} c_2 + \beta_1 A^{n_2-1} c_2 + \dots + \beta_{n_2} c_2 = 0 \quad (7.50)$$

where $\beta_i = (-1)^i P_i(\lambda_{n_1+1}, \dots, \lambda_{n_1+n_2})$, $i = 1, 2, \dots, n_2$.

It follows that n_2 additional columns go into making up T , and an n_2 dimensional companion matrix containing the modes $(\lambda_{n_1+1}, \dots, \lambda_{n_1+n_2})$ is created in \hat{A} .

(iii) $\{\lambda^{(1)}\} \subset \{\lambda^{(2)}\}$: The only difference between the results of this case and those of (ii) is that the ω_i in (7.41) are different from zero. Effects of this difference are only noticeable in \hat{A} as shown in (7.42). The transformation T and \hat{C} are not influenced by the values of the ω_i .

Continuing the process of constructing T the same type of comments as above can be made. It is only necessary to compare the set of modes influenced by a particular column of C with the set of mode considered up to that point. Knowledge of the spectral properties of the system permits \hat{A} , \hat{C} , and T to be written down immediately.

The idea of using the spectral properties of the system to gain insight into the technique presented by Anderson and Luenberger is a development of this report. If this information is available it greatly extends the usefulness of their technique. However, it has already been shown that knowledge of the systems eigenvalues and reciprocal basis vectors is sufficient to enable one to use M.S.T. which is more flexible. Therefore, M.S.T. will be compared to the technique originally proposed by Anderson and Luenberger.

Many of the points for comparison are presented in the previous section. For these points discussion is limited primarily to the properties of the canonical form developed by Anderson and Luenberger.

(1) A necessary condition for achieving the representation in (2.6) is that the system be completely controllable. For uncontrollable systems the process of constructing the transformation matrix T comes to a halt before the matrix has n columns.

If the partially constructed T matrix is augmented with additional columns chosen to make it nonsingular, then it becomes a pseudo canonical transformation matrix. The corresponding pseudo canonical system matrix is similar in form to (7.42). All of the controllable modes of the system are contained in companion matrices. Thus control laws can be determined to change the controllable modes of the system.

(2) In general the canonical transformation must be determined even if only one controllable mode is to be moved. Actually once the companion matrix containing this mode is constrained to be part of \hat{A} by the columns of T already determined the rest of T can be filled up arbitrary. As long as a nonsingular T is constructed control laws may be found to move the modes contained in the companion matrices present in \hat{A} . In order to design certain modes into particular companion matrices it is necessary to know the spectral expansions of the columns of C .

(3) Although this technique is not limited to single-input systems it places very strong constraints on the use of the control elements. A component of control is only employed to alter the modes of its associated companion matrix.

The contents of the companion matrices depend on the ordering of the columns in C . If the first column of C can influence every mode, i.e. the pair $[A, c_1]$ is completely controllable, then

the canonical form reduces to a single-input system. The method delegates the control of a mode to the first column of C found that can influence it. This rules out many designs in which a combination of control elements alter the same modes. It also may create poor designs by having modes changed by components of control with little influence over them.

Another restriction is that each companion matrix of odd order must contain at least one real mode to preserve the realness of the system. In M.S.T. there is no analogous restriction. Therefore, any real pair of modes may be converted to a complex conjugate pair to achieve some design objective. An example is presented at the end of this section to indicate how helpful this may be.

(4) The corresponding comments of the previous section are directly applicable here.

(5) Assuming that the spectral approach is not taken the major computational difficulty associated with the method is the determination of linear dependence among a set of vectors. This occurs because the columns of T are composed of linear combinations of linearly independent elements of the array (7.37). A modification of the Gram-Schmidt orthogonalization procedure is proposed by the authors to alleviate this problem.

It should be noted that an algorithm like the Leverrier

algorithm can not be employed because the parameters associated with the companion matrices are not subsets of the coefficients of the characteristic equation of A . They are only equal in the case of single-input systems.

Two examples are presented to illustrate some problems with this technique.

Example 7.51

Consider the system

$$\dot{x} = Ax + Cm \quad (7.52)$$

where

$$A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, \quad C = \begin{bmatrix} 100 & 1 \\ 10 & 50 \\ 1 & 100 \end{bmatrix}.$$

Regardless of the ordering of the columns of C the effective canonical form has the single-input representation

$$\dot{\hat{x}} = \hat{A} \hat{x} + \hat{C} m \quad (7.53)$$

where

$$\hat{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},$$

and $m = m_1$ or m_2 depending on the T chosen. Obviously better designs could be obtained if a combination of m_1 and m_2 is used to shift the modes.

Example 7.54

Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{C}m \quad (7.55)$$

where

$$\mathbf{A} = \begin{bmatrix} -20 & -9 & -10 \\ 2 & -1 & 2 \\ 18 & 9 & 8 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -8 \\ -2 & 8 \\ -1 & 8 \end{bmatrix}.$$

It is found that

$$\mathbf{T} = \begin{bmatrix} -8 & 12 & 2 \\ 8 & -20 & -2 \\ 8 & -2 & -1 \end{bmatrix}, \quad \mathbf{T}^{-1} = \frac{1}{8} \begin{bmatrix} 2 & 1 & 2 \\ -1 & -1 & 0 \\ 18 & 10 & 8 \end{bmatrix} \quad (7.56)$$

$$\hat{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -20 & -12 \end{bmatrix}, \quad \hat{\mathbf{C}} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}. \quad (7.57)$$

Therefore the modes of \mathbf{A} are given by

$$(s+1)(s^2+12s+20) = (s+1)(s+2)(s+10) = 0. \quad (7.58)$$

Assume that the mode at (-10) is acceptable, but the modes at (-1) and (-2) must be moved to the complex conjugate locations $(-1 \pm j)$.

Because of the restriction in the Anderson and Luenberger

method only one control law can be found to yield the desired distribution of modes. This control law is given by

$$\begin{aligned}
 m &= \begin{bmatrix} 0 & 18 & 10 \\ -9 & 0 & 0 \end{bmatrix} T^{-1} x & (7.59) \\
 &= \begin{bmatrix} 20.25 & 10.25 & 10.0 \\ -2.25 & -1.125 & -2.25 \end{bmatrix} x
 \end{aligned}$$

In order to use M.S.T. it is necessary to calculate the reciprocal basis vectors corresponding to the modes (-1) and (-2). Such a set can easily be obtained, for example let

$$v_1 = (2 \ 1 \ 2)^T \quad (7.60a)$$

$$v_2 = (1 \ 0 \ 1)^T \quad (7.60b)$$

The algorithm given in (5.38), with $\delta_2 = -\delta_1$, yields the control law

$$m = \begin{bmatrix} 0 & -1.0 & 0 \\ 0 & -0.125 & 0 \end{bmatrix} x \quad (7.61)$$

This is just one of many feedback laws generated by M.S.T. which realize the required modes.

CHAPTER VIII

APPLICATIONS

The material presented in the previous chapters is essentially devoted to the development of modal techniques. These concepts form the basis for an effective tool for the analysis and design of large complex systems. The modal approach strives to fill the gap between classical control theory, which is primarily used to determine the settings of arbitrarily specified compensators, and modern control theory in which a rigid design is obtained as a consequence of specifying, at times, arbitrary performance criteria. It is hoped that the modal techniques will serve as a systematic guide for the selection of the control system structure which best meets the true design objectives.

This chapter is concerned with the various roles modal concepts play in the analysis and synthesis of systems. Many different aspects of the theory are discussed along with specific applications to actual design problems.

A. State Estimation

In the design of modern control systems it is usually assumed that the state variables of the plant are measurable. One of the few design procedures that explicitly accounts for the fact that

certain states may not be available is presented in Section (5.D). For some applications, however, it is necessary to obtain estimates of inaccessible states that are required by the desired control law.

Luenberger [22] shows how the available system inputs and outputs may be used to construct an estimate of the system state vector. The device which reconstructs the state vector is called an 'observer' or 'asymptotic estimator.' The observer itself is a time-invariant linear system, with arbitrary modes, driven by the inputs and outputs of the system it observes. It is assumed that there is no noise in the system.

It is shown that the modes of the overall system, i.e. the controlled plant and observer, are composed of the union of two sets of modes. One set corresponds to the controlled plant, and the other to the observer. The composite system essentially consists of two systems in cascade. Therefore a direct application of the decomposition property of Section (4.E) enables the shifting of the modes of the overall system to be broken up into two smaller subproblems.

The problem of constructing an observer with arbitrary dynamics is shown here to be the dual of the modal control problem. Hence any of the methods presented in the preceding chapters may be employed to design the overall system.

In order to illustrate the design of such a system consider a

plant described by

$$\dot{x} = Ax + Cm \quad , \quad (A_{n \times n} , C_{n \times r}) \quad (8.1a)$$

$$y = Hx \quad , \quad (H_{e \times n}) \quad (8.1b)$$

Assume that a linear feedback system is designed by putting $m = Fx$. If the state is measurable then the closed-loop plant would be governed by $\dot{x} = [A + CF]x$, so the eigenvalues of $[A + CF]$ are the modes of the closed-loop system.

If the states are not available an estimate of them must be determined. Let the observer be described by

$$\dot{z} = Bz + Dy + Cm \quad , \quad (B_{n \times n} , D_{n \times e}) \quad (8.2)$$

Since the state x is replaced by the estimate of it given by z equations (8.1a) and (8.2) can be rewritten as

$$\dot{x} = Ax + CFz \quad (8.3a)$$

$$\dot{z} = (B + CF)z + DHx \quad (8.3b)$$

Subtracting (8.3a) from (8.3b) yields

$$\dot{z} - \dot{x} = Bz - (A - DH)x \quad (8.4)$$

Let the system matrix of the observer be defined by

$$B = A - DH \quad (8.5)$$

Then (8.4) becomes

$$\dot{z} - \dot{x} = B(z - x) \quad (8.6)$$

with the solution

$$z(t) = x(t) + e^{Bt}(z(0) - x(0)) . \quad (8.7)$$

Hence the error in the estimate is a function of the difference in the initial states of the plant and observer, and the dynamical behavior of e^{Bt} . For a stable B matrix the estimate of x asymptotically approaches x . Note that as long as any input into the system is also fed into the observer the error in the estimate is not affected.

In order to clearly display the dynamics of the overall system described by (8.3) it is convenient to use the nonsingular transformation of state defined by

$$\begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} I & 0 \\ -I & I \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \quad (I \text{ is } nxn \text{ identity matrix}) \quad (8.8)$$

The new representation of the system is

$$\begin{bmatrix} \dot{e} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} B & 0 \\ CF & A+CF \end{bmatrix} \begin{bmatrix} e \\ x \end{bmatrix} . \quad (8.9)$$

This representation clearly illustrates that the modes of the overall system are composed of the modes of the observer, B , and

the modes of the controlled plant, $A + CF$. It is seen that the modes of the controlled plant are identical to those obtained when the true states are used in the feedback law. Thus the only effect of the observer is the addition of more modes to the system.

The modes of the observer are essentially arbitrary. In addition to effecting the dynamics of the overall system these modes determine the accuracy of the estimator as shown in (8.7). Since A and H are given as part of the plant the specification of D uniquely determines the observer through (8.5). Therefore it is important to select a matrix D which gives desirable properties to B . Presumably D would be chosen to insure the stability of the system, and make transients die out quickly.

The procedure for determining a matrix D to create arbitrary dynamics in B has already been treated. In some sense this is the dual of the control problem. Consider the system

$$\dot{e} = A^T e - H^T m . \quad (8.10)$$

If the pair (A, H) of the given plant is completely observable, then the system (8.10) is completely controllable. Assuming this to be true a feedback law $m = D^T e$ may be determined to create arbitrary modes in the closed-loop system

$$\dot{e} = (A^T - H^T D^T) e \doteq B^T e . \quad (8.11)$$

The system matrix of this system is the transpose of the system

matrix of the required observer, and thus has the same modes. Therefore the required matrix D has been found. An illustration of the overall system may be found in Figure 4.

Note that if the eigenvalues, eigenvectors, and reciprocal basis vectors of the given plant A are initially determined, this information may be used to design both the control system and observer.

B. Modelling

One of the basic problems confronting the design engineer is the large dimensionality of the models required for a complete system description. In many applications, however, a complete system description is unnecessary, and an empirically determined approximate model corresponding to the dominant part of the system response works quite well.

To avoid the empirical determination of approximate models Davison [6] proposes a direct analytical method for simplifying linear dynamic systems. The method is based on neglecting the modes of the system farthest from the origin. In the paper Davison raises the very important and difficult question of how to predict the error caused by using a reduced system, but does not attempt to answer it.

The model proposed by Davison is derived below by taking a slightly more general approach. It is felt that this approach gives more insight into the formulation of the reduced model. In addition

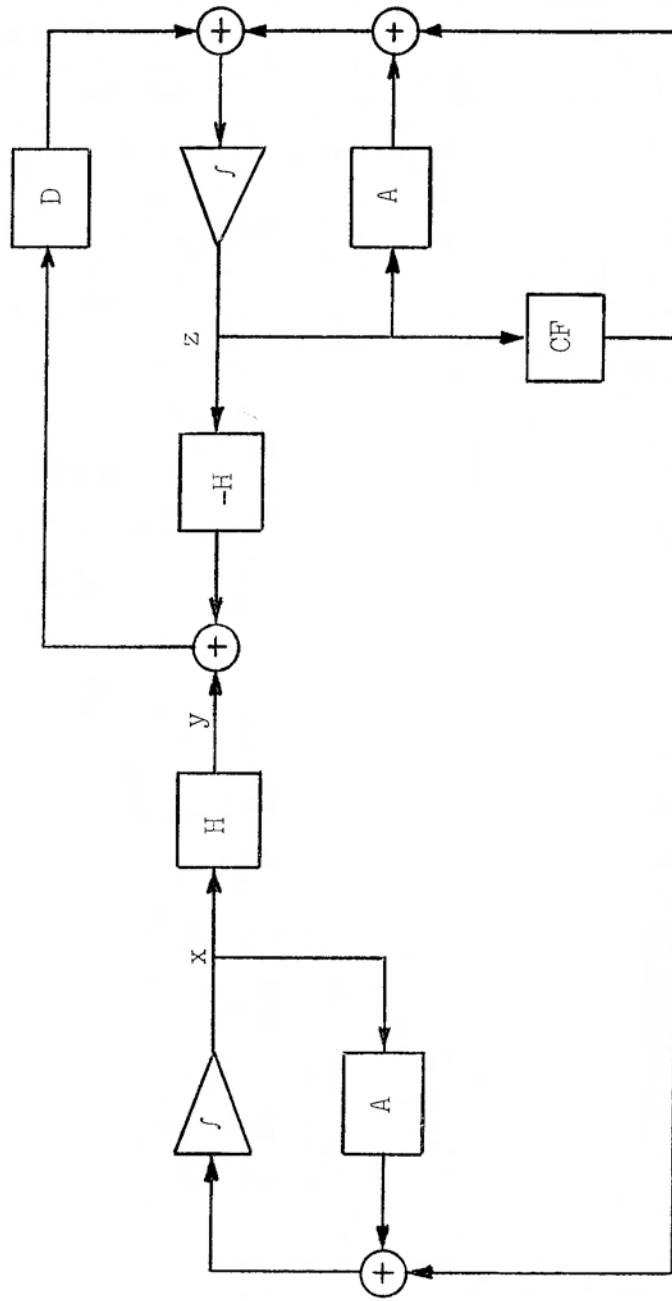


Figure 4. Controlled Plant and Observer

expressions for the errors incurred by employing the approximate model are derived along with suggestions for reducing them.

Consider the system described by

$$\dot{x} = Ax + Cm, \quad (A_{n \times n}, C_{n \times r}) \quad (8.12)$$

The canonical form of this system is

$$\dot{z} = Az + U^{-1}Cm, \quad (x = Uz) \quad (8.13)$$

where it is assumed that feedback has already been employed, if necessary, to isolate a set of d dominant modes so that

$$A = \text{diagonal} [\lambda_1, \lambda_2, \dots, \lambda_n] \quad (8.14)$$

where $\lambda_n < \dots < \lambda_{d+1} < \lambda_d < \dots < \lambda_1 < 0$.

It is desired to approximate the n^{th} order system (8.12) with the d^{th} order system

$$\dot{x}^d = \bar{A}x^d + \bar{C}m, \quad (\bar{A}_{d \times d}, \bar{C}_{d \times r}) \quad (8.15)$$

having the following properties

- (i) The components of the d -dimensional state vector form a subset of the components of the n -dimensional state vector x

- (ii) The modes of \bar{A} are $\lambda_1, \lambda_2, \dots, \lambda_d$, i.e. the dominant modes of A .
- (iii) The response of x^d in the reduced model is identical to the response of x^d in the original system when the higher order modes are not excited.

For notational simplicity assume that x^d is composed of the first d states of x , and define

$$x^d = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{bmatrix}, \quad x^s = \begin{bmatrix} x_{d+1} \\ x_{d+2} \\ \vdots \\ x_n \end{bmatrix}, \quad z^d = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_d \end{bmatrix}, \quad z^s = \begin{bmatrix} z_{d+1} \\ z_{d+2} \\ \vdots \\ z_n \end{bmatrix}. \quad (8.16)$$

The nonsingular transformation of state, i.e. U the matrix composed of the eigenvectors of A , which relates x and z can be partitioned to yield

$$\begin{bmatrix} x^d \\ x^s \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} z^d \\ z^s \end{bmatrix}. \quad (8.17)$$

The crux of the derivation is based upon the assumption that the contribution to the response of the higher order modes is insignificant compared to the contribution of the dominant modes, and therefore may be ignored. Mathematically this assumption takes the form of the equation

$$z^S(t) \equiv 0 . \quad (8.18)$$

Substituting (8.18) into (8.17) it is found that neglecting the higher order modes causes the suppressed states , x^S , to be linearly dependent on the states forming x^d . This relation is expressed by

$$x^S = U_{21}U_{11}^{-1} x^d . \quad (8.19)$$

Comment. There is no direct relationship between choosing the d dominant modes associated with the first d canonical state variables and the selection of states from x^d . The only restriction imposed on the choice of the elements of x^d is that the corresponding submatrix U_{11} be nonsingular. At least one such selection always exists. Therefore engineering judgment should be used to choose one that makes good physical sense.

The relation (8.19) is now used to find \bar{A} . Consider the free system in partitioned form, i. e.

$$\begin{bmatrix} \dot{x}^d \\ \dot{x}^S \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^d \\ x^S \end{bmatrix} . \quad (8.20)$$

It follows that

$$\dot{x}^d = A_{11}x^d + A_{12}x^S . \quad (8.21)$$

Substituting (8.19) into (8.21) and simplifying yields

$$\dot{x}^d = \bar{A} x^d \quad (8.22)$$

where $\bar{A} = [A_{11} + A_{12} U_{21} U_{11}^{-1}]$

In order to gain more insight into the formulation of \bar{A} consider the partitioned form of (A.9(i))

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \begin{bmatrix} \Lambda_d & 0 \\ 0 & \Lambda_s \end{bmatrix}. \quad (8.23)$$

The first partitioned row may be expanded to give

$$A_{11}U_{11} + A_{12}U_{21} = U_{11}\Lambda_d \quad (8.24)$$

Multiplying on the right by U_{11}^{-1} indicates that

$$\bar{A} = A_{11} + A_{12}U_{21}U_{11}^{-1} = U_{11}\Lambda_d U_{11}^{-1}. \quad (8.25)$$

The matrix \bar{C} of (8.15) may be determined from a consideration of the forced response of the canonical states in z^d . This response, obtained from (8.13) with $z^d(0) = 0$, is given as

$$z^d(t) = \int_0^t e^{\Lambda_d(t-\tau)} (U^{-1})_d C_m(\tau) d\tau \quad (8.26)$$

where $(U^{-1})_d$ denotes the first d rows of U^{-1} . Note also that from (A.4) it follows that

$$(U^{-1})_d = (V^T)_d = \begin{bmatrix} v_1^T \\ \vdots \\ v_d^T \end{bmatrix} . \quad (8.27)$$

Under assumption (8.18) the response of $x^d(t)$ can be written as

$$x^d(t) = U_{11} z^d(t) . \quad (8.28)$$

Therefore, with the aid of (8.26) and (8.28)

$$x^d(t) = \int_0^t e^{\bar{A}(t-\tau)} U_{11} (V^T)_d C m(\tau) d\tau . \quad (8.29)$$

The forced solution obtained from the reduced model (8.15) can be expressed as

$$x^d(t) = \int_0^t e^{\bar{A}(t-\tau)} \bar{C} m(\tau) d\tau . \quad (8.30)$$

Comparing (8.29) with (8.30) yields the desired relation

$$\bar{C} = U_{11} (V^T)_d C . \quad (8.31)$$

Recall that in previous chapters a matrix $P^T = V^T C$ is defined. Since $V^T = U^{-1}$ it follows that $C = U P^T$ or

$$c_k = \sum_{i=1}^n p_{ki} u_i , \quad k = 1, 2, \dots, r . \quad (8.32)$$

The p_{ki} are influence coefficients, and when properly normalized give a measure of the influence the k^{th} actuating vector, c_k , has over the i^{th} mode. With this notation expression (8.31) can be rewritten as

$$\bar{c} = U_{11}(P^T)_d . \quad (8.33)$$

An examination of (8.33) reveals that only that part of the actuating matrix coupled to the dominant modes is retained. Thus the steady state response, in addition to the transient response, associated with the higher order modes is absent in the response of the reduced system. The error in response caused by using the approximate model is now derived. It is convenient to invoke the property of superposition so that the free and forced response can be considered independently.

Free Response

(i) Original System

The free response of the first d states in system (8.12) is

$$(x(t))_d = \sum_{i=1}^n \langle v_i, x(0) \rangle e^{\lambda_i t} (u_i)_d \quad (8.34)$$

where $(\cdot)_d$ denotes the first d elements. With the eigenvalues distinct the eigenvectors, u_i , form a basis in n -dimensional space. Therefore, let the initial state be expanded as

$$x(o) = \sum_{i=1}^n \alpha_i u_i \quad (8.35)$$

Then,

$$(x(t))_d = \sum_{i=1}^n \alpha_i e^{\lambda_i t} (u_i)_d \quad (8.36)$$

(ii) Approximate Model

The free response of the approximate model (8.15) is

$$x^d(t) = \sum_{i=1}^d \langle v_i^d, (x(o))_d \rangle e^{\lambda_i t} u_i^d \quad (8.37)$$

where for $i = 1, 2, \dots, d$, the u_i^d are the columns of U_{11} , and the v_i^d are the rows of U_{11}^{-1} .

It follows from (8.35) that

$$x^d(o) = \sum_{i=1}^d \alpha_i u_i^d + \sum_{k=d+1}^n \alpha_k (u_k)_d \quad (8.38)$$

Since the u_i^d , $i = 1, 2, \dots, d$, are linearly independent they form a basis in d -dimensional space. Therefore, the following expansions are possible.

$$(u_k)_d = \sum_{i=1}^d \beta_{ki} u_i^d, \quad k = d+1, d+2, \dots, n \quad (8.39)$$

Substitute (8.39) into (8.38)

$$x^d(o) = \sum_{i=1}^d (\alpha_i + \sum_{k=d+1}^n \alpha_k \beta_{ki}) e^{\lambda_i t} u_i^d \quad (8.40)$$

(iii) Error in the Free Response

The instantaneous error is defined to be the absolute value of the difference between the model response, $x^d(t)$, and the response of the corresponding states in the original system, $(x(t))_d$.

$$\begin{aligned} \text{error} &= |x^d(t) - (x(t))_d| \\ &= \left| \sum_{i=1}^d \sum_{k=d+1}^n \alpha_k \beta_{ki} e^{\lambda_i t} u_i^d - \sum_{i=d+1}^n \alpha_i e^{\lambda_i t} (u_i)_d \right| \end{aligned} \quad (8.41)$$

where $\alpha_k \doteq \langle v_k, x(o) \rangle$

$$\beta_{ki} \doteq \langle (u_k)_d, v_i^d \rangle .$$

The second term is the "pure" error of the excited higher order modes which are neglected in the approximate system. Unfortunately when the higher order modes are excited an additional error is incurred as shown by the first term. This part of the error will not die out quickly since it involves the dominant modes. If the initial state, $x(o)$, lies along an eigenvector corresponding to a higher order mode, and this eigenvector lies close to a dominant eigenvector both α_k and β_{ki} will be large

causing a large error. Note that when none of the higher order modes are excited the error is zero because $\alpha_{d+1}, \dots, \alpha_n$ are zero.

Forced Response

(i) Original System

The forced response of the first d states in system (8.12) is

$$\begin{aligned} (x(t))_d &= \sum_{i=1}^n \int_0^t \langle v_i, C_m(\tau) \rangle e^{\lambda_i(t-\tau)} d\tau (u_i)_d \\ &= \sum_{i=1}^n \int_0^t \sum_{k=1}^r \langle v_i, c_{k^i} m_k(\tau) \rangle e^{\lambda_i(t-\tau)} d\tau (u_i)_d . \end{aligned} \quad (8.42)$$

Substitute (8.32) into (8.42)

$$(x(t))_d = \sum_{i=1}^n \int_0^t \sum_{k=1}^r p_{ki} m_k(\tau) e^{\lambda_i(t-\tau)} d\tau (u_i)_d . \quad (8.43)$$

(ii) Approximate Model

The forced response of the approximate model (8.15) is

$$\begin{aligned} x^d(t) &= \sum_{i=1}^d \int_0^t \langle v_i^d, \bar{C}_m(\tau) \rangle e^{\lambda_i(t-\tau)} d\tau u_i^d \\ &= \sum_{i=1}^d \int_0^t \sum_{k=1}^r \langle v_i^d, \bar{c}_{k^i} m_k(\tau) \rangle e^{\lambda_i(t-\tau)} d\tau u_i^d . \end{aligned} \quad (8.44)$$

Substitute (8.32) into (8.44)

$$x^d(t) = \sum_{i=1}^d \int_0^t \sum_{k=1}^r p_{ki} m_k(\tau) e^{\lambda_i(t-\tau)} d\tau u_i^d. \quad (8.45)$$

(iii) Error in Forced Response

Again only the error in the state components produced by the approximate model is considered.

$$\text{error} = |x^d(t) - (x(t))_d| \quad (8.46)$$

$$= \left| \sum_{i=d+1}^n \int_0^t \sum_{k=1}^r p_{ki} m_k(\tau) e^{\lambda_i(t-\tau)} d\tau (u_i)_d \right|$$

where $p_{ki} = \langle c_k, v_i \rangle$.

This error can also be written as

$$\text{error} = \left| \sum_{i=d+1}^n \int_0^t \langle v_i, C m(\tau) \rangle e^{\lambda_i(t-\tau)} d\tau (u_i)_d \right|. \quad (8.47)$$

In order to illustrate the resulting steady state error assume that each component of $m(t)$ is a unit step function. Then (8.46) yields

$$\text{error} = \left| \sum_{i=d+1}^n \sum_{k=1}^r \frac{p_{ki} (1 - e^{\lambda_i t})}{\lambda_i} (u_i)_d \right| \quad (8.48)$$

Since the higher order modes die out quickly the error (8.48) soon approaches the steady state error

$$(\text{error})_{\text{SS}} = \left| \sum_{i=d+1}^n \sum_{k=1}^r \frac{p_{ki}}{\lambda_i} (u_i)_d \right| . \quad (8.49)$$

Note that the p_{ki} are the same parameters employed throughout the report in the study of controllability and mode shifting. They were originally introduced in the Jordan canonical form, (2.2), and are utilized again in the definition of \bar{C} , (8.33). Hence when a dominate set of modes is created by means of mode shifting it is easy to get a quick check on the validity of the associated approximate model.

Geometrically, to reduce the error it is desired to make the λ_i , ($i = d+1, \dots, n$) very large, and the projections of the corresponding c_k on the v_i , i.e., p_{ki} , very small. Unfortunately these goals conflict with the additional goal of trying to keep the elements of the $(u_i)_d$, ($i = d+1, \dots, n$) small, making some compromise necessary.

C. Locating the Modes

This section is explicitly concerned with the problem of specifying mode locations. Previous sections have already touched upon the subject by providing guides into the selection of modes for designing a class of transfer function matrices, synthesizing a rapid asymptotic state estimator, and achieving greater accuracy from a reduced order dominant mode model of a large linear system. In general an exact specification of the mode locations is not

realistic because of the many other factors involved in a practical design. Therefore attention is focussed on specifying the mode locations in a qualitative way. The exact locations are best determined on an application by application basis.

The mode shifting approach forms the basis for an effective design tool. As in other techniques some trial and error must usually be incorporated before a final design is obtained. Also, like any other tool, it is not a panacea, and must be used judiciously. However, in its realm of application, it is felt that the additional insight obtained by taking this approach, in which the modes and feedback gains of the system are in direct view during the design, provides valuable guidance to the designer. Several different objectives are considered below to illustrate how the ideas associated with the mode shifting approach may be employed.

1. Stabilization

Consider a large complex plant described by

$$\dot{x} = Ax \quad , \quad (A \text{ is } nxn) \quad . \quad (8.50)$$

Control inputs may be conveniently applied to this plant at r available terminals. Thus in effect r manipulated control inputs can be employed which are coupled to the plant through their respective actuating vectors, c_k , ($k = 1, 2, \dots, r$) . Hence the complete system description is

$$\dot{x} = Ax + Cm, \quad (C \text{ is } n \times r). \quad (8.51)$$

Assume that the plant is unstable. This implies that at least one mode of A has a positive real part. Let λ_1 be the only unstable mode, and the objective be to shift λ_1 into the left half of the complex plane by employing feedback of the form $m = Fx$. It is assumed that only λ_1 is to be changed by m .

The rules for the mode shifting procedure can be listed as follows.

(i) Calculate λ_1 . Note that as a result of the modal decomposition property it is only necessary to find the mode that is to be changed regardless of the system's complexity. If the system has a special structure then additional savings in computation may occur as shown in Section (4.E)

(ii) Compute v_1 from $A^T v_1 = \lambda_1 v_1$. Normalization of v_1 is not necessary, but if the maximum element of v_1 is made equal to unity a quick check on the maximum gain required to shift λ_1 to γ_1 can be easily obtained.

(iii) Compute $p_{k1} = \langle c_k, v_1 \rangle$, $k = 1, 2, \dots, r$. Recall only those control variables, m_k , whose associated $p_{k1} \neq 0$ can be employed to shift λ_1 .

(iv) If only one control variable, say m_1 , is to be used then from (5.13)

$$m_i = \frac{(\gamma_1 - \lambda_1)}{p_{i1}} \langle v_1, x \rangle . \quad (8.52)$$

With v_1 normalized as suggested in (ii) the maximum gain is $(\gamma_1 - \lambda_1)/p_{i1}$. It is obvious that the control variable associated with the largest p_{i1} permits the greatest movement of λ_1 for a given maximum allowable gain. (See Section (5.B) for a discussion of "measure of controllability").

If the maximum gain required to achieve a desired γ_1 exceeds the allowable specifications then more than one control variable may be used to shift λ_1 . A general expression for m is given by (5.16) as

$$m = g_0 \frac{(\gamma_1 - \lambda_1)}{\langle p_1, g_0 \rangle} \langle v_1, x \rangle \quad (8.53)$$

where g_0 is an r -dimensional vector specifying the ratio of control elements. Letting g_{i0} equal zero indicates that m_i is not used. Appendix E shows that the particular set of ratios given by the rule

$$g_{k0} = (\text{sign } p_{k1}) , \quad k = 1, 2, \dots, r \quad (8.54)$$

minimizes the absolute value of gain required to achieve the desired γ_1 .

2. Classical Transient Criteria

In some design problems it is difficult to formulate a meaningful performance functional. It is common, however, for the designer to have a qualitative idea of the desired system behavior. When the transient response is of prime importance classical criteria as rise time, damping, time constant, etc., play a fundamental role in determining the acceptability of a system design.

A great deal of work has been performed to relate the transient response of a single-input single-output system to the pole-zero pattern of its transfer function. Such works include a paper by Elgred and Stephens [8] which considers the effects of closed-loop transfer function pole-zero locations on the transient response of a linear control system, and the classical paper of Graham and Lathrop [13] in which standard pole-zero patterns are proposed that satisfy various error criteria. For multi-input systems these results can be used to supply an initial specification of the mode distribution, and trial and error employed to compensate for the effect of the zeros if necessary. As a design aid the approximate model technique of Section (8.B) may be very helpful in locating the dominant modes to yield an acceptable response. In most cases the final design will depend on the actual numerical values of the plant parameters. The problem of specifying mode locations in conjunction with realizing them to yield a desired response in an actual plant is discussed by Wawrzyniak [39].

Two examples illustrating the modal technique are presented below to indicate the usefulness of the approach in the direct synthesis of a controller.

Example 8.55 Design of an Autopilot to Augment Lateral Stability

The goal is to design a constant linear feedback control system to produce the desired stability characteristics in the lateral motion of a high-performance aircraft. Interactions between the control deflections of the ailerons and rudder on the rolling and yawing moments of the aircraft generally complicate the design procedure. An advantage of the modal approach is that interactions are taken care of automatically. Both the structure and parameters of the controller are found which yield the desired distribution of modes.

A direct "brute force" method was used by Oehman and Suddath [28] to determine the required control law. The more general approach of Section (4.D) accomplishes the same task, and permits design parameters to be changed with greater ease. A brief description outlining the steps to the solution of the problem by the latter approach follows.

(i) Obtain a state representation of the system. For this system the state vector consists of the four quantities: angle of sideslip, angle of roll, and the derivatives of the angles of roll and yaw. Fortunately these states are obtainable from

measurable quantities of the aircraft.

(ii) Choose the control variable(s). The rudder and aileron deflections are available. It was decided to use a fixed ratio of control variables based on the relative authorities of the rudder and ailerons for augmentation purposes. The heuristic employed to derive this ratio is such that if the augments system is saturated by a large disturbance the ratio of the control deflections has the chosen value.

(iii) Determine the controllable modes. Proposition (4.21) provides a convenient check, and yields information (the α_{i0}) which is needed to shift the modes. Since every mode is controllable for the assumed ratio any distribution of modes may be realized in the controlled system.

(iv) Select the distribution of modes which yields the desired stability characteristics of the controlled system. The authors state that the modes were selected to satisfy military handling-qualities specifications found in [2].

(v) Determine the required control law by the method presented in Section (4.D).

The solution to the numerical problem considered is shown to yield satisfactory stability characteristics, and requires reasonable feedback gains. If changes have to be made designs

based on a different ratio of control elements or a different specification of modes would have to be considered. Such design changes are easily incorporated into the algorithm of Section (4.D). However, in the direct approach taken by Oehman and Suddath the inversion of an $n \times n$ (4×4 in this example) is required each time the control ratio is changed. Furthermore the compactness of the expression in (4.D), in which the control ratio is isolated, yields more insight into the effects of different ratios on the controller.

Example 8.56 Modal Control of a Boiler

The problem considered is the design of a steam pressure controller in a La Mont oil-fired boiler which will:

- (i) enable the system to respond quickly and smoothly to a step demand in steam flow, and
- (ii) maintain the steam pressure at a constant value.

Following the lead of Rosenbrock [32] the approach taken by Ellis and White [9] is to achieve the rapid response by modal control. Feedforward is used to eliminate steady-state offset in steam pressure.

The starting point for the design of the steam pressure controller is an eight state description of the boiler which already includes an air flow and fuel flow controller. It should

be noted that if integral action is necessary to achieve an acceptable steady state solution it may be incorporated into the system causing the number of states to increase by one. Naturally, to preserve the dynamic effects of the modal controller it is necessary to design it after the other feedback loops are connected to the plant.

In a boiler of this type the air flow-fuel flow ratio is an important factor in determining the efficiency of the boiler. Keeping this fact in mind the only reasonable control variable is the set point of the airflow controller. In addition to being an easily accessible input terminal of the system the application of the control signal at this point does not upset the air-fuel ratio. It is also found that the actuating vector associated with this control variable provides excellent control over the dominant mode.

An investigation of the modes of the system reveals that the slowest mode, -0.001926 , is nearly fifty times slower than the next mode, -0.09224 . Modal control is employed to move the first mode towards the second in an effort to speed up the response of the system. The method derived by Ellis and White to achieve the shift yields the control law of (5.13). Unfortunately the weighting vector is given by

$$\begin{aligned} \mathbf{v}^T = & [0.8083, 0.1200, -0.0070, 0.4041, -0.0018, \\ & -0.0066, 0.4108, -0.0086] \end{aligned} \quad (8.57)$$

and therefore requires the measurement of all eight state variables. Because of the low order of magnitude of several of the terms the control

$$m = g \langle v, x \rangle \quad (8.58)$$

is approximated by

$$\hat{m} = g(0.8083x_1 + 0.1200x_2 + 0.4041x_4 + 0.4108x_7) . \quad (8.59)$$

The state variables x_1 , x_2 , x_4 , and x_7 correspond to the drum pressure, range pressure, and integral action signals of the air and fuel flow controllers respectively. Measurements of these states are readily available, thus a feedback law of the type (8.59) is physically realizable.

Theoretically, if the v of (8.57) is used, the g required to move the first mode to the second is given by

$$g = \frac{\gamma_1^{-\lambda_1}}{p_1} = \frac{-0.09224 + 0.001926}{-0.4002} = 0.2258 . \quad (8.60)$$

Increasing the value of g up to 0.1 causes a corresponding increase in the speed of response. For $g = 0.1$, if the true v is used, the first mode moves to -0.04234 . However, since the approximate control (8.59) is employed, it is found by analog simulation that increasing g beyond 0.1 does not result in any significant improvement in response. This can be explained by the

fact that the approximate control law moves other modes in addition to the first, and does not move the first mode exactly as predicted.

Since this control problem has also been studied by analog and root locus techniques it provides an ideal example for comparing the various approaches. A comparison of the modal controller with the best conventional three-term steam pressure controller illustrates some of the advantages of modal control which are listed below.

(i) For the same disturbance input the maximum departure of the range pressure from its reference value is less when modal control is used, and the effect of the disturbance persists for a much shorter time.

(ii) Under modal control the steam flow has a much smoother, dead-beat response than the conventional controller.

Graphs are presented in the reference to pictorially illustrate these results.

3. Sensitivity

The sensitivity of the modes with respect to changes in the system parameters is considered in this section. A relation is derived which relates small deviations in the parameters to changes in the modes. In addition a procedure is presented to help guide

the selection of mode locations to achieve small sensitivities.

The modes in question are those of the controlled system described by the n^{th} order equation

$$\dot{x} = Ax . \quad (8.61)$$

This system may be derived by

(i) linearizing a nonlinear system about an equilibrium point, or

(ii) considering the open-loop system $\dot{x} = A^{(o)}x + Cm$ with the control $m = Fm^{\lambda}$. In this case $A = A^{(o)} + CF$.

Assume that A has distinct modes, and its spectral decomposition yields the eigenvalues, eigenvectors, and reciprocal basis vectors given by $(\lambda_1, \dots, \lambda_n), (u_1, \dots, u_n)$, and (v_1, \dots, v_n) respectively. The relation describing how the eigenvalues of A change when it takes on the increment ΔA appears in [10], and is presented below. Recall

$$Au_i = \lambda_i u_i . \quad (8.62)$$

Thus

$$[\Delta A]u_i + A(\Delta u_i) = \lambda_i(\Delta u_i) + (\Delta \lambda_i)u_i . \quad (8.63)$$

Multiplying through by v_i^T yields

$$v_i^T [\Delta A] u_i + v_i^T A (\Delta u_i) = \lambda_i v_i^T (\Delta u_i) + \Delta \lambda_i \quad (8.64)$$

since $v_i^T u_i = 1$. The final step uses the fact that $v_i^T A = \lambda_i v_i^T$ from (A.9) to get the result

$$\Delta \lambda_i = v_i^T [\Delta A] u_i \quad (8.65)$$

This result may be used to approximate the partial derivative of the eigenvalue λ_i with respect to a change in a parameter α by

$$\frac{\partial \lambda_i}{\partial \alpha} \approx \frac{\Delta \lambda_i}{\Delta \alpha} = \frac{v_i^T [\Delta A] u_i}{\Delta \alpha} \quad (8.66)$$

The sensitivity function, $S_\alpha^{\lambda_i}$, which is defined to be the ratio of the percentage change in λ_i to the percentage change in α assuming all other changes are zero can also be readily obtained. In particular,

$$S_\alpha^{\lambda_i} = \frac{\lambda_i \cdot \Delta \lambda_i / \lambda_i}{\Delta \alpha / \alpha} = \frac{\alpha v_i^T [\Delta A] u_i}{(\Delta \alpha) \lambda_i} \quad (8.67)$$

It is interesting to note that the quantities required to determine the sensitivity of the modes are sufficient to derive the control law to shift them (if they are controllable). Thus sensitivity analysis and mode shifting are two tools that may conveniently be used together. For example, if it is found that a

likely change in a system parameter will cause the system to go unstable the modes may be shifted to other locations which are either less sensitive to the expected change or deep enough into the left half of the complex plane to insure stability.

In many applications only approximate values are known for certain parameters. If the system is insensitive to these parameters then the nominal values may be adequate. Otherwise better estimates must be obtained or several case studies must be performed in order to obtain meaningful information about the behavior of the system.

To indicate the use of sensitivity concepts two general applications are discussed below.

Sensitivities of Large, Multiple-Loop Control Systems

An interesting study in sensitivity analysis, which also indicates the feasibility of the approach on large systems, is discussed by Van Ness, et al. [38]. The object of the investigation is to analyze the load-frequency control system of an interconnected power network. An example consisting of eight interconnected hydroelectric generating stations is considered. When linearized the system yields a 51st order equation of the form (8.61). The sensitivity of the eigenvalues with respect to 59 parameters is calculated to gain insight into the behavior of the system.

The authors claim that although traditional methods are not

helpful the sensitivity approach proves to be a very powerful tool for analysis. They cite results which agree very well with known facts about the system behavior. Other results indicate the existence of strong modal interaction between certain generators, i.e. some modes seem to be sensitive only to the parameters of certain sets of generators.

Sensitivity as a Guide for the Selection of Mode Locations

For a completely controllable single-input system

$$\dot{x} = A^{(o)}x + cm, \text{ (where } A^{(o)} \text{ has distinct modes)}$$

(8.68)

Morgan [26] derives sensitivity functions which relate the change in the location of any mode to a change in particular elements of the open-loop system matrix, $A^{(o)}$, or the actuating vector c . Unfortunately these sensitivity functions are rather complicated expressions which are functions of every mode and the transformation matrix, T defined in (7.31), which takes the representation (8.68) to phase-variable canonical form (2.5).

Morgan's results are listed below for completeness.

$$S_{a_{kl}}^{\lambda_i} = \frac{a_{kl}^{(o)} \langle f_l^T, \rho_i \rangle \langle \tau_i, g_k \rangle}{(\Delta a_{kl}^{(o)}) \lambda_i \left[\frac{d}{ds} \det[sI - A^{(o)}] \right]_{s=\lambda_i}} \quad (8.69)$$

$$S_{c_k}^{\lambda_i} = \frac{c_k \langle \tau_i, g_k \rangle \det[\lambda_i I - A^{(o)}]}{(\Delta c_k) \lambda_i \left[\frac{d}{ds} \det[sI - A^{(o)}] \right]_{s=\lambda_i}} \quad (8.70)$$

where

f_ℓ = the ℓ^{th} row of the matrix T ,

$\rho_i = [1, \lambda_i, \lambda_i^2, \dots, \lambda_i^{n-1}]^T$,

$\tau_i = [(-1)^{n-1} P_{n-1}(\lambda|\lambda_i), (-1)^{n-2} P_{n-2}(\lambda|\lambda_i), \dots, 1]$.

g_k = the k^{th} column of the matrix T^{-1}

λ_i = i^{th} mode of the closed-loop system.

It should be stressed that expressions of the type above serve only as a rough guide to the selection of the closed-loop modes. Some of the reasons why this approach is limited are based on the following facts.

(i) The derivation of (8.65) assumes that ΔA is a differential perturbation and hence is not accurate for large ΔA .

(ii) Although the assumed perturbation is arbitrary only one can be investigated at a time.

(iii) The sensitivity can not be specified arbitrarily because of conflicting factors, such as constraints on the gains.

In a multi-input system there is not in general a one to one

correspondence between the distribution of closed-loop modes and the feedback control law. Because of this additional freedom the use of sensitivity functions to determine the closed-loop mode locations is even more difficult to apply than in the single-input case. For this reason only a special problem is treated. This problem is motivated by stability considerations as discussed in Section (8.C.1) .

Consider the system

$$\dot{x} = A^{(o)}x + Cm \quad (8.71)$$

where the distinct eigenvalues of $A^{(o)}$ are $(\lambda_1, \lambda_2, \dots, \lambda_n)$. The problem is to use a "fixed ratio of control elements" controller to shift the mode λ_1 to a (positive) value γ_1 which yields a specified sensitivity for a given $\Delta A^{(o)}$.

Since only the first mode is to be shifted the control law takes the form

$$m = (\gamma_1 - \lambda_1)g_o \langle v_1^{(o)}, x \rangle \quad (8.72)$$

where

γ_1 is to be determined to yield the specified sensitivity

g_o is the assumed ratio of control elements

$$\alpha_{10} = \langle p_1^{(o)}, g_o \rangle = \sum_{k=1}^r \langle v_1^{(o)}, c_k \rangle g_{ko} = 1$$

Recall that

$$v_1^{(1)} = v_1^{(0)}, \text{ and} \quad (8.73)$$

$$u_1^{(1)} = (\gamma_1 - \lambda_1) \sum_{i=1}^n \frac{\alpha_{i0}}{\gamma_1 - \lambda_i} u_i^{(0)}. \quad (8.74)$$

The required sensitivity function is

$$S_{\alpha}^{\gamma_1} = \frac{\alpha v_1^{(1)T} [\Delta A^{(0)}] u_1^{(1)}}{(\Delta \alpha) \gamma_1} \quad (8.75)$$

where α , $(\Delta \alpha)$, $[\Delta A^{(0)}]$ and the value of $S_{\alpha}^{\gamma_1} (= S^0)$ are specified. New variables are now defined to isolate the effects of the variable parameters.

Define

$$e^T \doteq v_1^{(0)T} [A^{(0)}] \quad (8.76a)$$

$$\beta_i \doteq \langle e, u_i^{(0)} \rangle, \quad i = 1, 2, \dots, n. \quad (8.76b)$$

Then γ_1 must be chosen to satisfy $S_{\alpha}^{\gamma_1} = S^0$, where

$$\frac{(\Delta \alpha)}{\alpha} S_{\alpha}^{\gamma_1} = (\gamma_1 - \lambda_1) \sum_{i=1}^n \frac{\alpha_{i0} \beta_i}{\gamma_1 - \lambda_i}. \quad (8.77)$$

The solution to the above problem is not straightforward, and usually requires an iterative approach. However, since this serves only as a guide, the exact solution is not necessary. Once a nominal value of γ_1 is selected the effect on the sensitivity

of changes in the ratio of control elements can easily be obtained. Thus a guide to both the value of γ_1 and the ratio of controls to realize it satisfying some sensitivity criterion can be obtained.

4. Optimal Regulator: Quadratic Performance Index

Consider a completely controllable and completely observable plant ^{defined} discussed by

$$\dot{x} = Ax + Cm, \quad x(0) = x_0 \quad (8.78a)$$

$$y = Hx \quad (8.78b)$$

where A , C , and H are constant matrices of dimension $(n \times n)$, $(n \times r)$, and $(e \times n)$ respectively. Let the performance index be defined by

$$J(x_0, m) = \frac{1}{2} \int_0^{\infty} (y^T Q y + m^T R m) dt \quad (8.79)$$

where Q and R are appropriately selected positive definite weighting matrices. The optimal regulator problem can then be stated as, find a control m to minimize the performance index J for arbitrary x_0 .

The equations which generate the solution are well known, and are reviewed below. A Hamiltonian function is formed from which

the necessary conditions, which in this case are also sufficient, are obtained. Let the Hamiltonian function be defined as

$$H = \frac{1}{2}(x^T H^T Q H x + m^T R m) + \mu^T (Ax + Cm) \quad (8.80)$$

where μ is referred to as the adjoint, costate, or Lagrange multiplier vector. Then the necessary conditions for an optimum are

$$\frac{\partial H}{\partial m} = Rm + C^T \mu = 0 \quad (8.81a)$$

$$\dot{x} = \frac{\partial H}{\partial \mu} = Ax + Cm \quad (8.81b)$$

$$\dot{\mu} = -\frac{\partial H}{\partial x} = -H^T Q H x - A^T \mu \quad (8.81c)$$

Kalman [17] shows that the solution to the problem can be obtained by setting

$$\mu = Kx \quad (8.82)$$

where K is a symmetric ($n \times n$) matrix satisfying the matrix Riccati equation.

$$-\dot{K} = KA + A^T K - KCR^{-1}C^T K + H^T Q H. \quad (8.83)$$

A numerical solution to (8.83) can be computed by starting with $K(0) = 0$, and letting $t \rightarrow -\infty$. Then the optimal control law is obtained from (8.81a) as

$$\dot{m}^0 = Fx, \quad (\dot{F} = -R^{-1}C^TK) . \quad (8.84)$$

It should be noted that the system is really optimized for an impulse input. In other words J is minimized for an arbitrary initial state, x_0 , under the assumption that no inputs enter the system. Thus only the dynamical properties of the transient response of x and m are optimized. For step inputs there will be steady state offset errors.

The presence of steady state errors does not represent an inconsistency in the optimization theory because the performance index J is only guaranteed to be finite if the system is completely controllable. A step input has the effect of introducing an additional state, which is uncontrollable, into the representation of the system. If zero steady state error is required then integrators must be included in the system to be optimized or feedforward incorporated.

For single-input systems Kalman [19] rigorously considers the question of when linear control systems are optimal. The surprising result that optimality does not imply stability is shown for a class of systems that are not completely observable. More generally, even in the case of a completely observable system, the matrix Q must be chosen to make sure that the effect of undesirable modes is felt in the performance functional.

With the weighting matrices Q and R selected the regulator problem is completely specified, and the solution uniquely determined. In practice many sets of weighting matrices are chosen, and the corresponding optimal systems are evaluated to see if the responses are satisfactory. This requires that the matrix differential equation (8.83) be solved many times. In order to avoid the computational burden associated with the repeated solution of (8.83) the following approach may be taken to help select more desirable weighting matrices.

Taking the Laplace transform of equations (8.81), and using (8.81a) to eliminate m yields

$$\begin{bmatrix} sI - A & CR^{-1}C^T \\ -H^T Q H & -sI - A^T \end{bmatrix} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} = \begin{bmatrix} x(o) \\ -u(o) \end{bmatrix} . \quad (8.85)$$

Letov [21] shows that for completely controllable systems the modes of the optimal closed-loop system matrix

$$A^O = A + CF \quad (8.86)$$

are such that they satisfy

$$\Delta^O(s)\Delta^O(-s) = \det \begin{bmatrix} sI - A & CR^{-1}C^T \\ -H^T Q H & -sI - A^T \end{bmatrix} = 0 \quad (8.87)$$

where $\Delta^0(s) = \det[sI - A^0]$. After some algebraic manipulations it is found that the roots of (8.87) are given by

$$\det[I + R^{-1}T^T(-s)QT(s)] = 0 \quad (8.88)$$

where $T(s) = H[sI - A]^{-1}C$, i.e. the open-loop transfer matrix.

Rynaski and Whitbeck [33], and Tyler and Tuteur [36] employ a generalization of Chang's root square locus approach [4] as an aid in the selection of the weighting matrices to help create a desirable transient response. However, the results from this approach are rather limited. Rynaski and Whitbeck perform a series of nested root locus plots to obtain the parameters of Q and R , while Tyler and Tuteur consider the asymptotic properties of diagonal weighting matrices.

If the system has only a single-input then (8.87), with $R = 1$, reduces to

$$\Delta^0(s)\Delta^0(-s) = \Delta(s)\Delta(-s) + c^T\Gamma^T(-s)Q\Gamma(s)c, \quad (8.89)$$

where $\Gamma(s) = \text{adj}[sI - A]$. Since the system is completely controllable the location of the modes uniquely determines the required control law. Therefore the mode shifting technique may be used to derive the optimal control law without the necessity of solving the Riccati equation (8.83). Bass [3] employs this

approach utilizing his own mode shifting algorithm, and claims that it is more efficient than the Riccati approach.

A class of multi-input systems in which the mode shifting techniques yield an optimal solution are systems with the constraint of having a fixed ratio of control elements. An example of such a system has already been considered in Section (8.C.2). However, in that application the modes were not explicitly chosen to be optimal with respect to a performance functional.

Consider the system (8.78) again, but require that the elements of m be proportional to a specified vector g_0 . An effective single-input system can then be formed with C replaced by

$$\bar{c} = \sum_{k=1}^r g_{ko} c_k, \quad (g_{10} = 1). \quad (8.90)$$

It is assumed that g_0 is chosen to preserve the controllability of the system, i.e. the pair (A, \bar{c}) is completely controllable. Let the performance index be defined by

$$J = \frac{1}{2} \int_0^{\infty} (y^T Q y + m_1^2) dt. \quad (8.91)$$

Then the optimal location of the modes for this problem is obtained by solving (8.89) with c replaced by \bar{c} . These modes uniquely determine a control law, m , with the specified ratio of elements.

5. Heuristic for Constrained Gain Problem

In some control systems it is necessary to constrain the values of the feedback gains. These "constrained gain" problems may arise from power constraints, the desire to eliminate the need for certain measurements, or the fact that when certain values of gain are exceeded the linear model breaks down. One particular application in which the problem arises is cited by Rynaski, et al. [34].

"A problem of importance to optimal booster control system design is the so-called constrained gain problem."

One approach to the problem, when working with the optimal regulator formulation of the previous section, is to try many sets of weighting matrices until an acceptable design is obtained. This trial and error approach may be very costly because there are no known algorithms for adjusting the weighting matrices to reduce the resulting gains. Thus succeeding designs may not yield any improvement over the original.

Rynaski, et al. [34] try to solve the constrained gain problem analytically by formulating a revised optimal regulator problem. However, their results indicate that an analytical solution is unfeasible. As a feasible method they suggest a search on the gains of a solution obtained by the usual optimal regulator approach. The search is performed by simulating the controlled system on a

computer and varying the gains to minimize the performance index.

In general there are two serious problems associated with the above approach. First, the system is only optimized with respect to the initial state assumed in the simulation. Unlike a true variational approach the optimality of the system designed by such a procedure is strongly dependent on the initial state. Secondly, for large systems the feedback parameters may be strongly interacting making it difficult to adjust them to achieve a minimum of the performance index. Also there is no reason to believe that at the true minimum all of the constrained gains are at their upper boundaries.

A heuristic is now considered which allows the designer to shift the modes in a recursive manner while adjusting the feedback gains. The approach taken does not guarantee to produce a control law that satisfies the constraints on the gains. In fact, the method may be very difficult to apply in certain situations. For these reasons only a special case is considered to present the key idea.

For the sake of discussion assume that a feedback law has been obtained by some means as

$$m = Fx \quad (8.92)$$

$$\text{where } F = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ \vdots & \vdots & & \vdots \\ f_{r1} & f_{r2} & \dots & f_{rn} \end{bmatrix} .$$

Let the goal be to reduce some of the elements of F . The method considered here uses each element of m to adjust the corresponding row of F . Hence the problem can be considered as r independent problems. The k^{th} element of m is

$$m_k = \langle f_k, x \rangle \quad (8.93)$$

where f_k^T is the k^{th} row of F .

If additional feedback $\langle v, x \rangle$ is employed through the k^{th} actuating vector, then the effective k^{th} element of control is

$$\bar{m}_k = \langle \bar{f}_k, x \rangle \quad (8.94)$$

where $\bar{f}_k = f_k + v$.

Certainly if v is arbitrary then \bar{f}_k can be chosen to satisfy the constraints on the gains. However, an arbitrary v would in general not yield satisfactory response characteristics. In order to retain some control over the response of the system the additional feedback is selected to recursively change one real mode or a complex pair at a time.

The rule for moving the i^{th} real mode is given by (5.13)

as

$$m_k = \frac{\gamma_i - \lambda_i}{p_{ki}} \langle v_i, x \rangle . \quad (8.95)$$

After some algebraic manipulation, assuming that p'_{ki} , and p''_{ki} are not equal to zero, the rules for moving the complex pair of modes λ_i and λ_i^* are given by

$$m_k = \frac{2(\gamma'_i - \lambda'_i)}{p'_{ki}} \langle v'_i, x \rangle \quad (8.96a)$$

$$(\gamma''_i)^2 = (\lambda''_i)^2 + \frac{2\lambda''_i p''_{ki}}{p'_{ki}} (\gamma'_i - \lambda'_i) - (\gamma'_i - \lambda'_i)^2 \quad (8.96b)$$

or

$$m_k = \frac{2(\gamma'_i - \lambda'_i)}{p''_{ki}} \langle v''_i, x \rangle \quad (8.97a)$$

$$(\gamma''_i)^2 = (\lambda''_i)^2 + \frac{2\lambda''_i p'_{ki}}{p''_{ki}} (\lambda_i - \gamma_i) - (\lambda'_i - \gamma'_i)^2 \quad (8.97b)$$

depending on whether or not the real or imaginary part of v_i is used. Note that by requiring that only the real or imaginary part of v_i be used the new complex pair of modes does not have independently specified real and imaginary parts.

The first step of the procedure is to investigate the reciprocal basis vectors to see if one with a useful ratio can be found. The second step is to move its corresponding mode to create

the desired scale factor. If an acceptable vector can not be found the method can be extended to move several modes simultaneously. In this way a linear combination of the corresponding reciprocal basis vectors can be employed to change f_k in one step.

Example 8.98

Assume that the control law

$$m = 3x_1 + 2x_2$$

has been found to create the system

$$\dot{x} = Ax$$

$$\text{where } A = \begin{bmatrix} -2 & 1 \\ -2 & -4 \end{bmatrix} \text{ and } c = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

It may be verified that

$$\lambda_1 = \lambda_2^* = -3 + j, \quad p_1 = p_2^* = \frac{1}{2} + j$$

$$u_1 = u_2^* = \begin{bmatrix} -1 + 0j \\ 1 - j \end{bmatrix}, \quad v_1 = v_2^* = \frac{1}{2} \begin{bmatrix} -1 + j \\ 0 + j \end{bmatrix}.$$

Let the goal be to reduce the feedback coefficient of x_1 to unity. If additional control of the type (8.96a) is used then the increment to m is

$$(\gamma_1' - \lambda_1')[-2x_1 + 0x_2] .$$

Hence a solution is found by moving the real part of the complex pair to -2. The imaginary part of the revised modes has a magnitude of two. Thus

$$\bar{m} = x_1 + 2x_2 \quad \text{and} \quad \bar{A} = \begin{bmatrix} 0 & 1 \\ -8 & -4 \end{bmatrix} .$$

CHAPTER IX

SUMMARY AND SUGGESTIONS FOR FUTURE RESEARCH

In this final chapter the material in the thesis is summarized according to chapters, and extensions are suggested for future research.

A. Summary

The introductory chapter provides background by discussing the motivation that led to this research. A statement of the problem considered and the nature of the results obtained is also included.

Chapter 2 presents a mathematical description of the various representations of the class of systems considered.

Chapter 3 develops the essential modal theory. An alternate characterization of Kalman's concepts of controllability and observability is presented. With these new characterizations questions related to the controllability and observability of the modes are answered. The properties of existence and uniqueness of modal controllers are also established. In addition a special class of systems is considered which has the property that mode shifting may be performed without disturbing the zeros of the associated

transfer function matrix.

Chapter 4 develops the modal decomposition property that allows the selective movement of the modes. The basic mode shifting technique is derived which enables any number of controllable modes to be moved while the others remain fixed. A class of systems is considered for which a modal controller is easily obtained for the simultaneous movement of an arbitrary number of modes. Systems amenable to state decomposition, which ease the computational requirements associated with the design of modal controllers, are presented. Also an approximation is discussed which allows weakly coupled systems to effectively be decomposed.

Chapter 5 develops a powerful recursive design procedure which allows the control system to be designed by an evolutionary process. This procedure is recursive in the sense that a selected number of modes are moved to desired locations at each iteration while the others remain fixed. A method is also presented for designing modal controllers which do not require the measurement of the entire state vector.

Chapter 6 treats the case of repeated modes which for simplicity is not discussed in Chapters 4 and 5. A numerical example illustrates the procedure.

Chapter 7 compares the mode shifting technique developed in

this thesis to other methods for shifting modes. Some of the advantages of the new technique are illustrated by means of numerical examples.

Chapter 8 considers various applications of the modal concepts developed. It is shown that the design of an asymptotic state estimator (observer) is intimately related to the problem of modal control. The use of an observer to reconstruct the states when they are not accessible is also presented. A class of models, based on a dominate mode approximation, is considered to gain insight into their construction and behavior. In addition, the location of the modes is discussed with respect to stabilization, classical transient criteria, sensitivity, a quadratic performance index, and the constrained gain problem.

B. Suggestions for Future Research

The material of the preceding chapters develops a theory of modal control for the class of time-invariant linear dynamical systems. As mentioned in the introduction this research was undertaken with the hope that it would provide a framework for later work in multilevel theory. In particular, it is hoped that this technique will prove useful in the control of large multivariable systems. Some important areas for future study along these lines are listed below.

- (1) In the study of large multivariable systems it is

advantageous to employ some form of decomposition. Although some methods for decomposition are presented in this work they require special properties of the system. A general method for decomposition should be investigated. Perhaps the concept of aggregation can be employed to group states whose spectral decompositions are in some sense similar.

(2) Another method for reducing dimensionality is suggested by Pearson's model filtering approach. Work must be done to define what actually constitutes the dominant attributes of a system. The class of models discussed in the text uses the magnitude of the mode to define dominance. It is shown, however, that the spectral decomposition of the actuating matrix also plays a fundamental role in determining the effectiveness of the model.

(3) Once a system is decomposed the local controllers must be given goals. More work must be performed to relate the design objectives to the distribution of the modes and the means for achieving it when the control to do so is not unique. Several heuristics have already been suggested in this thesis to help answer this difficult problem.

(4) The concepts of modal controllability and observability are intuitively appealing and useful in the class of systems considered. A formulation of analogous concepts for different classes of systems would also be very useful.

APPENDIX A

The properties listed here are similar to those derived by Desoer [7]. Differences between the two results arise from different definitions of the scalar product. In this appendix the matrix A is restricted by the

A.1 Assumption. A is a real ($n \times n$) matrix with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$.

A.2 Eigenvectors, u_i , can be found by the defining relations $Au_i = \lambda_i u_i$, $i = 1, 2, \dots, n$ such that

- (i) λ_i real implies that u_i real, and
- (ii) $\lambda_\ell = \lambda_k^* = \lambda_k' - j\lambda_k''$ implies that $u_\ell = u_k^* = u_k' - ju_k''$.

Assume that such eigenvectors are chosen.

A.3 The eigenvectors, u_i ($i = 1, 2, \dots, n$), are linearly independent.

A.4 Let the $n \times n$ matrix $V = [v_1 : v_2 : \dots : v_n]$ be defined by $V^T = U^{-1}$ where $U = [u_1 : u_2 : \dots : u_n]$, i.e. the columns of U are eigenvectors of A .

A.5 The vectors v_k , $k = 1, 2, \dots, n$, form the reciprocal basis

to the u_i , $i = 1, 2, \dots, n$, and satisfy the relations

$$\langle u_i, v_k \rangle^\dagger = u_i^T v_k = \delta_{ik} \quad (i, k = 1, 2, \dots, n) .$$

A.6 Let the eigenvectors be defined as in (A.2), then

(i) λ_i real implies v_i real

(ii) $\lambda_\ell = \lambda_k^*$ implies $v_\ell = v_k^*$

A.7 Let $\lambda_2 = \lambda_1^*$, then for $k = 3, 4, \dots, n$

(i) λ_k real implies

$$\langle v_1', u_k \rangle = 0 \quad , \quad \langle v_1'', u_k \rangle = 0$$

(ii) λ_k complex implies

$$\langle v_1', u_k' \rangle = 0 \quad , \quad \langle v_1'', u_k' \rangle = 0$$

$$\langle v_1', u_k'' \rangle = 0 \quad , \quad \langle v_1'', u_k'' \rangle = 0 .$$

That is, the real vectors v_1' and v_1'' are orthogonal to the real and imaginary parts of every vector of the subspace spanned by u_3, u_4, \dots, u_n .

[†]Note that this scalar product is not the usual complex inner product ($\langle u_i, v_k \rangle \doteq u_i^{*T} v_k$). This form is more convenient for the development of the material presented in the body of the text.

A.8 Let $\lambda_2 = \lambda_1^*$, then

$$\langle v_1', u_1' \rangle = 1/2, \quad \langle v_1'', u_1' \rangle = 0$$

$$\langle v_1', u_1'' \rangle = 0, \quad \langle v_1'', u_1'' \rangle = -1/2.$$

Thus if v_i is replaced by $2v_i$ and the inner product $\langle v_i, u_k \rangle = v_i^{*T} u_k$ is used, then v_1' and v_1'' would constitute a reciprocal basis to u_1' and u_1'' , i.e.

$$\langle v_1', u_1' \rangle = 1, \quad \langle v_1'', u_1' \rangle = 0$$

$$\langle v_1', u_1'' \rangle = 0, \quad \langle v_1'', u_1'' \rangle = 1.$$

A.9 Let $\Lambda = \text{diagonal } [\lambda_1, \lambda_2, \dots, \lambda_n]$, then

$$(i) \quad AU = UA \quad \leftrightarrow \quad Au_i = \lambda_i u_i, \quad i = 1, 2, \dots, n$$

$$(ii) \quad A = UAV^T \quad \text{or} \quad \Lambda = V^T A U$$

$$(iii) \quad A^T V = V \Lambda \quad \leftrightarrow \quad A^T v_i = \lambda_i v_i, \quad i = 1, 2, \dots, n. \quad \text{Thus}$$

the reciprocal basis vectors are also eigenvectors of A^T .

$$A.10 \quad A = \sum_{i=1}^n \lambda_i u_i v_i^T$$

$$A.11 \quad e^{At} = \sum_{i=1}^n e^{\lambda_i t} u_i v_i^T$$

Thus the solution to $\dot{x} = Ax + Cm$, $x(0) = x_0$, can be expressed as

$$x(t) = \sum_{i=1}^n \left\{ \langle v_i, x_0 \rangle e^{\lambda_i t} + \int_0^t e^{\lambda_i(t-\tau)} \langle v_i, Cm(\tau) \rangle d\tau \right\} u_i$$

APPENDIX B

Some basic properties of determinants employed in the text are listed below.

B.1 The value of the determinant is unchanged if the rows and columns of the matrix are interchanged.

B.2 If two rows (or two columns) of a square matrix are interchanged, the sign of the determinant is changed.

B.3 If all elements of one row (or one column) of a square matrix are multiplied by a number k , the determinant is multiplied by k .

B.4 If each element in one row (or one column) is expressed as the sum of two terms, then the determinant is equal to the sum of two determinants, in each of which one of the two terms is deleted in each element of that row (or column).

B.5 If to the elements of any row (column) are added k times the corresponding elements of any other row (column), the determinant is unchanged.

B.6 Let $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$ where A_{11} and A_{22} are square,

then $\det A = (\det A_{11})(\det A_{22})$.

APPENDIX C

Denote the characteristic equation of the matrix

$$\begin{bmatrix} \lambda_1 + \delta_1 \alpha_1 & \delta_2 \alpha_1 & \dots & \delta_n \alpha_1 \\ \delta_1 \alpha_2 & \lambda_2 + \delta_2 \alpha_2 & & \delta_n \alpha_2 \\ \vdots & \vdots & & \vdots \\ \delta_1 \alpha_n & \delta_2 \alpha_n & & \lambda_n + \delta_n \alpha_n \end{bmatrix} \quad \text{by} \quad (\text{C.1})$$

$$s^n + f_1 s^{n-1} + f_2 s^{n-2} + \dots + f_n = 0. \quad (\text{C.2})$$

Note that the matrix (C.1) corresponds to the matrix used to define the f_i coefficients in (4.24) and (5.81) except for a slight simplification in notation.

In the derivation of (C.2) it is convenient to assume that $\alpha_n \neq 0$. The final result is independent of this assumption. Recall that a necessary condition for the mode λ_n not to be an eigenvalue of (C.1) is that $\delta_n \alpha_n \neq 0$.

Define

$$D_n = \det \begin{bmatrix} s-\lambda_1 - \delta_1 \alpha_1 & -\delta_2 \alpha_1 & \dots & -\delta_n \alpha_1 \\ -\delta_1 \alpha_2 & s-\lambda_2 - \delta_2 \alpha_2 & \dots & -\delta_n \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_1 \alpha_n & -\delta_2 \alpha_n & \dots & s-\lambda_n - \delta_n \alpha_n \end{bmatrix} \quad (C.3)$$

It follows from (B.4) that D_n may be written as

$$D_n = \det \begin{bmatrix} s-\lambda_1 - \delta_1 \alpha_1 & -\delta_2 \alpha_1 & \dots & -\delta_n \alpha_1 \\ -\delta_1 \alpha_2 & s-\lambda_2 - \delta_2 \alpha_2 & \dots & -\delta_n \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_1 \alpha_n & -\delta_2 \alpha_n & \dots & -\delta_n \alpha_n \end{bmatrix} + \quad (C.4)$$

$$\det \begin{bmatrix} s-\lambda_1 - \delta_1 \alpha_1 & -\delta_2 \alpha_1 & \dots & -\delta_{n-1} \alpha_1 & -\delta_n \alpha_1 \\ -\delta_1 \alpha_2 & s-\lambda_2 - \delta_2 \alpha_2 & \dots & -\delta_{n-1} \alpha_2 & -\delta_n \alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\delta_1 \alpha_{n-1} & -\delta_2 \alpha_{n-1} & s-\lambda_{n-1} - \delta_{n-1} \alpha_{n-1} & -\delta_n \alpha_{n-1} \\ 0 & 0 & 0 & s-\lambda_n \end{bmatrix}$$

With the aid of (B.5) the first determinant on the right hand side of (C.4) can be transformed into

$$\begin{bmatrix} s-\lambda_1 & 0 & \dots & 0 & 0 \\ 0 & s-\lambda_2 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & s-\lambda_{n-1} & \vdots \\ -\delta_1 \alpha_n & -\delta_2 \alpha_n & -\delta_{n-1} \alpha_n & -\delta_n \alpha_n \end{bmatrix} \quad (C.5)$$

by multiplying the n^{th} row by α_i/α_n , and subtracting it from the i^{th} row for $i=1,2,\dots,n-1$. Inspection of (C.4) and (C.5) then yields the recursive relation

$$D_n = (s-\lambda_1)(s-\lambda_2) \dots (s-\lambda_{n-1})(-\delta_n \alpha_n) + (s-\lambda_n)D_{n-1}. \quad (C.6)$$

The expression for D_{n-1} in terms of D_{n-2} can be obtained by

replacing n with $n-1$ in (C.6)). Substitution of this expression back into (C.6) yields

$$\begin{aligned}
 D_n &= (s-\lambda_1)(s-\lambda_2)\dots (s-\lambda_{n-1})(-\delta_n\alpha_n) + & (C.7) \\
 & (s-\lambda_1) \dots (s-\lambda_{n-2})(-\delta_{n-1}\alpha_{n-1})(s-\lambda_n) + \\
 & (s-\lambda_n)(s-\lambda_{n-1})D_{n-2} \quad .
 \end{aligned}$$

Continued recursion on (C.6) with the terminal condition

$$D_1 = s-\lambda_1-\delta_1\alpha_1 \quad (C.8)$$

finally yields

$$\begin{aligned}
 D_n &= (s-\lambda_1) \dots (s-\lambda_{n-1})(-\delta_n\alpha_n) + & (C.9) \\
 & (s-\lambda_1) \dots (s-\lambda_{n-2})(-\delta_{n-1}\alpha_{n-1})(s-\lambda_n) + \dots + \\
 & (-\delta_1\alpha_1)(s-\lambda_2) \dots (s-\lambda_n) + \\
 & (s-\lambda_1) \dots (s-\lambda_n) \quad .
 \end{aligned}$$

Expanding (C.9) and collecting like terms gives the desired polynomial

$$s^n + f_1s^{n-1} + f_2s^{n-2} + \dots + f_{n-1}s + f_n \quad (C.10)$$

where

$$f_i = (-1)^i [P_i(\lambda) + \sum_{k=1}^n \delta_k \alpha_k P_{k-1}(\lambda | \lambda_k)], \quad i = 1, 2, \dots, n.$$

The P functions are defined by

$P_i(\lambda_1, \lambda_2, \dots, \lambda_n)$ is the sum of the products, taken i at a time, of the elements from the set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$

(C.11a)

$P_i(\lambda)$ is a shortened notation which is used when the set of λ 's under discussion is clear.

(C.11b)

$P_i(\lambda | \lambda_k)$ denotes $P_i(\lambda)$ with $\lambda_k = 0$.

(C.11c)

$P_i(\lambda | \lambda_k, \lambda_\ell)$ denotes $P_i(\lambda)$ with $\lambda_k = \lambda_\ell = 0$.

(C.11d)

$P_0(\lambda) \doteq 1$.

(C.11e)

$P_i(\lambda) \doteq 0$, when i exceeds the number of elements in the set defined by λ .

(C.11f)

APPENDIX D-2

It is sufficient to prove that the matrix

$$R = \begin{bmatrix} 1 & 1 & \dots & 1 \\ P_1(\lambda|\lambda_1) & P_1(\lambda|\lambda_2) & \dots & P_1(\lambda|\lambda_n) \\ \vdots & \vdots & & \vdots \\ P_{n-1}(\lambda|\lambda_1) & P_{n-1}(\lambda|\lambda_2) & \dots & P_{n-1}(\lambda|\lambda_n) \end{bmatrix} \quad (D.1)$$

has an inverse if and only if the modes $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct. Note that to simplify the notation the set

$\{\lambda_1, \lambda_2, \dots, \lambda_{n_m}\}$ of (4.28) is replaced with the set

$\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

The proof is accomplished by performing a set of rank preserving operations on R which take it to a matrix (Vandermonde) that is known to be nonsingular if and only if the modes are distinct. The transformation takes place in $(n-1)$ steps. At each step a row of R , starting with the second, is made to match its counterpart in the Vandermonde matrix.

Step 1. Multiply the first row of R by $P_k(\lambda)$ and subtract it from the $(k+1)^{th}$ row for $k = 1, 2, \dots, n-1$. After utilizing relation (4.30) and changing the sign of rows two through n the

the remaining relation
 $P_i(\lambda|\lambda_k) = P_i(\lambda) - \lambda_k^{i-1} P_i(\lambda_k)$ $i = 1, 2, \dots, n, k = 1, 2, \dots, n$
 $P_0(\lambda|\lambda_k) = 1$
 and changing the sign of rows 2 through n the

resulting matrix is

$$\begin{bmatrix} 1 & \dots\dots\dots 1 \\ \lambda_1 & \dots\dots\dots \lambda_n \\ \lambda_1^{P_1}(\lambda|\lambda_1) & \dots\dots\dots \lambda_n^{P_1}(\lambda|\lambda_n) \\ \vdots & \vdots \\ \lambda_1^{P_{n-2}}(\lambda|\lambda_1) & \dots\dots\dots \lambda_n^{P_{n-2}}(\lambda|\lambda_n) \end{bmatrix} \tag{D.2}$$

Note that except for the first row each element of the matrix (D.2) is the product of the mode corresponding to that column and the element that was previously in the row above it.

Step 2. Multiply the second row of (D.2) by $P_k(\lambda)$ and subtract it from the $(k+2)^{th}$ row for $k = 1, 2, \dots, n-2$. After utilizing $\frac{1}{n-2}$ relation (4-30) and changing the sign of rows ~~three~~³ through n the resulting matrix is

$$\begin{bmatrix} 1 & \dots\dots\dots 1 \\ \lambda_1 & \dots\dots\dots \lambda_n \\ \lambda_1^2 & \dots\dots\dots \lambda_n^2 \\ \lambda_1^{2P_1}(\lambda|\lambda_1) & \dots\dots\dots \lambda_n^{2P_1}(\lambda|\lambda_n) \\ \vdots & \vdots \\ \lambda_1^{2P_{n-3}}(\lambda|\lambda_1) & \dots\dots\dots \lambda_n^{2P_{n-3}}(\lambda|\lambda_n) \end{bmatrix} \tag{D.3}$$

Continuing in this manner after the i^{th} step the resulting

matrix is given by/

$$\begin{array}{l}
 (i+1) \text{ row} \rightarrow \left[\begin{array}{ccc}
 1 & \dots & 1 \\
 \lambda_1 & \dots & \lambda_n \\
 \lambda_1^2 & \dots & \lambda_n^2 \\
 \vdots & & \vdots \\
 \lambda_1^i & \dots & \lambda_n^i \\
 \lambda_1^{i P_1}(\lambda|\lambda_1) & \dots & \lambda_n^{i P_1}(\lambda|\lambda_n) \\
 \vdots & & \vdots \\
 \lambda_1^{i P_{n-(i+1)}}(\lambda|\lambda_1) & \dots & \lambda_n^{i P_{n-(i+1)}}(\lambda|\lambda_n)
 \end{array} \right] \quad (3.4)
 \end{array}$$

Step (i+1). Multiply the $(i+1)^{\text{th}}$ row of (3.4) by $P_k(\lambda)$ and subtract it from the $(k+i+1)^{\text{th}}$ row of (3.4) for $k = 1, 2, \dots, (n-i-1)$. After utilizing ^{the recursion} relation (4.30) and changing the sign of rows $(i+2)$ through n the resulting matrix is

$$\begin{array}{l}
 (i+2) \text{ row} \rightarrow \left[\begin{array}{ccc}
 1 & \dots & 1 \\
 \lambda_1 & \dots & \lambda_n \\
 \lambda_1^2 & \dots & \lambda_n^2 \\
 \vdots & & \vdots \\
 \lambda_1^{i+1} & \dots & \lambda_n^{i+1} \\
 \lambda_1^{i+1 P_1}(\lambda|\lambda_1) & \dots & \lambda_n^{i+1 P_1}(\lambda|\lambda_n) \\
 \vdots & & \vdots \\
 \lambda_1^{i+1 P_{n-(i+2)}}(\lambda|\lambda_1) & \dots & \lambda_n^{i+1 P_{n-(i+2)}}(\lambda|\lambda_n)
 \end{array} \right] \quad (3.5)
 \end{array}$$

Finally after (n-1) steps the resulting matrix is given by the Vandermonde matrix

$$v_a = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{bmatrix} \quad (B.6)$$

The determinant of the Vandermonde matrix v_a is

$$\det[v_a] = \prod_{i>k} (\lambda_i - \lambda_k) \quad (B.7)$$

so that v_a is nonsingular if and only if the λ_i are all distinct.

As a result of properties ^{of determinants} (B.3) and (B.5) it follows that $\det[R] \neq 0$. Actually it can be shown that

$$\det[R] = (-1)^{\frac{n(n-1)}{2}} \det[v_a] \quad (B.8)$$

APPENDIX E

A typical element of the control vector (5.16) is

$$m_i^{(1)} = \frac{g_{i0}}{\langle p_1^{(0)}, g_0 \rangle} [(\gamma_1 - \lambda_1) \langle v_1^{(0)}, x \rangle], \quad i = 1, 2, \dots, r. \quad (\text{E.1})$$

Note that $\langle p_1^{(0)}, g_0 \rangle = \sum_{i=1}^r g_{i0} \langle v_1^{(0)}, c \rangle$.

Therefore $v_1^{(0)}$ can be multiplied by any arbitrary scalar without changing the feedback control law. Let $v_1^{(0)}$ be normalized so that $\langle v_1^{(0)}, v_1^{(0)} \rangle = 1$. Furthermore assume that $\|g_0\| = |g_{10}|$, i.e. the first element of g_0 has the largest absolute value.

The measure of controllability is then given by

$$\left| \frac{\langle p_1^{(0)}, g_0 \rangle}{g_{10}} \right|. \quad (\text{E.2})$$

It can be seen that this quantity is the absolute value of the inverse of the gain multiplying the fixed part of the first component of the control vector.

Obviously to maximize the measure of controllability, and hence minimize the absolute value of the required feedback gain it must be true that

$$g_{k0} = |\alpha_k| (\text{sign } p_{k1}^{(0)}) , k = 1, 2, \dots, r . \quad (\text{E.3})$$

It is to be shown that $\alpha_k = 1$, $k = 1, 2, \dots, r$.

With $g_{k0} = \text{sign } p_{k1}^{(0)}$, $k = 1, 2, \dots, r$, the expression in (E.2) reduces to

$$\sum_{k=1}^r |p_{k1}^{(0)}| . \quad (\text{E.4})$$

Since $|\alpha_1| \geq |\alpha_k|$, by assumption, it follows readily that

$$\sum_{k=1}^r |p_{k1}^{(0)}| \geq \frac{1}{|\alpha_1|} \sum_{k=1}^r |\alpha_k p_{k1}^{(0)}| . \quad (\text{E.5})$$

Hence the rule for obtaining the ratios given in (5.18) does indeed maximize the measure of controllability of the first mode.

APPENDIX F

$$\begin{aligned} \text{Recall } \lambda_2 &= \lambda_1^* = \lambda_1' - j\lambda_1'' \\ g_2 &= g_1^* = g_1' - jg_1'' \\ p_2 &= p_1^* = p_1' - jp_1'' \\ q_{12} &= q_{11}^* = q_{11}' - jq_{11}'' \\ \gamma_1 &\text{ is real .} \end{aligned}$$

Therefore,

$$(q_{11}g_1 + q_{12}g_2) = 2 \operatorname{Re}(q_{11}g_1) = 2(q_{11}'g_1' - q_{11}''g_1'') . \quad (\text{F.1})$$

Substitute (F.1) into (5.62) with $i = 2$

$$\langle (p_1' - jp_1''), 2(q_{11}'g_1' - q_{11}''g_1'') \rangle = (q_{11}' - jq_{11}'')(\gamma_1 - \lambda_1' + j\lambda_1'') . \quad (\text{F.2})$$

Decompose (F.2) into real and imaginary parts

$$2 \langle p_1', (q_{11}'g_1' - q_{11}''g_1'') \rangle = q_{11}'(\gamma_1 - \lambda_1') + q_{11}''\lambda_1'' \quad (\text{F.3a})$$

$$2 \langle p_1'', (q_{11}'g_1' - q_{11}''g_1'') \rangle = q_{11}''(\gamma_1 - \lambda_1') + q_{11}'\lambda_1'' \quad (\text{F.3b})$$

In matrix form

$$\begin{bmatrix} 2 \langle p_1', g_1' \rangle + \lambda_1' - \gamma_1 & -2 \langle p_1', g_1'' \rangle - \lambda_1'' \\ 2 \langle p_1'', g_1' \rangle + \lambda_1'' & -2 \langle p_1'', g_1'' \rangle + \lambda_1' - \gamma_1 \end{bmatrix} \begin{bmatrix} q_1' \\ q_1'' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (\text{F.4})$$

The determinant of the matrix in (F.4) is equal to the expression on the left side of equation (5.49) with $s = \gamma_1$. Hence equations (F.3) are dependent and an infinite set of solutions exist for q_1' and q_1'' . Therefore, q_{11}' can arbitrarily be set equal to unity, and q_{11}'' found from (F.3a).

One solution for (F.3) is then

$$q_{11} = q_{11}' + jq_{11}'' = 1 + j \left[\frac{\lambda_1' - \gamma_1 + 2 \langle p_1', g_1' \rangle}{\lambda_1'' + 2 \langle p_1', g_1'' \rangle} \right]. \quad (\text{F.5})$$

APPENDIX G

THE LEVERRIER ALGORITHM

A discussion of the Leverrier algorithm along with other computational procedures for finding the eigenvalues and eigenvectors of a matrix may be found in Faddeev [10]. The method presented below is actually a modification of Leverrier's algorithm proposed by D. K. Faddeev.

An example is also presented to illustrate how the eigenvectors and reciprocal basis vectors can easily be obtained when the eigenvalues are distinct.

The Algorithm

$$\text{Recall } [sI-A]^{-1} = \frac{\text{adj}[sI-A]}{\det[sI-A]} .$$

$$\text{Let } \det [sI-A] = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n$$

$$\text{adj } [sI-A] = Is^{n-1} + R_1 s^{n-2} + R_2 s^{n-3} + \dots + R_{n-1} .$$

The algorithm is, for $k = 1, 2, \dots, n$

$$a_k = -\left(\frac{1}{k}\right) \text{ trace } AR_{k-1} , \quad R_0 \doteq I$$

$$R_k = AR_{k-1} + a_k I .$$

The condition $R_n = 0$ is used as a check on the computation.

Example

$$\text{Let } A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

The Leverrier algorithm is used to verify the result

$$[sI-A]^{-1} = \frac{1}{s^2 + 2s + 5} \begin{bmatrix} s+1 & -4 \\ 1 & s+1 \end{bmatrix} .$$

Use of the algorithm yields

$$a_1 = -\text{trace } A = 2$$

$$R_1 = A + 2I = \begin{bmatrix} 1 & -4 \\ 1 & 1 \end{bmatrix}$$

$$a_2 = -\frac{1}{2} \text{ trace } [AR_1] = 5$$

$$R_2 = AR_1 + 5I = 0$$

$$\text{adj}[sI-A] = Is + R_1 = \begin{bmatrix} s+1 & -4 \\ 1 & s+1 \end{bmatrix}$$

$$\det[sI-A] = s^2 + a_1s + a_2 = s^2 + 2s + 5 .$$

The eigenvalues of A satisfy $\det[sI-A] = 0$, and are found to be $\lambda_1 = -1 - 2j$; and $\lambda_2 = \lambda_1^*$. The eigenvectors and

reciprocal basis vectors can now easily be obtained from $\text{adj}[sI-A]$.

Since the eigenvalues are distinct each column of $\text{adj}[\lambda_1 I-A]$ is proportional to u_1 and each row proportional to v_1 . In the same manner u_2 and v_2 could be obtained from $\text{adj}[\lambda_2 I-A]$, but (A.2) and (A.6) make this unnecessary.

$$\text{adj}[\lambda_1 I-A] = \begin{bmatrix} -2j & -4 \\ 1 & -2j \end{bmatrix} .$$

A set of eigenvectors and reciprocal basis vectors is given by

$$u_1 = u_2^* = \begin{bmatrix} -2j \\ 1 \end{bmatrix} , \quad v_1 = v_2^* = -\frac{1}{8} \begin{bmatrix} -2j \\ -4 \end{bmatrix} .$$

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