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EXACT SOLUTION TO LYAPUNOV'S EQUATION USING ALGEBRAIC METHODS

by

Theodore Euclid Djaferis

This report is based on the unaltered thesis of Theodore E. Djaferis, submitted in partial fulfillment of the degree of Master of Science at the Massachusetts Institute of Technology in January 1977. The research has been supported by ERDA under Grant ERDA-E(49-18)-2087. The computational work was done using the computer system MACSYMA developed by the Math Lab group at M.I.T. The Math Lab group is supported by the Defence Advanced Research Projects Agency, work order 2095, under Office of Naval Research Contract No. N00014-75-C-0661.

Electronic Systems Laboratory
Department of Electrical Engineering and Computer Science
Massachusetts Institute of Technology
Cambridge, Massachusetts 02139

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THEODORE EUCLID DJAFERIS

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ABSTRACT

Being able to obtain the solution to the Lyapunov Equation is very important in many areas of Control Theory. By applying the usual methods of solution only an approximate solution can be obtained. In this thesis we present algorithms for obtaining the exact solution to Lyapunov's Equation, using Algebraic Methods.

THESIS SUPPERVISOR: Sanjoy K. Mitter
TITLE: Professor of Electrical Engineering and Computer Science

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THEODORE EUCLID DJAFERIS

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Signature of Author ....Theodore Djaferis............
Department of Electrical Engineering and Computer Science
January 4, 1977

Certified by ....
Thesis Supervisor

Accepted by ....
Chairman, Departmental Committee on Graduate Students

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Chapter 1

Intruduction

1.1 General Remarks

In recent years there has been impressive progress in the theoretical understanding of the structure, representation and control of linear multivariable systems. In contrast,workers in the field have paid very little attention to the computational aspects of systems problems. This does not mean that algorithms for the solution of systems problems have not been developed. But most of the algorithms that have been proposed have never been seriously studied as far as stability, convergence and similar issues are concerned.

In this thesis we undertake a study of solution methods for Lyapunov's equation

$$PA + A'P = -Q \tag{1.1}$$

using the methods of modern algebra. The emphasis is on the use of finite algebraic procedures which are easily implemented on a digital computer and which lead to an explicit solution to the problem.

1.2 Importance of Equation

It is well known that this is an important equation in the study of stability of linear finite dimensional time-invariant systems. If Q is symmetric and positive definite and if A is a stability matrix (real parts of eigen-values of A strictly negative) then the unique positive definite solution to (1.1) is given by the convergent integral.

$$P = \int_0^\infty e^{A't} \cdot Q \cdot e^{At} dt. \tag{1.2}$$

[4]

In Optimal Control it is frequently desired to evaluate quadratic integrals of the form

$$J = \int_0^\infty x'(t) \cdot Q \cdot x(t) dt \tag{1.3}$$

under the constraint that x(t) satisfies

$$\dot{x}(t) = Ax(t) \qquad x(0) = c$$

If P is the solution of equation (1.1) we have that

$$J = c' \cdot P \cdot c. \tag{1.4}$$

Stochastic control is another area of importance in the evaluation of covariance matrices in filtering and estimation for continuous systems.

The need for solving this equation also arises when one uses Newton's Method to solve the Algebraic Riccati equation

$$PA + A'P + c'c - PBR^{-1}B'P = 0 \tag{1.5}$$

where R is positive definite.

If (A,B) is a controllable pair and (A,C) an observable pair then there exists a unique positive definite solution P to (1.5).

In [10] it is shown that if P_k, k=0, 1,2... is the unique positive definite solution of the linear algebraic matrix equation

$$A_k'P_k + P_kA_k + c'c + L_k'RL_k = 0 \tag{1.6}$$

where recursively,

$$L_k = R^{-1}B'P_{k-1} \qquad k = 1,2,\ldots$$

$$A_k = A - BL_k$$

where L_0 is chosen such that the matrix $A_0 = A - BL_0$ is a stability matrix then

i) $P \leq P_{k+1} \leq P_k \leq \ldots \qquad k = 0, 1, 2, \ldots$

ii) $\lim_{k \to \infty} P_k = P$

Equation (1.6) with $k = 0, 1, 2, \ldots$ is a Lyapunov equation.

1.3 Methods of Solution

The Lyapunov equation has many areas of application and therefore a great deal of effort has been put in both the theoretical as well as its computational aspects. There have been devised several methods of solution which can broadly be characterized as either Direct, Transformation or Numerical. An exposition accompanied by error analysis of several such methods is contained in [1, 2].

The basic drawback with such methods is the fact that the solution obtained is an approximate one. This becomes frustrating when the problem is ill-conditioned. Furthermore if a Riccati equation is to be solved which requires the solution of several Lyapunov equations the matter becomes even more complicated. Not only is the solution an approximate one but nothing is said about the accuracy of the approximation.

The need for improvement is quite evident and in certain cases demanded. In this thesis we have developed new algorithms for obtaining the exact solution of the Lyapunov equation.

1.4 Summary of Thesis

Let $A'P + PA = -Q$ be a Lypunov equation with A being a stability matrix and both A and Q n dimensional matrices with real entries. Let $R[x,y]$ be the ring of polynomials in x and y over the reals R, and $\mathcal{M}$ be the set of all nxn square matrices over the reals. The solution P of this equation is given by

$$P = f_A(q(x,y),Q)$$

where $q(x,y)$ in $\mathbf{R}[x,y]$

and $f_A: \mathbf{R}[x,y] \times \mathcal{M} \longrightarrow \mathcal{M}$ defined as

$$f_A(h(x,y),M) = \sum_{j,k} h_{jk} (A')^j \cdot M \cdot (A)^k$$

This method is based on an important paper by KALMAN [9]. Kalman's concern was the characterization of polynomials whose zeros lie in certain algebraic domains (and the unification of the ideas of Hermite and Lyapunov). In this thesis we clarify and complete some ideas contained in the paper and extend the results by showing that the same ideas lead to finite algorithms for the solution of Linear Matrix Equations.

The thesis is divided into four chapters. In chapter 2 we introduce the algebraic structure in which we will be working and provide proofs of several theorems related to a linear matrix equation. This chapter provides the basis for chapter 3 where the computational algorithms are presented. In chapter 4 we list the computer programs used in implementing the algorithms and present several numerical examples. In chapter 5 we present some generalizations and extensions.

Chapter 2

Algebraic Structure

2.1 Introduction

This chapter provides the theoretical basis on which our method for solving the Lyapunov Equation lies.

There are two main themes. The first one is the association of a unique matrix with every polynomial in R[x,y] and the notion of a positive polynomial. Lemmata (2.1),(2.2),(2.3) and part (iii) of Lemma (2.4) refer to this idea. The above four Lemmata are stated in section (2.2) but their proof is presented in Appendix A.

The second theme is that of the action f_A which is examined in section (2.3).

The above two themes are used in proving the two theorems in section (2.4), which are related to the Lyapunov Equation.

2.2 Four Lemmata from the Theory of Matrices and Polynomials

Let R be the field of real numbers R[x] the ring of polynomials in x over R and R[x,y] the ring of polynomials in x and y over R. The elements of R[x] are denoted as p(x) and the elements of R[x,y] as h(x,y). R[x] is a subring of R[x,y].

Suppose that p(x,y) is in R[x,y] and l(z) is the column vector

$$l(z) = \begin{bmatrix} 1 \\ z \\ \cdot \\ \cdot \\ \cdot \\ z^{n-1} \end{bmatrix}$$

where n is one plus the largest power of p(x,y), in either x or y.

Then we can write

$$p(x,y) = 1'(y)\cdot C(p)\cdot 1(x)$$

for some unique nxn matrix $C(p)=(a_{ij})$. (The element a_{ij} is the coefficient of the term $x^{j-1}\cdot y^{i-1}$ in $p(x,y)$). If n is allowed to take a value larger than the one defined above for any particular $p(x,y)$ the uniqueness of $C(p)$ is lost.

We therefore can associate a unique matrix $C(p)$ with any polynomial $p(x,y)$. The reason behind this association is the intent of assigning polynomials to value classes.

Definition 2.1. A polynomial $p(x,y)$ in $R[x,y]$ is positive if and only if $C(p)$ is (i) symmetric and (ii) positive definite.

Let Φ denote the ideal $(\varphi(x), \varphi(y))$ in $R[x,y]$.

$$\Phi = \left\{ g(x,y) \;\middle|\; \begin{array}{l} g(x,y) = a(x,y)\varphi(x)+b(x,y)\varphi(y) \text{ for any} \\ a(x,y), b(x,y) \text{ in } R[x,y] \end{array} \right\}$$

Let $R[x,y]/\Phi$ denote the associated quotient ring. The elements of $R[x,y]/\Phi$ will be thought of as cosets or as equivalence classes (whichever is more advantageous at a given situation) denoted as $\Phi+ p(x,y)$ or $[p(x,y)]$ respectively. We shall denote by $p(x,y)\bmod\Phi$ the polynomial of minimal degree in the equivalence class $[p(x,y)]$.

Let $R_m(x)$ denote the vector space over R of all polynomials of degree less than m in $R[x]$.

Lemma 2.1 . Let $p(x,y)$ be a polynomial in $R[x,y]$ with $C(p)$ being an mxm matrix . Then $p(x,y)$ is positive if and only if there exist polynomials $\pi_1(x),\ldots.\pi_m(x)$ such that

$$p(x,y) = \sum_{i=1}^{m} \pi_i(x)\pi_i(y)$$

where $\{\pi_i(x)\}$ are a basis for $R_m(x)$[1].

Definition 2.2. Two polynomials $a(x)$, $b(x)$ in $R[x]$ are called relatively prime if there exist polynomials $T_u(x)$ and $\lambda_u(x)$ such that $T_u(x)a(x) + \lambda_u(x)b(x) = u$ where u is a unit in $R[x]$.

Lemma 2.2. Let n be the degree of $\varphi(x)$. If $p(x,y) \bmod \Phi$ is positive of degree $n-1$ in both x and y then $(\sigma(x)\sigma(y)p(x,y)) \bmod \Phi$ is positive of degree $n-1$ in x and y, if and only if $\sigma(x)$ and $\varphi(x)$ are relatively prime.

Lemma 2.3. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be complex numbers which are distinct and have positive real parts. Then the nxn matrix $\Lambda_n = \left(\frac{1}{\lambda_i + \bar{\lambda}_j}\right)$ is hermitean ($\Lambda_n = \Lambda_n^*$ where (*) the hermitean adjoint) positive definite.

Definition 2.3. A polynomial $g(x,y)$ is called symmetric if $C(g)$ is a symmetric matrix.

A polynomial $g(x,y)$ is symmetric if $g^+(x,y) = g(x,y)$ where $g^+(x,y)$ is that polynomial obtained from $g(x,y)$ by interchanging x and y.

---

[1]Lemmata 2.1, 2.2 and 2.3 correspond to Lemmata 2, 3 and Main Lemma in [9] respectively, 2.1 and 2.3 being the same, with the idea of 3 being borrowed from KALMAN [9], to arrive at the statement of Lemma 2.2. Lemma 2.4 captures the essential idea of the Theorem in [9]. In Kalman's paper only sketches of proofs are given. Here we provide complete proofs.

Lemma 2.4. Let A be an nxn stability matrix with $\varphi_2(x) = \det(Ix-A)$ and let $\Phi = (\varphi_2(x), \varphi_2(y))$. Define

$$\varphi_1(x) = \varphi_2(-x) \tag{2.1}$$

$$P_\varphi(x,y) = \frac{\varphi_2(x)\cdot\varphi_2(y)-\varphi_1(x)\varphi_1(y)}{x+y} \tag{2.2}$$

i) Polynomials $\varphi_1(x)$, $\varphi_2(x)$ are relatively prime. That is there exist $T_u(x)$, $\lambda_u(x)$ in $R[x,y]$ such that

$$T_u(x)\varphi_1(x) + \lambda_u(x)\varphi_2(x) = u \tag{2.3}$$

where u is a unit in $R[x,y]$.

ii) $P_\varphi(x,y)$ is an element of $R[x,y]$

iii) Let $q_u(x,y) = T_u(x)T_u(y)P_\varphi(x,y) \bmod \Phi$ (2.4)

Then $q_u(x,y)$ is positive of degree n-1 in both x and y.

2.3 Defining the action f_A

Let A be some nxn matrix over R with $\varphi(x) = \det(Ix-A)$ being its characteristic polynomial. Let $\mathcal{M}$ be the set of all nxn matrices over R.

We define the action f_A: $R[x,y] \times \mathcal{M} \to \mathcal{M}$ in the following manner.

$$f_A(h(x,y),M) = \sum_{j,k} h_{jk}(A')^j \cdot M \cdot (A)^k \tag{2.6}$$

These are some properties of this map.

i) $f_A(u,M) = uM$ (u a unit in $R[x,y]$)

ii) $f_A(g(x,y) + h(x,y),M) = f_A(g(x,y),M) + f_A(h(x,y),M)$

iii) $f_A(g(x,y)q(x,y),M) = f_A(g(x,y),f_A(q(x,y),M))$

$= f_A(q(x,y),f_A(g(x,y),M))$

iv) $f_A(h(x,y),M) = f_A(h \bmod \Phi, M)$

v) $f_A(h(x,y),M_1+M_2) = f_A(h(x,y),M_1) + f_A(h(x,y),M_2)$

Property i) follows directly from the definition.

Property ii) is shown as follows:

Let $p(x,y) = g(x,y) + h(x,y)$

$$p_{ij} = g_{ij} + h_{ij}$$

$$\begin{aligned} f_A(p(x,y),M) &= \sum_{ij} p_{ij}\ (A')^{i} \cdot M \cdot (A)^{j} \\ &= \sum_{ij} (g_{ij} + h_{ij})\ ((A')^{i} \cdot M \cdot (A)^{j}) \\ &= \sum_{ij} g_{ij}\ (A')^{i} \cdot M \cdot (A)^{j} \\ &\quad + \sum_{ij} h_{ij}\ (A')^{i} \cdot M \cdot (A)^{j} \\ &= f_A(g(x,y),M) + f_A(h(x,y),M) \end{aligned}$$

Property iii) is shown as follows:

Let $p(x,y) = g(x,y)q(x,y)$

$$p_{jk} = \sum_{\substack{i+l=j \\ h+m=k}} g_{ih}q_{lm}$$

$$\begin{aligned} f_A(p(x,y),M) &= \sum_{jk} p_{jk}\ (A')^{j} \cdot M \cdot (A)^{k} \\ &= \sum_{jk} \left(\sum_{\substack{i+l=j \\ h+m=k}} g_{ih}q_{lm} \right) (A')^{j} \cdot M \cdot (A)^{k} \end{aligned}$$

$$f_A(g(x,y),M) = \sum_{jk} g_{ih}\ (A')^{i} \cdot M \cdot (A)^{h}$$

$$f_A(q(x,y),M) = \sum_{lm} q_{lm} \; (A')^{l}\cdot M\cdot (A)^{m}$$

Now

$$f_A(g(x,y),\; f_A(q(x,y),M) = \sum_{ih} g_{ih}(A')^{i}\left(\sum_{lm} q_{lm}(A')^{l}\cdot M\cdot (A)^{m}\right)\cdot (A)^{h}$$

$$= \sum_{ih}\left(\sum_{lm} g_{ih}q_{lm} \; (A')^{i+l}\cdot M\cdot (A)^{m+h}\right)$$

suppose that we write this sum differently

let $j = i+l$ $\qquad k = m+h$

Then

$$f_A(g(x,y),f_A(q(x,y),M)) = \sum_{jk}\left(\sum_{\substack{i+l=j\\ h+m=k}} g_{ih}q_{lm} \; (A')^{j}\cdot M\cdot (A)^{k}\right)$$

$$= f_A(p(x,y),M)$$

similarly $f_A(p(x,y),M) = f_A(q(x,y),f_A(g(x,y),M))$

Property iv) is shown as follows:

Let $h(x,y) = h_1(x,y)\varphi(x) + h_2(x,y)\varphi(y) + r(x,y)$

This is obtained by first dividing $h(x,y)$ by $\varphi(x)$ and following that dividing the remainder by $\varphi(y)$. This means that the degree of $r(x,y)$ in both x and y is less than n. This decomposition of $h(x,y)$ is unique, and we also have that

$$r(x,y) = h \bmod \Phi$$

$$f_A(h(x,y),\; M) = f_A(h_1(x,y)\varphi(x)+h_2(x,y)\varphi(y)+r(x,y),M)$$

$$= f_A(h_1(x,y),f_A(\varphi(x),M))+f_A(h_2(x,y),f_A(\varphi(y),M))$$

$$+f_A(r(x,y),M)$$

$$f_A(\varphi(x),\; M) = M\cdot\varphi(A) = 0$$

$$f_A(\varphi(y),\; M) = \varphi(A')\cdot M = 0$$

by the Cayley-Hamilton Theorem.

Therefore

$$f_A(h(x,y),M) = f_A(h \bmod \Phi, M)$$

Property v) is shown as follows:

$$f_A(h(x,y),\ M_1+M_2) = \sum_{ij} h_{ij}\ (A')^i(M_1+M_2)\ A^j$$

$$= \sum_{ij} h_{ij}\ (A')^i M_1 A^j + h_{ij}(A')^i M_2 A^j$$

$$= f_A(h(x,y),M_1) + f_A(h(x,y),M_2)$$

The definition of f_A paves the way for the construction of a particular module. Define the product (*) between cosets $\Phi + h(x,y)$ and nxn matrices M by:

$$(\Phi+h(x,y))*\ M = \sum_{ij} h_{ij}\ (A')^i M A^j$$

with the outcome in $\mathcal{M}$.

Property iv) ensures that the product is well defined since it does not matter which element in $\Phi+h(x,y)$ we use.

Square nxn matrices under addition form an abelian group.

Property v) makes certain that

$$\Phi+\ h(x,y)*\ (A+B) = (\Phi+h(x,y))*A+(\Phi+h(x,y)*B.$$

Property iii) ensures that

$$(\Phi+h(x,y))*\ [(\Phi+g(x,y))*M] = [(\Phi+h(x,y))(\Phi+g(x,y))]\ *M.$$

And property ii) ensures that

$$[(\Phi+h(x,y))+(\Phi+g(x,y))]\ *M = (\Phi+h(x,y))*M + (\Phi+\ g(x,y))*M.$$

The ring $R[x,y]/\Phi$ has a nuit element $\Phi+1$ and we have from property i) that

$$(\Phi+1)*M = M.$$

The above can be summarized in

<u>Lemma 2.5</u>. The set $\mathcal{M}$ of square nxn matrices is a module over the quotient ring $R[x,y]/\Phi$.

Even though Lemma 2.5 will not be explicitly called upon in any of the subsequent proofs it none the less gives great insight in what is essentially taking place and the rationale behind this method of approach to the solution of

$$PA + A'P = -Q .$$

The matrix P is operated on by the matrix A. This can be expressed as

$$(\Phi + (x+y))* P = PA + A'P = -Q.$$

Suppose that a multiplicative inverse of element $\Phi+(x+y)$ is found in $R[x,y]/\Phi$ denoted by $\Phi+(x+y)^{-1}$ such that

$$(\Phi +(x+y))\cdot(\Phi +(x+y)^{-1}) = \Phi+1$$

We would then have the following:

$$(\Phi+(x+y)^{-1})*[\Phi +(x+y)*P] = (\Phi+(x+y)^{-1})*(-Q)$$

Because of the properties mentioned above this can be written as

$$[(\Phi+(x+y)^{-1})\cdot(\Phi+(x+y)]*P = (\Phi+(x+y)^{-1})*(-Q)$$

and therefore

$$P=(\Phi+(x+y)^{-1})*Q .$$

2.4 <u>Algebraic proofs of two theorems related to a Linear Matrix System</u>.

We now have all the necessary algebraic construction to prove the following two theorems.

<u>Theorem 2.1</u>. Let A be an nxn square matrix over the reals. A is a stability matrix if and only if for any symmetric positive definite matrix Q there exists a unique symmetric positive

definite solution P to the matrix equation

$$PA + A'P = -Q \tag{2.7}$$

<u>Theorem 2.2.</u> Let A be an nxn square matrix over the reals. If A is a stability matrix and (A,C) is an observable pair then the matrix equation

$$PA + A'P = -C'C \qquad \text{(C is pxn)} \tag{2.8}$$

has a unique symmetric positive definite solution P.

<u>Proof of Theorem 2.1.</u> Suppose that A is an nxn stability matrix. We claim that for any Q_1

$$P = \frac{1}{u^2} \cdot f_A(q_u(x,y),Q_1)$$

is the unique solution of $PA + A'P = -Q_1$, where f_A is defined as in (2.6) and $q_u(x,y)$ as in (2.4). Using the properties of action f_A we have

$$\begin{aligned}
PA + A'P &= \frac{1}{u^2} \cdot (f_A(q_u(x,y),Q_1)\cdot A + A'\cdot f_A(q_u(x,y),Q_1)) \\
&= \frac{1}{u^2} \cdot (f_A(x,f_A(q_u(x,y),Q_1)) \\
&\qquad + f_A(y,f_A(q_u(x,y),Q_1))) \\
&= \frac{1}{u^2} \cdot (f_A((x+y),f_A(q_u(x,y),Q_1))) \\
&= \frac{1}{u^2} \cdot (f_A((x+y)q_u(x,y),Q_1)) \\
&= \frac{1}{u^2} \cdot (f_A((x+y)q_u(x,y)\bmod\Phi,Q_1)) \\
&= \frac{1}{u^2} \cdot (-u^2\cdot Q_1) = Q_1
\end{aligned}$$

Uniqueness follows by observing that the linear operator $L:R^{n^2} \longrightarrow R^{n^2}$ defined by

$$L(P) = PA + A'P$$

is onto since no restriction was placed on Q_1. This implies that L is one-one.

We now show that P is positive definite.

Since $q_u(x,y)$ is positive (Lemma 2.4) this implies that (Lemma 2.1) there exist polynomials $\{\pi_i(x)\}$ such that

$$q_u(x,y) = \sum_{i=1}^{n} \pi_i(x)\pi_i(y)$$

where $\{\pi_i(x)\}$ is a basis for $R_n(x)$.

Therefore

$$\begin{aligned} P &= \frac{1}{u^2} \; f_A(q_u(x,y),Q) \\ &= \frac{1}{u^2} \; f_A\Big(\sum_{i=1}^{n} \pi_i(x)\pi_i(y),Q\Big) \\ &= \frac{1}{u^2} \sum_{i=1}^{n} \pi_i(A')\cdot Q\cdot \pi_i(A) \end{aligned}$$

Since Q is symmetric from the uniqueness of the solution P we also have P being symmetric. Since $Q > 0$ we have from the last expression that P is at least positive semi-definite.

Suppose therefore, that there exists an n-vector $z \neq 0$ such that $z'Pz=0$. this implies that $\pi_i(A)\cdot z=0$ for all $1 \leq i \leq n$. The polynomials $\{\pi_i(x)\}$ form a basis for $R_n(x)$. Therefore there exist constants $k_1,k_2,\ldots k_n$ such that

$$\sum_{i=1}^{n} k_i\pi_i(x) = 1$$

$$\Longrightarrow \qquad f_A\Big(\sum_{i=1}^{n} k_i\pi_i(x),\; I\Big)=I \text{ (I nxn identity matrix)}$$

$$\Longrightarrow \qquad \sum_{i=1}^{n} k_i\pi_i(A) = I$$

$$\Rightarrow \sum_{i=1}^{n} k_i \pi_i(A) \cdot z = I \cdot z$$

Since $\pi_i(A) = 0$ for all i, the left hand side of the above equality is zero. This is a contradiction since I is positive definite. Therefore P is positive definite.

Suppose now that for any symmetric positive definite matrix Q there exixts a symmetric positive definite solution P of (2.8).

Let z be some eigenvector corresponding to the eigenvalue λ.

$$- \bar{z}' \cdot Q \cdot z < 0 \qquad (\bar{z} \text{ denotes complex conjugate})$$

$$\Longrightarrow \quad \bar{z}'(PA+A'P)z < 0$$

$$\Longrightarrow \quad \bar{z}'P(\lambda z) + (\bar{\lambda}\bar{z}')Pz < 0$$

$$\Longrightarrow \quad (\lambda+\bar{\lambda})\ \bar{z}'Pz < 0$$

Since $P > 0$ this implies that $\lambda+\bar{\lambda} < 0$ (ie that $Re(\lambda) < 0$). Therefore A is a stability matrix. This completes the proof of Theorem 2.1.

<u>Proof of Theorem 2.2.</u>

Suppose that A is an nxn stability matrix. Using Lemma 2.4 this implies that

$$q_u(x,y) = T_u(x)T_u(y)P\varphi(x,y) \mod \Phi$$

is positive. By Lemma 2.1 $q_u(x,y)$ can be written as:

$$q_u(x,y) = \sum_{i=1}^{n} \pi_i(x)\pi_i(y)$$

with $\{\pi_i(x)\}$ being a basis for $R_n(x)$. In a way similar to the proof of theorem 2.1 the solution P of (2.8) exists and can be written as :

$$P = \frac{1}{u^2} \; f_A(q_u(x,y), C'C)$$

$$= \frac{1}{u^2} \sum_{i=1}^{n} \pi_i(A') \; C'C\pi_i(A).$$

Since $C'C \geq 0$ we have that for any n-vector z and $1 \leq i \leq n$

$$z' \cdot \pi_i \; (A')C'C\pi_i(A)z = \left\| C\pi_i(A)z \right\| \geq 0$$

where $\|z\| = (\sum_{i=1}^{n} z_i^2)^{\frac{1}{2}}$. This means that $P \geq 0$.

Suppose then that there exists $z \neq 0$ such that $z' \cdot P \cdot z = 0$.

This implies that

$$\left\| C\pi_i(A)z \right\| = 0 \quad \text{for } 1 \leq i \leq n$$

$$\Longrightarrow \quad C\pi_i(A)z = 0 \quad \text{for } 1 \leq i \leq n$$

Since $\{\pi_i(x)\}$ are a basis for $R_n(x)$ there exists an nxn matrix K such that:

$$K \cdot \begin{bmatrix} \pi_1(x) \\ \pi_2(x) \\ \vdots \\ \cdot \\ \\ \pi_n(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ \cdot \\ \cdot \\ \cdot \\ x^{n-1} \end{bmatrix}$$

which is shorthand notation for the n equations

$$k_{i1}\pi_1(x) + k_{i2}\pi_2(x) + \ldots + k_{in}\pi_n(x) = x^{i-1}$$

$$\text{for } 1 \leq i \leq n$$

with $(k_{i1}, k_{i2}, \ldots k_{in})$ being the i^{th} row of K.

Now then

$$f_A(k_{i1}\pi_1(x) + k_{i2}\pi_2(x) + \ldots + k_{in}\pi_n(x), \; I\;) = A^{i-1}$$

$$\Longrightarrow \quad \sum_{j=1}^{n} k_{ij} \cdot C \cdot \pi_j(A) = CA^{i-1} \qquad 1 \leq i \leq n$$

by multiplying both sides by C.

Define the operator $H : R^n \to R^{n \cdot p}$ by:

$$H(w) = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \cdot w$$

Since (A,C) is an observable pair the null space of H is $\{0\}$.

Since $C \cdot \pi_i(A) = 0$, $1 \le i \le n$, this implies

$$\sum_{j=1}^{n} k_{ij}\, C\, \pi_j(A) = 0 \quad \text{for all} \quad 1 \le i \le n$$

$\Rightarrow \quad H(z) = 0$.

This is a contradiction since $z \neq 0$ and the null space of H is $\{0\}$. This completes the proof of Theorem 2.2.

Theorem 2.2 is not an if and only if statement. But adding the condition that matrix C'C is invertible we have

Lemma 2.6. Let A be an nxn square matrix, over the reals. Let P be the unique positive definite solution of the matrix equation

$$PA + A'P = -C'C \tag{2.9}$$

where C'C is invertible. Then A is a stability matrix and (A,C) is an observable pair.

Proof: We have that the eigenvalues of C'C are non-negative. Since C'C is non-singular this implies that none of them is zero and threfore C'C is positive definite. It then follows as in the proof of Theorem 2.1 that A is stability matrix.

We now show that (A,C) is observable.

The solution **P** of (2.9) can be written as:

$$\mathbf{P} = \frac{1}{u^2} \cdot f_A\,(q_u(x,y),\ C'C)$$

where $q_u(x,y)$ as in (2.4), is positive. From Lemma (2.1) we have that there exists $\{\pi_i(x)\}$ which is a basis for $R_n(x)$ and

$$q_u(x,y) = \sum_{j=1}^{n} \pi_j(x)\pi_j(y)$$

$$\Longrightarrow \quad P = \frac{1}{u^2} \cdot f_A\Big(\sum_{j=1}^{n} \pi_j(x)\pi_j(y),\ C'C\Big)$$

$$= \frac{1}{u^2} \cdot \sum_{j=1}^{n} f_A(\pi_j(x)\pi_j(y),\ C'C)$$

$$= \frac{1}{u^2} \cdot \sum_{j=1}^{n} \pi_j(A')C'C\pi_j(\mathbf{A})$$

Since $P > 0$ we have that

$$z'Pz = \sum_{j=1}^{n} z'\pi_j(A)'C'C\pi_j(A)z = \sum_{j=1}^{n} \|C\pi_j(A)\ z\| > 0$$

Therefore if $z \neq 0$ we must have $\|C\pi_j(A)z\| > 0$ for at least one j in the range $1 \leq j \leq n$. Suppose that $\|C\pi_k(A)z\| > o$ which implies that $C\pi_k(A)z \neq 0$.

Now $\{\pi_j(x)\}$ is a basis for $R_n(x)$, therefore there exists an invertible nxn matrix K such that:

$$\mathbf{K} \cdot \begin{bmatrix} \pi_1(x) \\ \pi_2(x) \\ \vdots \\ \pi_n(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x \\ \vdots \\ x^{n-1} \end{bmatrix}$$

The above represents n equations of the form

$$k_{i1}\pi_1(x) + k_{i2}\pi_2(x) + \ldots + k_{in}\pi_n(x) = x^{i-1}$$

with $(k_{i1}, k_{i2}, \ldots k_{in})$ being the i^{th} row of K.

Therefore:

$$f_A(k_{i1}\pi_1(x)+\ldots+k_{in}\pi_n(x), I) = A^{i-1}$$

$$\Longrightarrow \quad k_{i1}\pi_1(A) + k_{i2}\pi_2(A) + \ldots + k_{in}\pi_n(A) = A^{i-1}$$

Multiply both sides by C.

$$\Longrightarrow \quad k_{i1}C\cdot\pi_1(A) + k_{i2}C\cdot\pi_2(A) + \ldots + k_{in}C\cdot\pi_n(A) = CA^{i-1}$$

$$\text{for} \quad 1 \leq i \leq n$$

Let Λ be the matrix

$$\Lambda = \begin{bmatrix} k_{11}Ip & k_{12}Ip & \ldots & k_{1n}Ip \\ k_{21}Ip & k_{22}Ip & \ldots & k_{2n}Ip \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ k_{n1}Ip & k_{n2}\cdot Ip & \ldots & k_{nn}Ip \end{bmatrix}$$

where Ip is the pxp identity matrix. (Matrix C is pxn).

We then can write the above set of equations as:

$$\Lambda\cdot \begin{bmatrix} C\pi_1(A) \\ C\pi_2(A) \\ \cdot \\ \cdot \\ \cdot \\ C\pi_n(A) \end{bmatrix} = \begin{bmatrix} C \\ CA \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \end{bmatrix} = L$$

We can think of matrix L as a linear operator from R^n to $R^{n\cdot p}$. We wish to show that L is one- one, (i.e. that the null space of L is $\{ 0 \}$).

By construction matrix Λ is invertible since K is invertible, which means that if $w \neq 0$ an $n\cdot px1$ vector then $\Lambda\cdot w \neq 0$.

Let w be the vector:

$$w = \begin{bmatrix} C\pi_1(A)z \\ C\pi_2(A)z \\ \cdot \\ \cdot \\ \cdot \\ C\pi_n(A)z \end{bmatrix}$$

where $z \neq 0$ is an nx1 vector. We do have that $w \neq 0$ and therefore $\Lambda w \neq 0$. But

$$\Lambda \cdot w = \begin{bmatrix} C \cdot z \\ CA \cdot z \\ \cdot \\ \cdot \\ \cdot \\ CA^{n-1} \cdot z \end{bmatrix} = L \cdot z$$

which implies that the null space of L is $\{0\}$ and that (A,C) is an observable pair.

Chapter 3

Computational Algorithms

3.1 Introduction

The proof of Theorem 2.1 is constructive and purely algebraic. It therefore gives great insight into how a computational algorithm should be constructed, for obtaining the solution P of an equation of the form

$$PA + A'P = -Q \tag{3.1}$$

where A is an nxn stability matrix. The algorithm so constructed, basically involves obtaining $\varphi_2(x)$ the characteristic polynomial of A. Using the Extended Euclidean algorithm a polynomial $T_u(x)$ as in (2.3) can be obtained. Having these polynomials, the polynomial $P_\varphi(x,y)$, $q_u(x,y)$ and the solution P are formed.

By restricting the field of interest R, to that of the rational numbers F, the procedure for obtaining the exact solution of (3.1) is fully implementable, using the remarkable facilities provided by the computer programming system MACSYMA available at M.I.T.

Three algorithms are presented here, the Rational, Integer, and Modular, which are based on the constructive proof of Theorem (2.1).

MACSYMA (Project MAC's SYmbolic MAnipulation System) is a large computer programming system used for performing symbolic as well as numerical mathematical computations. This would easily allow us to make parametric studies.

3.2 The Rational Algorithm

This algorithm is a mere implementation of the steps outlined in the proof of Theorem (2.1).

R_1) Obtain $\varphi_2(x)$, the characteristic polynomial of A.

R_2) Set $P_\varphi(x,y) = \dfrac{\varphi_2(x)\varphi_2(y) - \varphi_1(x)\varphi_1(y)}{x + y}$

R_3) Using the Extended Euclidean **Algorithm** obtain $T_u(x)$ and u.

R_4) Set $q_u(x,y) = T_u(x)T_u(y)P_\varphi(x,y)\bmod\Phi$

R_5) Form $P_u = f_A(q_u(x,y),Q)$

R_6) Set $P = \dfrac{1}{u^2} \cdot P_u$

3.3 The Integer Algorithm

Multiplying A and Q in (3.1) by a suitable positive integer an equivalent Lyapunov equation

$$PA_1 + A_1'P = -Q_1 \tag{3.2}$$

is obtained with A_1, Q_1 having integer entries. Suppose that $\varphi_2'(x)$ is the characteristic polynomial of A_1. It is clear that $\varphi_2'(x)$ has integer coefficients and it can therefore be considered as an element of $Z[x,y]$ (the ring of polynomials in x and y over the Integers).

Let

$$\varphi_1'(x) = \varphi_2'(-x)$$

$$P_\varphi'(x,y) = \frac{\varphi_2'(x)\varphi_2'(y) - \varphi_1'(x)\varphi_1'(y)}{x + y} \tag{3.3}$$

We claim that $P_\varphi'(x,y)$ is an element of $Z[x,y]$. Suppose that n is odd. It is clear that for n=1 or n=3

$$x + y \mid x^n + y^n$$

and that the quotient is an element of $Z[x,y]$. Suppose then that for all $m \leq n-1$ we have that $x+y \mid x^{2m+1}+y^{2m+1}$ and that the quotient is an element of $Z[x,y]$. Show that hypothesis is true for $m=n$.

$$x^{2n+1}+y^{2n+1} = (x^2+y^2)(x^{2n-1}+y^{2n-1}) - x^2y^2(x^{2n-3}+y^{2n-3})$$

From the induction hypothesis we therefore have that $x+y \mid x^{2n+1}+y^{2n+1}$ and that the quotient is an element of $Z[x,y]$. For the case when n is even we have that

$$x+y \mid x^n - y^n$$

and that quotient is an element of $Z[x,y]$. Following the proof of Lemma 2.4 ii) we have that $P'_\varphi(x,y)$ is an element of $Z[x,y]$.

It is also clear that there exist polynomials $T'_u(x)$, $\lambda'_u(x)$ and integer u' such that

$$T'_u(x)\varphi'_1(x) + \lambda'_u(x)\varphi'_2(x) = u' \tag{3.4}$$

with $T'_u(x)$ $\lambda'_u(x)$ having integer coefficients.

Since the leading coefficient of $\varphi'_2(x)$ is unity division by $\varphi'_2(x)$ is possible. If we then let Φ' be the ideal $(\varphi'_2(x), \varphi'_2(y))$ in $Z[x,y]$ we have

$$q'_u(x,y) = T'_u(x)T'_u(y)P'_\varphi(x,y) \bmod \Phi'$$

being an element of $Z[x,y]$. Consequently

$$P^*_u = f_{A_1}(q'_u(x,y), Q_1) \tag{3.5}$$

has integer entries with the solution of (3.1) now being expressed as:

$$P = \frac{1}{(u')^2} \cdot P^*_u$$

In (3.4) it is required that polynomials $T'_u(x)$, $\lambda'_u(x)$ and integer u' be found such that (3.3) is satisfied. Existence can be shown in the following manner.

Let

$\varphi_2'(x) = a_0x^n + a_1x^{n-1}+\ldots+ a_n$. Define S to be the nxn matrix.

$$S = \begin{bmatrix} a_1 & a_0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & a_0 & 0 & 0 & \cdots & 0 \\ a_5 & a_4 & a_3 & a_2 & a_1 & a_0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{2n-1} & a_{2n-2} & \cdots & \cdots & \cdots & \cdots & \cdots & a_n \end{bmatrix} \quad (3.6)$$

where $a_k = 0$ for $k > n$ and $a_0 = 1$. Since $\varphi_2'(x)$ is a stability polynomial, S is positive definite (cf. BROCKETT). Since det S > 0 it is clear that for each allowable integer value of u' there exist unique polynomials $T_u'(x)$ $\lambda_u'(x)$ of degree less than n such that

$$T_u'(x)\varphi_1'(x) + \lambda_u'(x)\varphi_2'(x) = u'$$

If $T_u'(x) = d_1x^{n-1} + d_2x^{n-2} +\ldots+ d_n$ then

$$d_i = \frac{M_{ni}\cdot u'}{2 \det S} \qquad 1 \leq i \leq n$$

where M_{ni}=det S_{ni} with S_{ni} the $(n-1)x(n-1)$ matrix obtained from S by deleting the n^{th} row and i^{th} column.

By letting $u'=k\cdot(2 \det S)$, with k an integer greater than zero we have u' in Z and $T_u'(x)$, $\lambda_u'(x)$ in Z[x,y].

The Integer algorithm proceeds as follows.

I_1) Obtain A_1, Q_1

I_2) Find $\varphi_2'(x)$ the characteristic polynomial of A_1

I_3) Set $P_\varphi'(x,y) = \dfrac{\varphi_2'(x)\varphi_2'(y) - \varphi_1'(X)\varphi_1'(y)}{x + y}$

I_4) Find $T'_u(x)$ and u'

I_5) Set $q'_u(x,y) = T'_u(x)T'_u(y)P'_\varphi(x,y) \bmod \Phi'$

I_6) $P^*_u = f_{A_1}(q'_u(x,y), Q_1)$

I_7) Set $P = \frac{1}{(u')^2} P^*_u$

Doing all calculations in integer arithmetic may save time since greatest common divisor computations will not be performed in intermediate steps.

3.4 The Modular Algorithm

The integer algorithm paves the way for a modular approach to the solution. Suppose that p is a prime that does not divide $2 \cdot \det S$ with S defined in (3.6). If $A_1 = (a_{ij})$ and $Q_1 = (q_{ij})$ let

$$_pA = (a_{ij} \bmod p)$$

$$_pQ = (q_{ij} \bmod p)$$

both $_pA$ and $_pQ$ being considered as matrices over Z_p, the field of integers modulo p. Let $Z_p[x,y]$ be the ring of polynomials in x and y over Z_p.

Let

$$_p\varphi_2(x) = \det(Ix - {_pA}) \qquad {_p\varphi_2}(x) \text{ in } Z_p[x,y]$$

and $$_p\varphi_1(x) = {_p\varphi_2}(-x)$$

It can be easily shown that

$$_p\varphi_2(x) = \varphi'_2(x) \bmod p$$

$$_p\varphi_1(x) = \varphi'_1(x) \bmod p$$

where the notation $\varphi'_2(x) \bmod p$ means: reduce each coefficient of $\varphi'_2(x)$ modulo p considering the derived polynomial as an element of $Z_p[x,y]$.

Let

$$_pP_\varphi(x,y) = \frac{_p\varphi_2(x)\,_p\varphi_2(y) - {_p\varphi_1(x)}\,_p\varphi_1(y)}{x + y}$$

where $x+y$ is now thought as an element in $Z_p[x,y]$, the division done modulo p and $_pP_\varphi(x,y)$ being an element of $Z_p[x,y]$.

It follows that there exist polynomials $_pT_u(x)$, $_p\lambda_u(x)$ in $Z_p[x,y]$ and $_pu$ in Z_p such that:

$$_pT_u(x)\,_p\varphi_1(x) + {_p\lambda_u(x)}\,_p\varphi_2(x) = {_pu}$$

where:

$$_pT_u(x) = T'_u(x) \bmod p$$

$$_p\lambda_u(x) = \lambda'_u(x) \bmod p$$

$$_pu = u' \bmod p$$

Let $_p\Phi$ be the ideal $(_p\varphi_2(x), {_p\varphi_2(y)})$ in $Z_p[x,y]$

and

$$_pq_u(x,y) = {_pT_u(x)}\,_pT_u(y)\,_pP_\varphi(x,y) \bmod {_p\Phi}$$
$$= e_{00} + e_{10}y + e_{01}x + \ldots + e_{(n-1)(n-1)}x^{n-1}y^{n-1}$$

we have that

$$_pq_u(x,y) = q'_u(x,y) \bmod p$$

Let

$$_pP_u = \sum_{jk} e_{kj}(_pA')^k \quad {_pQ} \quad (_pA)^j$$

with all operations done modulo p.

If

$$P^*_u = (g_{ij}) \text{ in (3.5) then}$$
$$_pP_u = (g_{ij} \bmod p).$$

Now if $_pP_u$, $_pu$ are obtained for a sufficient number of primes, the Chinese Remainder Theorem (cf. Knuth) can be used to find P^*_u and u' making it possible to obtain the solution

$$P = \frac{1}{(u')^2} \cdot P_u^*$$

The Chinese Remainder Theorem is used in the following manner. Let m_1 and m_2 be relatively prime so that $m_1 > m_2$. Let $u_1 = u \bmod m_1$ and $u_2 = u \bmod m_2$ where $0 \leq u < m_1 m_2$. If c,k are integers such that

$$c \cdot m_1 + k \cdot m_2 = 1$$

then

$$u = m_1([c \cdot (u_2 - u_1)] \bmod m_2) + u_1.$$

Suppose now that $m_1 = p_1 \cdot p_2 \cdot \ldots p_{n-1}$, $m_2 = p_n$ where p_n is the n^{th} prime used. If u is some integer for which we have u_1 and u_2 then we may obtain u mod $m_1 \cdot m_2$ by the above procedure.

The way by which we ensure that P_u^* has been constructed is, by checking element wise at each iteration whether $P_u^* \cdot A + A' P_u^* = -Q$.

The reason why the selected primes p must not divide $2 \cdot \det S$ is because this guarantees that ${}_p\varphi_1(x)$, ${}_p\varphi_2(x)$ are relatively prime over $Z_p[x,y]$.

Since considerable coefficient growth takes place in intermediate computations of the Integer Algorithm it may be advantageous to implement the Modular Algorithm.

The Modular Algortithm

M_1) Obtain ${}_pA, {}_pQ$

M_2) Let ${}_p\varphi_2(x) = \det(Ix - {}_pA)$

M_3) Set ${}_pP_\varphi(x,y) = \dfrac{{}_p\varphi_2(x)\,{}_p\varphi_2(y) - {}_p\varphi_1(x)\,{}_p\varphi_1(y)}{x + y}$

M_4) Obtain ${}_pT_u(x)$ p^u

M_5) Set ${}_pq_u(x,y) = {}_pT_u(x)\,{}_pT_u(y)\,{}_pP_\varphi \bmod_p \Phi$

M_6) Obtain ${}_pP_u$

M_7) Repeat steps M_1-M_6 for a sufficient number of primes and by use of the Chinese Remainder Theorem find P_u^*, u'.

M_8) Set

$$P = \frac{1}{(u')^2} \cdot P_u^*$$

Chapter 4

Computer Programs and Numerical Results

4.1 Introduction

The three algorithms presented in chapter 3 have been programmed on the extremely versatile computer programming system MACSYMA available here at M. I. T. Each algorithm has been programmed as a FUNCTION on MACSYMA. The function SLEAMR(N, PA, PQ) corresponds to the Rational Algorithm, the function SLEAMI (N, PA, PQ) corresponds to the Integer Algorithm and function SLEAMM (A,Q,PR,N,PA,PQ) to the Modular Algorthm. Evaluating each function at some arbitrary values of their arguements one obtains the solution of the corresponding Lyapunov Equation. We proceed now to explain this in more detail. (SLEAM stands for, Solution of Lyapunov Equation using Algebraic Methods.)

4.2 The Function SLEAMR

Purpose:

The value of this function is the solution of the Lyapunov Equation

$$PA + A'P = Q \qquad (4.1)$$

where A and Q have rational entries, with A being a stability matrix and Q symmetric.

The arguements of the function

N = the dimension of the A matrix

PA = the A matrix

PQ = the Q matrix

By evaluating SLEAMR at N, PA=A and PQ=Q (ie SLEAMR(N, A, Q)) one obtains as the value of this function the solution of (4.1). This is done using the Rational Algorithm.

The definition of function SLEAMR(N, PA, PQ) is shown in Table(4.1)

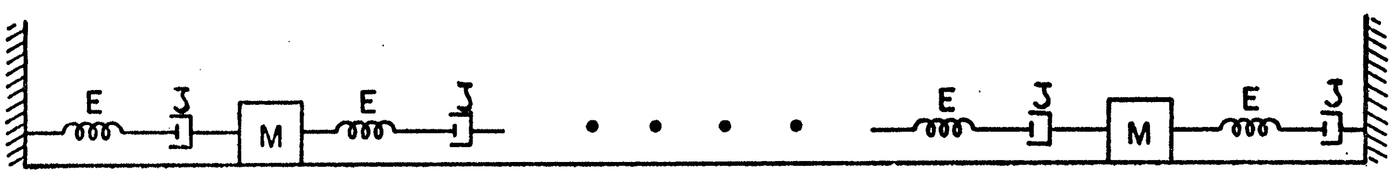

Figure 4.1

```
(D10) SLEARR(N, PA, PQ) := BLOCK([], RATMX : TRUE, PF2X : RAT(DETERMINANT(X IDENT(N) - PA)),

PF1X : EV(PF2X, X = - λ), PF2Y : EV(PF2X, X = Y), PF1Y : EV(PF2X, X = - Y), PPF :

QUOTIENT(PF2X PF2Y - PF1X PF1Y, X + Y, X, Y), RO : 0, ROP : 1, C : PF1X, D : PF2X, L1, QR : DIVIDE(C, D, λ),

R : RATEXPAND(LAST(QR)), Q : RATEXPAND(FIRST(QR)), IF R = 0 THEN (TU : RO, PU : D, GO(L3)) ELSE GO(L2), L2, C : D,

D : R, T : ROP, ROP : RO, RO : RATEXPAND(T - Q RO), GO(L1), L3, PQU : REMAINDER(- TU EV(TU, X = Y) PPF, PF2X, λ),

PQU : RATEXPAND(REMAINDER(PQU, PF2Y, Y)), PAT : TRANSPOSE(PA), CPQU : EMATRIX(N, N, RAT(LAST(PQU)), 1, 1),

Z : REVERSE(MAPLIST(NUMFACTOR, PQU)), FOR J THRU N DO (FOR I1 FROM N (J - 1) + 1 THRU N J DO CPQU                     :
                                                                                              J, I1 - N (J - 1)

RAT(Z  )), A1 : PQ, B1 : PQ, G2 : EMATRIX(N, N, 0, 1, 1), IF CPQU     = RAT(0) THEN G1 :
     I1                                                          1, 1

EMATRIX(N, N, 0, 1, 1) ELSE G1 : CPQU     PQ, FOR I1 FROM 2 THRU N DO (B1 : A1 . PA, A1 : COPYMATRIX(B1),
                                     1, 1

FOR J THRU I1 - 1 DO (IF CPQU      = RAT(0) THEN DJ : 0 ELSE G2 : CPQU      B1 + G2, B1 : PAT . B1),
                             J, I1                                  J, I1

                                                                            1
IF CPQU        = RAT(0) THEN DJ : 0 ELSE G1 : CPQU        B1 + G1), PGU : --- (G1 + G2 + TRANSPOSE(G2)))
       I1, I1                                    I1, I1                     2
                                                                          PU

(C11)
```

Table 4.1

4.3 The Function SLEAMI

Purpose:

The value of this function is the solution of the Lyapunov Equation

$$PA + A'P = Q \tag{4.2}$$

where A and Q have integer entries, with A being a stability matrix and Q symmetric.

The arguements of the function.

N = the dimension of the A matrix

PA = the A matrix

PQ = the Q matrix

By evaluating SLEAMI at N, PA=A,PQ=Q (ie SLEAMI(N, A, Q)) one obtains the solution of (4.1). This is done using the Integer Algorithm.

The definition of function SLEAMI (N, PA, PQ) is given in Table (4.2).

```
(D12) SLEAMI(N, PA, PQ) := BLOCK([], RATMX : TRUE, PF2X : RAT(DETERMINANT(X IDENT(N) - PA)),

PF1X : EV(PF2X, X = - X), PF2Y : EV(PF2X, X = Y), PF1Y : EV(PF2X, X = - Y), PPF :

                                                           N           N             N          N - 1
QUOTIENT(PF2X PF2Y - PF1X PF1Y, X + Y, X, Y), NP : N + 1, IF - = ENTIER(-) THEN K : - ELSE K : -----,
                                                           2           2             2            2

LA : [RAT(LAST(PF2X))], LB : [], FOR J THRU N DO (LA : APPEND(LA, [RATCOEF(PF2X, X, J)]), LB : APPEND(LB, [RAT(0)])),

MM : MATRIX(APPEND(REST(LA, NP - 2), REST(LB, 2))), FOR I1 FROM 2 THRU N DO (IF I1 <= K THEN (MA : NP - 2 I1,

MB : 2 I1, MM : ADDROW(MM, APPEND(REST(LA, MA), REST(LB, MB)))) ELSE (MA : N - 2 I1, MB : NP + MA,

MM : ADDROW(MM, APPEND(REST(LB, MB), REST(LA, MA))))), TU : RAT(DETERMINANT(MINOR(MM, N, N))),

                                                                                         N - L
PU : RAT(2) FIRST(LA) TU, FOR L FROM N - 1 STEP - 1 THRU 1 DO TU : RAT(DETERMINANT(MINOR(MM, N, L))) X      + TU,

PQU : REMAINDER(- TU EV(TU, X = Y) PPF, PF2X, X), PQU : RATEXPAND(REMAINDER(PQU, PF2Y, Y)), PAT : TRANSPOSE(PA),

CPQU : EMATRIX(N, N, RAT(LAST(PQU)), 1, 1), Z : REVERSE(MAPLIST(NUMFACTOR, PQU)),

FOR J THRU N DO (FOR I1 FROM N (J - 1) + 1 THRU N J DO CPQU                     : RAT(Z   )), A1 : PQ, B1 : PQ,
                                                           J, I1 - N (J - 1)            I1

G2 : EMATRIX(N, N, 0, 1, 1), IF CPQU     = RAT(0) THEN G1 : EMATRIX(N, N, 0, 1, 1) ELSE G1 : CPQU      PQ,
                                    1, 1                                                         1, 1

FOR I1 FROM 2 THRU N DO (B1 : A1 . PA, A1 : COPYMATRIX(B1), FOR J THRU I1 - 1 DO (IF CPQU      = RAT(0) THEN BJ :
                                                                                         J, I1

0 ELSE G2 : CPQU      B1 + G2, B1 : PAT . B1), IF CPQU       = RAT(0) THEN BJ : 0 ELSE G1 : CPQU       B1 + G1),
                J, I1                                 I1, I1                                     I1, I1

        1
PQU : --- (G1 + G2 + TRANSPOSE(G2)))
        2
      PU

(C13)
```

Table 4.2

4.4 The Function SLEAMM

Purpose:

The value of this funcion is the solution of the Lyapunov Equation

$$PA + A'P = Q \tag{4.3}$$

where A and Q have integer entries, with A being a stability matrix and Q symmetric.

The arguements of the function

N = the dimension of the A matrix

PA= A = the A matrix

PQ= Q = the Q matrix

PR = A LIST containing primes.

By evaluating SLEAMM at N, A, Q PR (ie SLEAMM (A, Q,PR, N, PA, PQ)) the solution of (4.3) is obtained as the value of the function. This is done using the Modular Algorithm. As the computation progresses an integer is printed out showing the number of primes used so far. One should make sure that PR contains enough primes for the computation.

A List of primes is given in Table(4.3). The definition of the function is given in Table (4.4).

```
        primes;
(D21) [34359737497, 34359737519, 34359737549, 34359737567, 34359737591, 34359737717, 34359737771, 34359737777,
34359737791, 34359737813, 34359737821, 34359737837, 34359737849, 34359737869, 34359737917, 34359738011, 34359738043,
34359738059, 34359738141, 34359738227, 34359738247, 34359738269, 34359738299, 34359738307, 34359738319, 34359738337,
34359738421, 34359738451, 34359738467, 34359738493, 34359738557, 34359738567, 34359738599, 34359738607, 34359738641,
34359738659, 34359738669, 34359738691, 34359738697, 34359738701, 34359738703, 34359738773, 34359738779, 34359738839,
34359738857, 34359738859, 34359738877, 34359738883, 34359738937, 34359738943]
(C22)
```

Table 4.3

```
     C8;
(C7) SELMAT(A, Q, PR, N, PA, PQ) := BLOCK([], I : 0, MATRIX : TRUE, AT : TRANSPOSE(A), LP1, I : I + 1,

MODULUS : P : PR , FOR J1 THRU N DO (FOR J2 THRU N DO (PA        : RAT(A        ), PQ        : RAT(Q        ))),
                I                                       J1, J2         J1, J2        J1, J2         J1, J2

PF2X : RAT(DETERMINANT(X IDENT(N) - PA)), PF1X : EV(PF2X, X = - X), PF2Y : EV(PF2X, X = Y), PF1Y : EV(PF2Y, Y = - Y),

                                                              N           N           N           N - 1
PPF : QUOTIENT(PF2X PF2Y - PF1X PF1Y, X + Y, X, Y), NP : N + 1, IF - = ENTIER(-) THEN K : - ELSE K : -----,
                                                              2           2           2             2

LA : [RAT(LAST(PF2X))], LB : [], FOR J THRU N DO (LA : APPEND(LA, [RATCOEF(PF2X, X, J)]), LB : APPEND(LB, [RAT(0)])),
MM : MATRIX(APPEND(REST(LA, NP - 2), REST(LB, 2))), FOR I1 FROM 2 THRU N DO (IF I1 <= K THEN (MA : NP - 2 I1,
MB : 2 I1, MM : ADDROW(MM, APPEND(REST(LA, MA), REST(LB, MB)))) ELSE (MA : N - 2 I1, MB : NP + MA,
MM : ADDROW(MM, APPEND(REST(LB, MB), REST(LA, MA))))), T0 : RAT(DETERMINANT(MINOR(MM, N, N))),
P0 : RAT(2) FIRST(LA) T0, IF P0 = RAT(0) THEN (PRINT(P, "IS AN UNLUCKY PRIME"), GO(LP1)) ELSE GO(LP2), LP2,

                                                                             N - L
FOR L FROM N - 1 STEP - 1 THRU 1 DO T0 : RAT(DETERMINANT(MINOR(MM, N, L))) X      + T0,
PQ0 : REMAINDER(- T0 EV(T0, X = Y) PPF, PF2X, X), PQ0 : RATEXPAND(REMAINDER(PQ0, PF2Y, Y)), PAT : TRANSPOSE(PA),
CPQ0 : EMATRIX(N, N, RAT(LAST(PQ0)), 1, 1), Z : REVERSE(MAPLIST(NUMFACTOR, PQ0)),
FOR J THRU N DO (FOR I1 FROM N (J - 1) + 1 THRU N J DO CPQ0                   : RAT(Z  )), A1 : PQ, B1 : PQ,
                                                           J, I1 - N (J - 1)         I1

G2 : EMATRIX(N, N, 0, 1, 1), IF CPQ0     = RAT(0) THEN G1 : EMATRIX(N, N, 0, 1, 1) ELSE G1 : CPQ0     PQ,
                                    1, 1                                                       1, 1

FOR I1 FROM 2 THRU N DO (B1 : A1 . PA, A1 : A1 . PA, FOR J THRU I1 - 1 DO (IF CPQ0      = RAT(0) THEN BJ :
                                                                                  J, I1

0 ELSE G2 : CPQ0      B1 + G2, B1 : PAT . B1), IF CPQ0       = RAT(0) THEN BJ : 0 ELSE G1 : CPQ0       B1 + G1),
                J, I1                                 I1, I1                                    I1, I1

PQ0 : G1 + G2 + TRANSPOSE(G2), FOR H1 THRU N DO (FOR H2 THRU N DO PQ0       : RATDISREP(PQ0       )),
                                                                     H1, H2                  H1, H2

P0 : RATDISREP(P0), IF I = 1 THEN GO(LP3) ELSE GO(LP4), LP4, R0 : 0, R0P : 1, C : P1, D : P, LP5,
QR : DIVIDE(C, D, X), R : LAST(QR), Q1 : FIRST(QR), IF R = 0 THEN GO(LP6) ELSE GO(LP7), LP7, C : D, D : R, T : R0P,
R0P : R0, R0 : T - Q1 R0, GO(LP5), LP6, FOR K1 THRU N DO (FOR K2 FROM K1 THRU N DO P1Q0        :
                                                                                     K2, K1

P1Q0        : P1 RATDISREP(RAT(R0 (PQ0        - P1Q0        ))) + P1Q0        ), P10 :
    K1, K2                            K1, K2        K1, K2              K1, K2

P1 RATDISREP(RAT(R0 (P0 - P10))) + P10, P1 : P P1, GO(LP8), LP3, P1 : P, P1Q0 : PQ0, P10 : P0, LP8, MODULUS : FALSE,

       1                                             1
IF ---- (P1Q0 . A + AT . P1Q0) = Q THEN (P1Q0 : ---- P1Q0, RETURN(P1Q0)) ELSE PRINT(I), GO(LP1))
       2                                             2
    P10                                          P10

(C8)
```

Table 4.4

4.5 A Numerical Example

The example corresponds to the evaluation of

$$G = \int_0^\infty x'(t) \cdot Q \cdot (t)dt$$

where $x(t)$ is a solution of

$$\dot{x}(t) = Ax(t) \qquad x(0) = c \qquad (4.4)$$

The system modeled by (4.4) is given in Figure (4.1)

The A matrix of a system with five blocks evaluated at $\zeta=1$, $E=1$ and $M=10000$ (a vaue assignment which forces the system to have characteristic roots close to the imaginary axis) the matrix Q, and the solution P of the equation $PA + A'P = Q$ are given in Tables (4.5), (4.5), (4.6), respectively.

a;

(C24)
$$\begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000} & \frac{1}{10000} & \frac{1}{10000} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{10000} & \frac{1}{10000} & -\frac{1}{5000} & -\frac{1}{5000}
\end{bmatrix}$$

(C25) q;

(C25)
$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

(C26)

Table 4.5

p;

(C26)

$$\begin{bmatrix}
-\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{12500}{3} & 0 & -\frac{10000}{3} & 0 & -2500 & 0 & -\frac{5000}{3} & 0 & -\frac{2500}{3} \\
0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{10000}{3} & 0 & -\frac{20000}{3} & 0 & -5000 & 0 & -\frac{10000}{3} & 0 & -\frac{5000}{3} \\
0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\
0 & -2500 & 0 & -5000 & 0 & -7500 & 0 & -5000 & 0 & -2500 \\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\
0 & -\frac{5000}{3} & 0 & -\frac{10000}{3} & 0 & -5000 & 0 & -\frac{20000}{3} & 0 & -\frac{10000}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \\
0 & -\frac{2500}{3} & 0 & -\frac{5000}{3} & 0 & -2500 & 0 & -\frac{10000}{3} & 0 & -\frac{12500}{3}
\end{bmatrix}$$

(C27)

Table 4.6

4.6 The Parametric Case

With some minor alterations to the function SLEAM(N, PA, PQ), the function PRMTRC (N, PA, PQ) was defined for the purpose of obtaining a parametric solution to equation $PA + A'P = -Q$. The definition of PRMTRC (N, PA, PQ) is given in Table (4.7).

The following example corresponds to the evaluation of

$$G = \int_0^{\infty} x'(t) \cdot Q \cdot x(t)dt$$

where $x(t)$ is a solution of

$$\dot{x}(t) = A\, x(t) \qquad\qquad x(0) = c\ .$$

The A matrix for a system as in Figure (4.1) with two blocks, the matrix Q and the parametric solution P are given in Table (4.8).

```
(C11) PMATRC(N, PA, PQ) := BLOCK([], RATMX : TRUE, DECLARE(N, CONSTANT, E, CONSTANT, Z, CONSTANT)),
PF2X : RATEXPAND(DETERMINANT(X IDENT(N) - PA)), PF1X : EV(PF2X, X = - X), PF2Y : EV(PF2X, X = Y),
PF1Y : EV(PF2X, X = - Y), PPF : QUOTIENT(PF2X PF2Y - PF1X PF1Y, X + Y, X, Y), NP : N + 1,
      N           N          N            N - 1
IF    - = ENTIER(-) THEN K : - ELSE K : -----, LA : [RAT(LAST(PF2X))], LB : [], FOR J THRU N DO (LA :
      2           2          2              2
APPEND(LA, [RATCOEF(PF2X, X, J)]), LB : APPEND(LB, [RAT(0)])), NM : MATRIX(APPEND(REST(LA, NP - 2), REST(LB, 2))),
FOR I1 FROM 2 THRU N DO (IF I1 <= K THEN (MA : NP - 2 I1, MB : 2 I1, NM : ADDROW(NM,
APPEND(REST(LA, MA), REST(LB, MB)))) ELSE (MA : N - 2 I1, MB : NP + MA, NM : ADDROW(NM,
APPEND(REST(LB, MB), REST(LA, MA))))), TO : RAT(DETERMINANT(MINOR(NM, N, N))), PO : RAT(2) FIRST(LA) TO,
                                                                       N - L
FOR L FROM N - 1 STEP - 1 THRU 1 DO TO : RAT(DETERMINANT(MINOR(NM, N, L))) X      + TO,
PQU : REMAINDER(- TO EV(TO, X = Y) PPF, PF2X, X), PQU : RATEXPAND(REMAINDER(PQU, PF2Y, Y)), PAT : TRANSPOSE(PA),
CPQU : IDENT(N), FOR KR FROM 2 THRU N DO (FOR KC FROM KR THRU N DO (CPQU       :
                                                                       KC, KR

                                KC - 1  KR - 1                                         KR - 1
CPQU       : RATCOEF(PQU, X       Y       ), IF KR = KC THEN PQU : PQU - CPQU      (X Y)       ELSE PQU :
    KR, KC                                                                   KR, KC

                               KC - 1  KR - 1    KR - 1  KC - 1
RATEXPAND(PQU - CPQU       (X       Y       + X       Y       )))), FOR KR FROM 2 THRU N DO (CPQU      :
                    KR, KC                                                                      1, KR

                               KR - 1                                 KR - 1   KR - 1
CPQU      : RATCOEF(PQU, X       ), PQU : RATEXPAND(PQU - CPQU      (X       + Y       ))), CPQU     : PQU, A1 : PQ,
    KR, 1                                                     1, KR                              1, 1

B1 : PQ, G2 : EMATRIX(N, N, 0, 1, 1), IF CPQU     = RAT(0) THEN G1 : EMATRIX(N, N, 0, 1, 1) ELSE G1 : CPQU     PQ,
                                             1, 1                                                       1, 1

FOR I1 FROM 2 THRU N DO (B1 : A1 . PA, A1 : COPYMATRIX(B1), FOR J THRU I1 - 1 DO (IF CPQU      = RAT(0) THEN BJ :
                                                                                         J, I1

0 ELSE G2 : CPQU      B1 + G2, B1 : PAT . B1), IF CPQU       = RAT(0) THEN BJ : 0 ELSE G1 : CPQU       B1 + G1),
                J, I1                                 I1, I1                                    I1, I1

        1
PQU : --- (G1 + G2 + TRANSPOSE(G2)))
        2
      PO

(C12)
```

Table 4.7

a;

(D23)

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{2E}{M} & -\frac{2Z}{M} & \frac{E}{M} & \frac{Z}{M} \\ 0 & 0 & 0 & 1 \\ \frac{E}{M} & \frac{Z}{M} & -\frac{2E}{M} & -\frac{2Z}{M} \end{bmatrix}$$

(C24) q;

(D24)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(C25) p;

(D25)/K/

$$\begin{bmatrix} -\frac{E}{2Z} & 0 & 0 & 0 \\ 0 & -\frac{M}{3Z} & 0 & -\frac{M}{6Z} \\ 0 & 0 & -\frac{E}{2Z} & 0 \\ 0 & -\frac{M}{6Z} & 0 & -\frac{M}{3Z} \end{bmatrix}$$

(C26)

Table 4.8

Chapter 5

Generalizations and Extentions

5.1 The Matrix Equation PA+BP=-C

We now employ the ideas developed in Chapter 2 to show

Lemma 5.1. Let A be an nxn matrix over the reals and B an mxm matrix over the reals, and C an mxn matrix over the reals. Let

$$\varphi_2(x) = \det(Ix-A)$$

$$\psi_2(x) = \det(Ix-B)$$

$$\varphi_1(x) = \varphi_2(-x)$$

$$\psi_1(x) = \psi_2(-x)$$

Suppose that $\psi_1(x)$ and $\varphi_2(x)$ are relatively prime such that

$$\lambda_u(x)\psi_1(x) + \mu_u(x)\varphi_2(x) = u$$

$$\lambda_u'(x)\psi_2(x) + \mu_u'(x)\varphi_1(x) = u$$

for $\lambda_u(x), \mu_u(x), \lambda_u'(x), \mu_u'(x)$ polynomials in $R[x,y]$ and u in R. And let

$$P_{\psi\varphi}(x,y) = \frac{\varphi_2(x)\psi_2(y) - \varphi_1(y)\psi_1(x)}{x+y}$$

i) $P_{\psi\varphi}(x,y)$ is an element of $R[x,y]$.

ii) Let $f_{BA} : R[x,y] \times \mathcal{MN} \to \mathcal{MN}$ be the action defined by

$$f_{BA}(g(x,y),M) = \sum_{jk} g_{jk} B^j M A^k$$

where $\mathcal{MN}$ is the space of all mxn matrices over the reals.

Let

$$q_u(x,y) = \lambda_u(x)\mu_u'(y)P_{\psi\varphi}(x,y) \bmod \Psi$$

where Ψ is the ideal $(\varphi_2(x),\psi_2(y))$ in $R[x,y]$.

Then

$$PA + BP = -C \tag{5.1}$$

has a unique solution given by

$$P = \frac{1}{u^2} f_{BA}(q_u(x,y),C)$$

Proof of i). Let

$$\varphi_2(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 = f_0 x^n + f_1 x^{n-1} + \cdots + f_n$$

$$\psi_1(y) = (-1)^m b_m y^{m-1} + (-1)^{m-1} b_{m-1} y^{m-1} + \cdots + b_1$$

$$= e_0 y^m + e_1 y^{m-1} + \cdots + e_m$$

In a similar manner to the proof of Lemma 2.4 , let

$$g(x,y) = \varphi_2(x)\psi_2(y) = \sum_{jk} g_{jk} x^k y^j$$

$$h(x,y) = \varphi_1(y)\psi_1(x) = \sum_{il} h_{il} x^l y^i$$

It is clear that $g_{jk} = a_k b_j$, $h_{il} = (-1)^{i+l} a_i b_l$

Let $b(x,y) = g(x,y) - h(x,y)$ which can be written as

$$b(x,y) = \sum_{\substack{jk \\ 0 \le j \le m \\ 0 \le k \le n}} g_{jk}\, x^k y^j - h_{kj}\, x^j y^k$$

$$= \sum_{jk} a_k b_j\, x^k y^j - (-1)^{k+j} a_k b_j\, x^j y^k \qquad (5.2)$$

Now if $k+j$ is even then the corresponding term in the above sum becomes:

if $k = \min(j,k)$

$$a_k b_j x^k y^k \, (y^{j-k} - x^{j-k})$$

if $j = \min(j,k)$

$$a_k b_j x^j y^j (x^{k-j} - y^{k-j})$$

And if $k+j$ is odd then the corresponding term in the above sum becomes:

if $k = \min(j,k)$

$$a_k b_j x^k y^k \, (y^{j-k} + x^{j-k})$$

if $j = \min(j,k)$

$$a_k b_j x^j y^j \ (x^{k-j} + y^{k-j})$$

But in any case $x+y$ will divide each term (as in Lemma 2.4) and the quotient of $b(x,y)$ divided by $x+y$ will be the sum of the quotients obtained by dividing each term in the sum (5.2) by $x+y$.

Proof of ii). The proof will proceed in three steps.

Step 1. We list some properties of action f_{BA}

i) $f_{BA}(u,M) = uM$ where u is a unit in $R[x,y]$

ii) $f_{BA}(g(x,y) + h(x,y), M) = f_{BA}(g(x,y),M)+f_{BA}(h(x,y),M)$

iii) $f_{BA}(g(x,y)h(x,y),M) = f_{BA}(g(x,y),f_{BA}(h(x,y),M))$

$\qquad = f_{BA}(h(x,y),f_{BA}(g(x,y),M))$

iv) $f_{BA}(g(x,y), M) = f_{BA}(g(x,y)\bmod\Psi,M)$

v) $f_{BA}(g(x,y),M+N) = f_{BA}(g(x,y),M) + f_{BA}(g(x,y),N)$

All the above are analogous to the properties of the action f_A and in the case when $B = A'$ then

$$f_{BA}(g(x,y,M)=f_A(g(x,y),M)$$

for all $g(x,y)$ in $R[x,y]$ and M in $\mathcal{M}$.

Properties i),ii) and v) are quite clear. We now show that property iii) holds.

Let

$$g(x,y) = \sum_{jk} g_{jk}x^k y^j \qquad h(x,y) = \sum_{il} h_{il}x^l y^i$$

$$q(x,y) = g(x,y)h(x,y) = \sum_{st} q_{st}x^t y^s$$

$$= \sum_{st} \Big(\sum_{\substack{i+j=s \\ k+l=t}} g_{jk}h_{il}\Big) x^t y^s$$

$$f_{BA}(q(x,y),M) = \sum_{st} q_{st}B^{s}MA^{t}$$

$$= \sum_{st} \Big(\sum_{\substack{i+j=s\\ k+l=t}} g_{jk}h_{il} \Big) B^{s}MA^{t}$$

$$f_{BA}(g(x,y),M) = \sum_{jk} g_{jk}B^{j}MA^{k}$$

$$f_{BA}(h(x,y),f_{BA}(g(x,y),M)) = \sum_{il} h_{il}B^{i}\Big(\sum_{jk} g_{jk}B^{j}MA^{k}\Big)A^{l}$$

$$= \sum_{il} \sum_{jk} h_{il}g_{jk}B^{i+j}MA^{k+l}$$

let $s=i+j$, $t=k+l$

$$= \sum_{st} \Big(\sum_{\substack{i+j=s\\ k+l=t}} h_{il}g_{jk}\Big) B^{s}MA^{t}$$

$$= f_{BA}(q(x,y),M)$$

We now show property iv).

Any polynomial $h(x,y)$ in $R[x,y]$ can be uniquely written as:

$$h(x,y) = a(x,y)\varphi_2(x) + b(x,y)\psi_2(y) + r(x,y)$$

where the degree of $r(x,y)$ is less than m in y and less than n in x , by first dividing $h(x,y)$ by $\varphi_2(x)$ and then dividing the remainder by $\psi_2(x)$. Therefore

$$f_{BA}(h(x,y),M) = f_{BA}(a(x,y)\varphi_2(x),M) + f_{BA}(b(x,y)\psi_2(y),M) + f_{BA}(r(x,y),M)$$

$$= f_{BA}(a(x,y),f_{BA}(\varphi_2(x),M)) + f_{BA}(b(x,y),f_{BA}(\psi_2(y),M)) + f_{BA}(r(x,y),M)$$

$$= f_{BA}(a(x,y),M\varphi_2(A)) + f_{BA}(b(x,y),\psi_2(B)M) + f_{BA}(r(x,y),M)$$

$$= f_{BA}(r(x,y),M) = f_{BA}(h(x,y) \bmod \Psi,M)$$

because of the Cayley-Hammilton Theorem.

Step 2. Since

$$q_u(x,y) = \lambda_u(x)\mu_u'(y)P_{\psi\varphi}(x,y)\bmod\Psi$$

we will have

$$\begin{aligned}(x+y)(\lambda_u(x)\mu_u'(y)P_{\psi\varphi}(x,y)) &= \lambda_u(x)\mu_u'(y)(\varphi_2(x)\psi_2(y)-\varphi_1(y)\psi_1(x))\\ &= \lambda_u(x)\mu_u'(y)\varphi_2(x)\psi_2(y)\\ &\quad - \lambda_u(x)\mu_u'(y)\varphi_1(y)\psi_1(x)\\ &= \lambda_u(x)\mu_u'(y)\varphi_2(x)\psi_2(y)\\ &\quad - (u-\mu_u(x)\varphi_2(x))(u-\lambda_u'(y)\psi_2(y))\\ &= \lambda_u(x)\mu_u'(y)\varphi_2(x)\psi_2(y)-u^2+u\lambda_u'(y)\psi_2(y)\\ &\quad + u\mu_u(x)\varphi_2(x)-\mu_u(x)\lambda_u'(y)\varphi_2(x)\psi_2(y)\end{aligned}$$

which implies that

$$(x+y)q_u(x,y))\bmod\Psi = -u^2.$$

Step 3. We now show that

$$P = \frac{1}{u^2} f_{BA}(q_u(x,y),C)$$

is the unique solution of (5.1).

$$\begin{aligned}PA + BP &= \frac{1}{u^2}(f_{BA}(q_u(x,y),C)A + Bf_{BA}(q_u(x,y),C))\\ &= \frac{1}{u^2}(f_{BA}(x,f_{BA}(q_u(x,y),C)) + f_{BA}(y,f_{BA}(q_u(x,y),C)))\\ &= \frac{1}{u^2}(f_{BA}(x+y,f_{BA}(q_u(x,y),C)))\\ &= \frac{1}{u^2}(f_{BA}((x+y)q_u(x,y),C))\\ &= \frac{1}{u^2}(f_{BA}((x+y)q_u(x,y)\bmod\Psi,C))\\ &= \frac{1}{u^2}(-u^2C) = -C\end{aligned}$$

Uniqueness follows by observing that the linear operator $L: R^{mn} \longrightarrow R^{mn}$ defined by

$$L(P) = PA + BP$$

is onto since no restriction was placed on C. This implies that L is one-one. This completes the proof of Lemma 5.1.

We have shown that $PA + BP = -C$ has a unique solution if $\psi_1(x)$ and $\varphi_2(x)$ are relatively prime where

$$\psi_2(x) = \det(Ix-B)$$

$$\varphi_2(x) = \det(Ix-A)$$

$$\psi_1(x) = \psi_2(-x)$$

The usual statement of this theorem [cf. Bellman] is as follows.

The equation $PA + BP = -C$ has a unique solution for all C if $\lambda_i+\mu_j \neq 0$ where λ_i are the characteristic roots of A and μ_i the characteristic roots of B.

We end this section by showing that these two statements are equivalent.

Assume that $\psi_1(x)$ and $\varphi_2(x)$ are relatively prime. Suppose then that there exist i,j such that $\lambda_i+\mu_j = 0$. This means that $\lambda_i = -\mu_j$ which implies that $\psi_1(x)$ and $\varphi_2(x)$ have at least one root in common. This in turn implies that $\psi_1(x)$ and $\varphi_2(x)$ have a nontrivial common divisor which is a contradiction.

Assume on the other hand that $\lambda_i+\mu_j \neq 0$ for all i,j. Suppose then that there exists a $k(x)$ of degree greater than or equal to one, such that $k(x) \mid \psi_1(x)$ and $k(x) \mid \varphi_2(x)$. This would imply that $\psi_1(x)$ and $\varphi_2(x)$ have at least one root in common which contradicts our initial assumption.

The above suggests an algorithm for obtaining the solution of equation (5.1). As in the case of the Lyapunov equation (3.1) Rational, Integer and Modular versions of the algorithm can be constructed in a similar manner.

Algorithm for solving equation $PA + BP = -C$.

A_1) Obtain $\varphi_2(x)$, $\psi_2(x)$ the characteristic polynomials of A and B respectively.

A_2) Set $P_{\psi\phi}(x,y) = \dfrac{\varphi_2(x)\psi_2(y) - \varphi_1(y)\psi_1(x)}{x + y}$

A_3) Using the Extended Euclidean Algorithm obtain $\lambda_u(x)$, $\lambda_u'(x)$, $\mu_u(x)$, $\mu_u'(x)$ and u.

A_4) Set $q_u(x,y) = \lambda_u(x)\mu_u'(y)P_{\psi\varphi}(x,y) \bmod \Psi$

A_5) Form $P_u = f_{BA}(q_u(x,y),c)$

A_6) Set $P = \dfrac{1}{u^2} \cdot P_u$

5.2 Conclusions

In closing we wish to comment on what has been accomplished by this thesis, point out some disadvantages associated with the method used in solving the Lyapunov equation and discuss several possibilities that can be persued in the future.

We have constructed purely algebraic algorithms for obtaining the exact solution of the Lyapunov equation. The algebraic structure on which the methods are based is quite rich and can further be exploited. The algorithms are quite simple requiring no obscure algebraic constructions, (the Extended Euclidean Algorithm providing a basis building block) and as demonstrated fully implementable on existing computers.

The price we had to pay for an exact solution takes the form of coefficient growth, creating space requirements. The critical parameters which dictate the amount of storage required, are: dimension of the A matrix as well as the size of the entries in both the A and the Q matrices. The problem of space has quite adequately been dealt with by the introduction of the Modular algorithm. But in doing so the excecution time is increased. In this thesis no serious time complexity evaluation is presented.

In most engineering situations an exact solution is not required, but merely a five or ten digit approximation. Existing methods completely neglect the question of accuracy in the approximation to the solution of the Lyapunov equation. Because of the nature of the method presented, which results in an exact solution, it is quite possible that a closer examination may reveal a scheme by which some control can be exercised on the accuracy of the approximation. As exhibited by the parametric example included in chapter 4 our method offers great possibilities for parametric studies.

We have extended the results and suggested algebraic methods of solution for the more general matrix equation

$$PA + BP = -C.$$

The Riccati equation did come under consideration and some less important 2x2 examples were soved by Newton's Method with our method being employed in the solution of the intermediate Lyapunov equations. The problem encountered hindering further progress was again that of coefficient growth. It was felt that that in order to attempt more realistic examples it would be wise to either first devise a method for obtaining appoximate solutions with controlled accuracy or re-examine the Riccati equation under the light of the present work.

Finally we have gained great insight from all this work. We feel that this is only the begining of a more serious study on the computational aspects of Control Theory.

APPENDIX

This Appendix contains the proofs of Lemmata (2.1), (2.2), (2.3) and (2.4) found in section (2.2).

Lemma 2.1. Let $p(x,y)$ be a polynomial in $R[x,y]$ with $C(p)$ being an $m\times m$ matrix. Then $p(x,y)$ is positive if and only if there exist polynomials $\pi(x),\ldots,\pi_m(x)$ such that

$$p(x,y) = \sum_{i=1}^{m} \pi_i(x)\pi_i(y)$$

where $\{\pi_i(x)\}$ are a basis for $R_m(x)$.

proof: Suppose that $p(x,y)$ is positive. This implies that $C(p)$ is positive definite and symmetric. From linear algebra [7] we have that

$$C(p) = V \cdot V'$$

for some real $m\times m$ matrix $V=(v_{ij})$. This implies that $\det V \neq 0$ and therefore V is invertible.

since
$$p(x,y) = 1'(y)C(p)1(x)$$
$$= (1'(y)\cdot V)\cdot(V'.(x))$$

let $\pi_1(y) = v_{11} + v_{21}y + v_{31}y^2 + \ldots + v_{m1}y^{m-1}$

and $\pi_i(y) = v_{1i} + v_{2i}y + \ldots + v_{mi}y^{m-1}$

$$1 \leq i \leq m$$

and we have

$$p(x,y) = \sum_{i=1}^{m} \pi_i(y)\cdot\pi_i(x)$$

Let $g(x)$ be a polynomial in $R_m(x)$.

$$g(x) = g_1 + g_2x + \ldots g_mx^{m-1}$$

Since V is inverible it has m linearly independent columns $\{v_1\}$ which form a basis for all vectors of length m.

We therefore have real numbers $\alpha_1 \alpha_2 \ldots \alpha_m$ such that

$$\begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_m \end{bmatrix} = \alpha_1 \cdot v_1 + \alpha_2 v_2 \quad \ldots \quad + \alpha_m v_m$$

and that

$$[1 \ x \ x^2 \ \ldots x^{m-1}] \begin{bmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ \cdot \\ g_m \end{bmatrix} = [1 \ x \ x^2 \ldots \ x^{m-1}] \begin{bmatrix} \alpha_1 \cdot v_1 \cdot + \alpha_2 v_2 + \ \ldots \alpha_m v_m \end{bmatrix}$$

which equivalently is written as:

$$g(x) = \alpha_1 \cdot \pi_1(x) + \alpha_2 \pi_2(x) + \ldots \ \alpha_m \cdot \pi_m(x)$$

$$= \sum_{i=1}^{m} \alpha_i \pi_i(x)$$

therefore $\{ \pi_i(x) \}$ form a basis of $R_m(x)$.

Suppose now that there exist polynomials $\pi_i(x), \pi_2(x), \ldots \pi_m(x)$ forming a basis for $R_m(x)$ such that

$$p(x,y) = \sum_{i=1}^{m} \pi_i(x) \cdot \pi_i(y)$$

$$\Longrightarrow \qquad = \sum_{i=1}^{m} [1 \ y \ldots y^{m-1}] \begin{bmatrix} \pi_{i1} \\ \pi_{i2} \\ \cdot \\ \cdot \\ \cdot \\ \pi_{im} \end{bmatrix} [\pi_{i1} \ldots \pi_{im}] \cdot \begin{bmatrix} 1 \\ x \\ \cdot \\ \cdot \\ \cdot \\ x^{m-1} \end{bmatrix}$$

where $\pi_i(x) = \pi_{i1} + \pi_{i2} x + \ldots \pi_{im} x^{m-1}$

We can therefore write $p(x,y)$ as:

$$p(x,y) = [1,\ y\ \dots\ y^{m-1}]\ \Pi \cdot \Pi' \begin{bmatrix} 1 \\ x \\ \cdot \\ \cdot \\ \cdot \\ x^{m-1} \end{bmatrix}$$

where the i^{th} column of $\Pi = (\pi_{ij})$ is

$$\pi_i = \begin{bmatrix} \pi_{i1} \\ \pi_{i2} \\ \cdot \\ \cdot \\ \cdot \\ \pi_{im} \end{bmatrix}$$

Since $\{\pi_i(x)\}$ form a basis we must have $\{\pi_i\}$ being linearly independent and $\det\Pi \neq 0$.

I also claim that the largest power of $p(x,y)$ in x or y is $m-1$. Since if we assume that there are no terms in $p(x,y)$ which are of degree $m-1$ in either x or y we must have

$$\sum_{i=1}^{m} \pi_{im} \cdot \pi_i = 0$$

implying that $\pi_{im} = 0$, $1 \leq i \leq m$, and therefore a contradiction to the hypothesis that $\{\pi_i\}$ are linearly independent.

This ensures that $C(p) = \Pi \cdot \Pi'$ and that it is symmetric and positive semidefinite.

Assume now that there exists some vector $z \neq 0$ such that

$$z'\Pi\Pi'z = 0$$

Since Π is inverible this cannot happen and therefore $C(p) = \Pi \cdot \Pi'$ is positive definite.

Lemma 2.2. Let n be the degree of $\varphi(x)$. if $p(x,y) \bmod \Phi$ is positive of degree $n-1$ in both x and y then $\sigma(x)\sigma(y)p(x,y) \bmod \Phi$ is positive of degree $n-1$ in x and y, if and only if $\varphi(x)$ and $\sigma(x)$ are relatively prime.

proof: The proof will proceed in three steps.

step 1. We first show that there exists a vector space isomorphism between $R_n(x)$ (the vector space over R of polynomials of degree less than n under addition) and the quotient space $R[x]/\varphi$(where $\varphi = (\varphi(x))$ considered as a vector space over R under addition. ($R[x]/\varphi$ is actually an algebra if we also include multiplicity.)

Let $t: R_n(x) \longrightarrow R[x]/\varphi$ be defined by

$$t(g(x)) = \varphi + g(x).$$

It is a vector space homomorphism since

$$t(\alpha_1 g_1,(x) + \alpha_2 g_2(x)) = \alpha_1 t(g_1(x) + \alpha_2 t(g_2(x))$$

Let $\varphi + g(x)$ be an element of $R[x]/\varphi$. if $g \bmod \varphi$ denotes the polynomial in $\varphi + g(x)$ of minimal degree (which must be less than n) we have

$$t(g \bmod \varphi) = \varphi + g \bmod \varphi = \varphi + g(x)$$

Let $g_1(x) \neq g_2(x)$ be elements in $R_n(x)$. Then it is clear that $\varphi + g_1(x) \neq \varphi + g_2(x)$ and this shows that t is an isomorphism.

step 2. We now show that if $\{\pi_i(x)\}$ $1 \leq i \leq n$ is a basis for $R_n(x)$ then $\{\sigma(x)\pi_i(x)\}$ is also a basis for $R_n(x)$ if and only if $\sigma(x)$, $\varphi(x)$ are relatively prime.

If $\{\pi_i(x)\}$ is a basis for $R_n(x)$ then $\varphi + \pi_i(x)$ is a basis for $R[x]/\varphi$.

Suppose that $\sigma(x)$, $\varphi(x)$ are relatively prime . This implies that there exists $\lambda(x)$ in $R_n(x)$ such that

$$(\varphi + \lambda(x)) \cdot (\varphi + \sigma(x)) = \varphi + 1$$

where $\varphi+1$ denotes the multiplicative indentity in $R[x]/\varphi$

For any coset $\varphi + a(x)$ there exist k_i in R such that

$$(\varphi + \lambda(x)) \cdot (\varphi + a(x)) = \sum_{i=1}^{n} k_i (\varphi + \pi_i(x))$$

$$= (\varphi + 1) \cdot \left(\sum_{i=1}^{n} k_i(\varphi + \pi_i(x))\right)$$

$$= (\varphi + \lambda(x)) \cdot \left(\sum_{i=1}^{n} k_i(\varphi + \sigma(x)\pi_i(x))\right)$$

$$(\varphi + \alpha(x)) = \sum_{i=1}^{n} k_i(\varphi + \sigma(x)\pi_i(x))$$

and therefore $\{\varphi + \sigma(x)\pi_i(x)\}$ is a basis for $R[x]/\varphi$. By step 1 we have that

$$\{(\sigma(x)\pi_i(x)) \bmod \varphi\} \text{ is basis for } R_n(x) .$$

Suppose that $\sigma(x)$, $\varphi(x)$ have a nontrivial factor in common, ie there exists $T(x)$ in $R_n(x)$, $(\varphi+T(x)) \neq 0$ such that

$$(\varphi + \tau(x)) \cdot (\varphi + \sigma(x)) = \varphi + 0$$

where $\varphi+0$ is the additive identity in $R[x]/\varphi$.

Suppose then that $\{\sigma(x)\pi_i(x) \bmod \varphi\}$ is a basis for $R_n(x)$.

$$(\varphi + \tau(x)) = (\varphi + x) \cdot (\varphi + 1)$$

$$= (\varphi + \tau(x)) \cdot \sum_{i=1}^{n} k_i(\varphi + \sigma(x)\pi_i(x) \bmod \varphi)$$

$$= \sum_{i=1}^{n} k_i(\varphi + \tau(x)\sigma(x)\pi_i(x))$$

$$= \varphi + o$$

which is a contradiction. This proves step 2.

Step 3. We now prove the lemma. If pmodΦ is positive then

$$\mathrm{pmod}\Phi = \sum_{i=1}^{n} \pi_i(y)\pi_i(x)$$

$$\Longrightarrow \quad \Phi + \mathrm{pmod}\Phi = \Phi + \left(\sum_{i=1}^{n} \pi_i(y)\pi_i(x)\right)$$

$$\Phi + \sigma(x)\sigma(y)(\mathrm{pmod}\Phi) = \Phi + \sum_{i=1}^{n} (\sigma(y)\pi_i(y))(\sigma(x)\pi_i(x))$$

$$\Phi + \sigma(x)\sigma(y)p(x,y) = \Phi + \sum_{i=1}^{n} (\sigma(y)\pi_i(y))(\sigma(x)\pi_i(x))$$

$$(\sigma(x)\sigma(y)p(x,y))\mathrm{mod}\Phi = \sum_{i=1}^{n} ((\sigma(y)\pi_i(y))(\sigma(x)\pi_i(x)))\mathrm{mod}\Phi$$

$$\sigma(x)\sigma(y)p(x,y)\mathrm{mod}\Phi = \sum_{i=1}^{n} \sigma(y)\pi_i(y)\mathrm{mod}\Phi\sigma(x)\pi_i(x)\mathrm{mod}\Phi$$

From Lemma 2.1 and step 2 $\sigma(x)\sigma(y)p(x,y)\mathrm{mod}\Phi$ will be positive of degree n-1 in x and y if and only if $\sigma(x)\pi_i(x)\mathrm{mod}\Phi$ form a basis of $R_n(x)$. This completes the proof of Lemma 2.2.

Lemma 2.3. Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be complex numbers which are distinct and have positive real parts. Then the nxn matrix $\Lambda_n = \left(\dfrac{1}{\lambda_i+\bar{\lambda}_j}\right)$ is hermitean ($\Lambda_n = \Lambda_n^*$ where (*) the hermitean adjoint) positive definite.

proof: We first show that if the mxm matrix $\Lambda_m = \left(\dfrac{1}{\lambda_i+\bar{\lambda}_j}\right)$ is positive definite so is the mxm matrix $S_m = \left(\dfrac{c_i\bar{c}_j}{\lambda_i+\bar{\lambda}_j}\right)$ provided that each $c_i \neq 0$.

Let S_1 be defined as:

$$S_1 = \begin{bmatrix} c_1 & 0 & 0 \ldots & 0 \\ 0 & c_2 & \ldots & 0 \\ 0 & 0 & c_3 \ldots & \\ & \vdots & & \\ 0 & 0 & \ldots & c_1 \end{bmatrix} \Lambda_1 \begin{bmatrix} \bar{c}_1 & 0 & 0 & \ldots & 0 \\ 0 & \bar{c}_2 & 0 & \ldots & 0 \\ 0 & 0 & \bar{c}_3 & \ldots & 0 \\ & \vdots & & & \\ 0 & 0 & 0 & \ldots & \bar{c}_1 \end{bmatrix}$$

where $1 \le l \le m$. In order for S_m to be positive definite we must have $\det S_l > 0$ for $1 \le l \le m$ (Sylvester Criterion).

$$\det S_l = \left(\prod_{i=1}^{l} c_i\right) \det \Lambda_l \left(\prod_{i=1}^{l} \bar{c}_i\right)$$

Since $|c_i| \neq 0$ and Λ_m is positive definite we have that

$$\det S_l > 0 \qquad \text{for} \quad 1 \le l \le m$$

This therfore ensures that if Λ_{n-1} is positive definite so is the matrix

$$K_{n-1} = \left(\frac{1}{\lambda_i+\bar{\lambda}_j} - \frac{\lambda_n+\bar{\lambda}_n}{(\lambda_i+\lambda_n)(\lambda_j+\lambda_n)}\right) = \left(\frac{\left(\frac{\lambda_i-\lambda_n}{\lambda_i+\bar{\lambda}_n}\right)\overline{\left(\frac{\lambda_j-\lambda_n}{\lambda_j+\bar{\lambda}_n}\right)}}{(\lambda_i+\bar{\lambda}_j)}\right)$$

We now prove the Lemma by induction on n.

It is true that $\Lambda_n = \Lambda_n^*$ for all n since

$$\Lambda_n^* = (c_{ij}) = \left(\overline{\frac{1}{\lambda_j+\bar{\lambda}_i}}\right) = \left(\frac{1}{\lambda_i+\bar{\lambda}_j}\right) = (a_{ij}) = \Lambda_n$$

where $c_{ij} = \bar{a}_{ji}$. It is clear that $\Lambda_1 > 0$ since

$$\frac{1}{\lambda_1+\bar{\lambda}_1} > 0$$

Suppose that $\Lambda_m > 0$ for all $m \le n-1$. Applying the Sylvester Criterion on Λ_n we see that all determinants of intermediate minors are positive, by the induction hypothesis. We just have to show that $\det\Lambda_n > 0$. By observing the structure of K_{n-1} and using elementary properties of determinants we now show that

$$\left(\frac{1}{\lambda_n+\bar{\lambda}_n}\right)\det K_{n-1} = \det \Lambda_n .$$

Let

$$b_i = \begin{bmatrix} \dfrac{1}{\lambda_1+\bar{\lambda}_i} \\ \dfrac{1}{\lambda_2+\bar{\lambda}_i} \\ \vdots \\ \dfrac{1}{\lambda_{n-1}+\bar{\lambda}_i} \end{bmatrix} \quad c_i = \begin{bmatrix} \dfrac{-(\lambda_n+\bar{\lambda}_n)}{(\lambda_1+\bar{\lambda}_n)(\bar{\lambda}_i+\lambda_n)} \\ \dfrac{-(\lambda_n+\bar{\lambda}_n)}{(\lambda_2+\bar{\lambda}_n)(\bar{\lambda}_i+\lambda_n)} \\ \vdots \\ \dfrac{-(\lambda_n+\bar{\lambda}_n)}{(\lambda_{n-1}+\bar{\lambda}_n)(\bar{\lambda}_i+\lambda_n)} \end{bmatrix} \quad a_n = \begin{bmatrix} \dfrac{1}{\lambda_1+\bar{\lambda}_n} \\ \dfrac{1}{\lambda_2+\bar{\lambda}_n} \\ \vdots \\ \dfrac{1}{\lambda_{n-1}+\bar{\lambda}_n} \end{bmatrix}$$

Then

$$K_{n-1} = [b_1+c_1,\ b_2+c_2, \ldots\ b_{n-1}+c_{n-1}]$$

and

$$\begin{aligned}\det K_{n-1} &= \det [b_1,\ b_2, \ldots\ b_{n-1}] + \det [c_1, b_2, b_3, \ldots b_{n-1}] \\ &\quad + \ldots + [\det b_1, b_2, \ldots\ b_{n-1}] \\ &= \det [b_1, b_2, \ldots\ b_{n-1}] \\ &\quad + \frac{\lambda_n+\bar{\lambda}_n}{\bar{\lambda}_1+\lambda_n} \det [a_n,\ b_2, b_3 \ldots b_{n-1}] \\ &\quad + \frac{\lambda_n+\bar{\lambda}_n}{\bar{\lambda}_1+\lambda_n} \det [b_1,\ -a_n, b_3, \ldots b_{n-1}] \\ &\quad + \ldots \\ &\quad + \frac{\lambda_n+\bar{\lambda}_n}{\bar{\lambda}_{n-1}+\lambda_n} \cdot \det [b_1,\ b_2, \ldots b_{n-2},\ -a_n]\end{aligned}$$

$$\Rightarrow \frac{1}{\lambda_n+\bar{\lambda}_n} \det K_{n-1} = \frac{1}{\bar{\lambda}_n+\lambda_n} \det [b_1, b_2, b_3, \ldots b_{n-1}]$$

$$+ \frac{1}{\bar{\lambda}_1+\lambda_n} \det [-a_n, b_2, \ldots b_{n-1}]$$

$$+ \frac{1}{\bar{\lambda}_2+\lambda_n} \det [b_1, -a_n, \ldots b_{n-1}]$$

$$+ \ldots$$

$$+ \frac{1}{\bar{\lambda}_{n-1}+\lambda_n} \det [b_1, b_2, \ldots b_{n-2}, -a_n]$$

$$= (-1)^{n-1} \frac{1}{\bar{\lambda}_1+\lambda_n} \det [b_2, b_3, \ldots b_{n-1}, a_n]$$

$$+(-1)^{n-2} \frac{1}{\bar{\lambda}_2+\lambda_n} \det [b_1, b_3, \ldots b_{n-1}, a_n]$$

$$+ \ldots$$

$$+ \frac{1}{\bar{\lambda}_n+\lambda_n} \det [b_1, b_2, \ldots b_{n-1}]$$

Expanding $\det \Lambda_n$ by the last row gives:

$$\det \Lambda_n = (-1)^{n+1} \frac{1}{\bar{\lambda}_1+\lambda_n} \det [b_2, b_3, \ldots b_{n-1}, a_n]$$

$$+ (-1)^{n+2} \frac{1}{\bar{\lambda}_2+\lambda_n} \det [b_1, b_3, \ldots b_{n-1}, a_n]$$

$$+ \ldots .$$

$$+ (-1)^{2n} \frac{1}{\bar{\lambda}_n+\lambda_n} \det [b_1, b_2, \ldots b_n]$$

and therefore $\quad \dfrac{1}{\bar{\lambda}_n+\lambda_n} \det K_{n-1} = \det \Lambda_n$

Since K_{n-1} is positive definite and λ_n has positive real part we have that

$$\det \Lambda_n > 0$$

and that $\Lambda_n > 0$.

We can also note as a consequence of this lemma that if $\lambda_1 \lambda_2 \ldots \lambda_n$ are complex numbers with negative real parts then the matrix $T_n = \left(\frac{-u^2}{\lambda_i + \bar{\lambda}_j}\right)$ where $u \neq 0$ is a real number is also positive definite. This completes the proof of the Lemma.

Lemma 2.4. Let A be an nxn stability matrix with $\varphi_2(x)$ det $(Ix - A)$ and let $\Phi = (\varphi_2(x), \varphi_2(y))$. Define

$$\varphi_1(x) = \varphi_2(-x) \tag{2.1}$$

$$P_\varphi(x,y) = \frac{\varphi_2(x)\varphi_2(y) - \varphi_1(x)\varphi_1(y)}{x + y}$$

i) Polynomials $\varphi_1(x), \varphi_2(x)$ are relatively prime. That is there exist $T_u(x), \lambda_u(x)$ in $R[x,y]$ such that

$$T_u(x)\varphi_1(x) + \lambda_u(x)\varphi_2(x) = u \tag{2.3}$$

where u is a unit in $R[x,y]$.

ii) $P_\varphi(x,y)$ is an element of $R[x,y]$.

iii) Let $q_u(x,y) = T_u(x)T_u(y)P_\varphi(x,y) \bmod \Phi$ (2.4)

Then $q_u(x,y)$ is positive of degree n-1 in both x and y.

Proof of i). Suppose that there exists a $k(x)$ of degree greater than or equal to 1 such that

$$k(x) \mid \varphi_1(x) \quad , \quad k(x) \mid \varphi_2(x)$$

$$\varphi_1(x) = l_1(x)k(x) \qquad \varphi_2(x) = l_2(x)k(x)$$

this implies that $\varphi_1(x)$ and $\varphi_2(x)$ have at least one common root. This cannot happen since $\varphi_1(x) = \varphi_2(-x)$ and $\varphi_2(x)$ is a stability polynomial. Therefore no such $k(x)$ exists and $\varphi_1(x), \varphi_2(x)$ are relatively prime.

Proof of ii).

Let

$$\varphi_2(x) = a_0 + a_1x + \ldots + a_nx^n$$

$$\varphi_1(x) = a_0 + (-1)a_1x + \ldots + (-1)^na_nx^n$$

Let

$$g(x,y) = \varphi_2(x)\varphi_2(y) = \sum_{jk} g_{jk}x^ky^j$$

$$h(x,y) = \varphi_1(x)\varphi_1(y) = \sum_{il} h_{il}x^l y^i$$

$$b(x,y) = g(x,y) - h(x,y) = \sum_{st} b_{st}x^t y^s$$

where $b_{st} = g_{st} - h_{st} = a_t a_s - (-1)^t a_t \cdot (-1)^s a_s$
$= a_t a_s - a_t a_s (-1)^{t+s}$

Let $s=t=m \quad 0 \leq m \leq n$

$b_{mm} = a_m a_m - (1)^{2m} a_m a_m = 0$ for all m.

Let $s=m$, $t=k$, $0 \leq m \leq n$, $0 \leq k \leq n$, $m \neq k$

$b_{mk} = a_m a_k - (-1)^{m+k} a_m a_k = 0$ if m+k even
$= 2a_m a_k$ if m+k odd

It can be shown by induction that

i) $x+y \mid x^m + y^m$ if m is odd

ii) $x+y \mid x^m - y^m$ if m is even .

With this in mind and that $b(x,y)$ is symmetric the division of $b(x,y)$ by $x+y$ is performed by summing the quotients obtained from the divisions of all terms of the form $q_{mk}x^k y^m + q_{km}x^m y^k$ by $x+y$.

Proof of iii).

The proof will proceed in three steps.

step 1: Assume that the eigen-values of A $\lambda_1, \lambda_2, \ldots \lambda_n$ are all distinct. Show that $q_u(x,y)$ is of degree n-1 in both x and y and that it is positive.

Since $P_\varphi(x,y)$ is symmetric so is

$$q_u(x,y) = (T_u(x)T_u(y)P_\varphi(x,y)) \bmod \Phi$$

On the other hand

$$(x+y)\cdot(T_u(x)T_u(y)P_\varphi(x,y)) = T_u(x)T_u(y)\ [\varphi_2(x)\varphi_2(y)-\varphi_1(x)\varphi_1(y)]$$
$$= T_u(x)T_u(y)\varphi_2(x)\varphi_2(y) - T_u(x)T_u(y)\varphi_1(x)\varphi_1(y)$$
$$= (T_u(x)T_u(y) - \lambda_u(x)\lambda_u(y))\varphi_2(x)\varphi_2(y)$$
$$+ u\lambda_u(x)\varphi_2(x) + u\lambda_u(y)\varphi_2(y) - u^2$$

which implies that

$$((x+y)\cdot q_u(x,y)) \bmod \Phi = -u^2 \qquad (2.5)$$

In order for this to happen the degree of $q_u(x,Y)$ in both x and y which is less than or equal to n-1, must actually be n-1.

On the other hand

$$(\lambda_i + \bar{\lambda}_j)\cdot q_u(\lambda_i, \bar{\lambda}_j) = -u^2$$

and therefore

$$q_u(\lambda_i, \bar{\lambda}_j) = \frac{-u^2}{\lambda_i+\bar{\lambda}_j}$$

Since we have assumed that the λ_i's are distinct then $l(\lambda_1), l(\lambda_2),\dots,l(\lambda_n)$ by the Van-dermonde determinant theorem must be linearly independent vectors.

We now wish to show that $C(q_u(x,y))$ is positive definite.

Let $z \neq 0$

$$\bar{z}'((q_u)\ z = (\sum_{i=1}^{n} \bar{k}_i l'(\bar{\lambda}_i))\, C(q_u) \sum_{j=1}^{n} k_j l(\lambda_j))$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{k}_i k_j\ l'(\bar{\lambda}_i)\ C(q_u) l(\lambda_j)$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{n} \bar{k}_i k_j q_u(\lambda_i, \bar{\lambda}_j)$$

$$= [\bar{k}_1, \bar{k}_2 \ldots \bar{k}_n] \quad K_n \quad \begin{bmatrix} k_1 \\ k_2 \\ \cdot \\ \cdot \\ \cdot \\ k_n \end{bmatrix}$$

with $K_n = (q_u(\lambda_i, \lambda_j)) = (\frac{-u^2}{\lambda_i + \bar{\lambda}_j})$

Since $z \neq 0$ not all of the k_i's are zero. Lemma 2.3 ensures us that K_n is positive definite and therefore

$$\bar{z}'C(q_u)z > 0 \quad \text{if } z \neq 0$$

making $C(q_u)$ positive definite.

<u>step 2</u>. Since $\varphi_1(x)\varphi_1(y)T_u(x)T_u(y)P\varphi(x,y) \bmod \Phi = P\varphi(x,y) \bmod \Phi$ we also have $P_\varphi(x,y) \bmod \Phi = P_\varphi(x,y)$ being positive as a consequence of Lemma 2.2.

<u>step 3</u>. Suppose now that the eigen-values of A are not distinct. Show that $q_u(x,y)$ is positive.

In order to simplify the notation we let

$$\psi(x) = \varphi_2(x)$$

$$\psi^+(x) = \varphi_1(x)$$

All we have to do in showing that $q_u(x,y)$ is positive is to show that $P_\varphi(x,y)$ is positive. Then by Lemma 2.2 it is assured that $q_u(x,y)$ is positive.

We prove that $P_\varphi(x,y)$ is positive by showing that it can be expresses as:

$$P_\varphi(x,y) = \sum_{i=1}^{n} \pi_i(x)\pi_i(y)$$

where $\{\pi_i(x)\}$ is a basis for $R_n(x)$.

Write

$$\psi(x) = \psi_1(x)\psi_2(x)\ldots\psi_s(x)$$

where each $\psi_i(x)$ has distinct zeros and

$$\psi_s \,\Big|\, \psi_{s-1} \qquad \psi_{s-1} \,\Big|\, \psi_{s-2}\ldots \qquad \psi_1 \,\Big|\, \psi.$$

We then have

$$P_\varphi(x,y) = \frac{\psi(x)\psi(y) - \psi^+(x)\psi^+(y)}{x+y}.$$

We know that the degree of $P_\varphi(x,y)$ is less than n in both x and y.

Let $\eta_j(x) = \psi_1^+(x)\psi_2^+(x)\ \ldots\psi_{j-1}^+(x)\cdot\psi_{j+1}(x)\ldots\psi_s(x)$

for $1 \le j \le s$.

If we let

$$P_{\psi_j}(x,y) = \frac{\psi_j(x)\psi_j(y) - \psi_j^+(x)\psi_j^+(y)}{x+y}$$

it can be shown that

$$P_\varphi(x,y) = \sum_{j=1}^{s} \eta_j(x)\eta_j(y)P_{\psi_j}(x,y)$$

by substituting in the expression what $\eta_j(x)$ and $P_{\psi_j}(x,y)$ are and cancelling terms.

From step 2 we know that each $P_{\psi_j}(x,y)$ is positive and therefore by Lemma 2.1

$$P_{\psi_j}(x,y) = \sum_{k=1}^{n_j} \pi_{jk}(x)\pi_{jk}(y)$$

where n_j is the degree of $\psi_j(x)$ and $\{\pi_{jk}(x)\}$ are a basis for $R_{n_j}(x)$.

Therefore

$$P_\varphi(x,y) = \sum_{j=1}^{s} \sum_{k=1}^{n_j} \eta_j(x)\pi_{jk}(x) \cdot \eta_j(y)\pi_{jk}(y)$$

We show that $\{\eta_j(x)\pi_{jk}(x)\}$ is a basis for $R_n(x)$.

Suppose that there exists real numbers $\dot{m}_{jk}$ not all zero such that

$$\sum_{j=1}^{s} \sum_{k=1}^{n_j} \dot{m}_{jk}\eta_j(x)\pi_{jk}(x) = 0$$

We can write this as

$$\sum_{j=1}^{s-1} \sum_{k=1}^{n_j} m_{jk}\eta_j(x)\pi_{jk}(x) = \sum_{k=1}^{n_s} \dot{m}_{sk}\eta_s(x)\pi_{sk}(x)$$

if all m_{sk} $1 \leq k \leq n_s$ are zero we can proceed by writing

$$\sum_{j=1}^{s-2} \sum_{k=1}^{n_j} m_{jk}\eta_j(x)\pi_{jk}(x) = \sum_{k=1}^{n_{s-1}} m_{s-1k}\eta_{s-1}(x)\pi_{s-1k}(x)$$

and continue. Suppose then that $j=s'$ is the first time that we encounter non zero elements in $\{m_{s'k}\}$ $1 \leq k \leq n_{s'}$. Then

$$(*) \qquad \sum_{j=1}^{s'-1} \sum_{k=1}^{n_{s'}} m_{jk}\eta_j(x)\pi_{jk}(x) = \sum_{k=1}^{n_{s'}} m_{s'k}\eta_{s'}(x)\pi_{s'k}(x).$$

Multiply both sides by $b_{s'}(x) = \psi_1(x)\psi_2(x)\ldots\psi_{s'-1}(x)$.

The right hand side of (*) can then be written as:

$p(x)\cdot\varphi(x)$ for some $p(x)$

and

$$b_{s'}(x) \sum_{k=1}^{n_{s'}} m_{s'k}\eta_{s'}(x)\pi_{s'k}(x) = p(x)\cdot\varphi(x)$$

if $p(x) = 0$ we then have that

$$\eta_{s'}(x) \cdot b_{s'}(x) \cdot \sum_{k=1}^{n_{s'}} m_{s'k}\pi_{s'k}(x) = 0$$

But since $\{\pi_{s'k}(x)\}$ are a basis for $R_{n_{s'}}(x)$ this would imply that $m_{s'k} = 0$, $1 \leq k \leq n_{s'}$ contradicting the assumption that $j=s'$ is the first such j for which not all $m_{s'k}=0$. Suppose then that $p(x) \neq 0$.

This would mean that

$$\varphi(x) \mid b_{s'}(x) \sum_{k=1}^{n_{s'}} m_{s'k}\eta_{s'}(x)\pi_{s'k}(x)$$

or that

$$\psi_{s'}(x) \mid (\psi_1^+(x) \cdot \psi_2^+(x) \ldots \psi_{s'-1}^+(x) \sum_{k=1}^{n_{s'}} m_{s'k}\pi_{s'k}(x) .$$

But $\psi_{s'}(x)$ and $\psi_1^+(x)\psi_2^+(x) \ldots \psi_{s'-1}^+(x)$ are relatively prime threfore

$$\psi_{s'}(x) \mid \sum_{k=1}^{n_{s'}} m_{s'k}\pi_{s'k}(x) .$$

Since the degree of $\sum_{k=1}^{n_{s'}} m_{s'k}\pi_{s'k}(x)$ is less than $n_{s'}$

this can only happen if $\sum_{k=1}^{n_{s'}} m_{s'k}\pi_{s'k}(x) = 0$ or

equivalently, when $m_{s'k} = 0$ for $1 \leq k \leq n_{s'}$.

This again leads to a contradiction since we have assumed that $j=s'$ is the first time we have $m_{s'k}$, $1 \leq k \leq n_{s'}$ not all being zero.

The process is repeated until all m_{jk} are shown to be zero contradicting our original assumption.

Therefore $\{\eta_j(x)\pi_{jk}(x)\}$ is a basis for $R_n(x)$, and $P_\varphi(x,y)$ is positive. Since $n=n_1+n_2+\ldots+n_s$ we also have that $P_\varphi(x,y)$ is of degree n-1 in both x and y.

This completes the proof of step 3 and the proof of Lemma 2.4.

REFERENCES

1. T. M. Athay "Numerical Analysis of the Lyapunov Equation With Application to Interconnected Power Systems", M.S. Thesis M.I.T. June 1976.

2. T. M. Athay and N. R. Sandell, "An Iterative Decoupling Solution Method For Large Scale Lyapunov Equations".

3. R. Bellman, Introduction To Matrix Analysis, McGraw-Hill, New York, 1960.

4. R. W. Brockett, Finite Dimensional Linear Systems, Wiley, New York, 1970.

5. W. S. Brown, "On Euclid's Algorithm and the Computation of Polynomial Greatest Common Divisors", A.C.M. Journal, Vol. 18, No.4, Octorber 1971.

6. W. S. Brown and J. F. Traub "On Euclid's Algorithm and the Theory of Subresultants", A.C.M. Journal, vol. 18, No. 4, October 1971.

7. I. N. Herstein, Topics in Algebra, Xerox, Lexington, Mass., 1964.

8. N. Jacobson, Basic Algebra 1, W.H. Freeman, San Francisco, 1974.

9. R. E. Kalman, "Algebraic Characterization of Polynomials Whose Zeroes Lie in Certain Algebraic Domains", Proc N.A.S., Mathamatics, Vol. 64, No. 3, November 1969.

10. D. L. Kleinman, "On An Iterative Technique for Riccati Equation Computations", IEEE Trans A.C., vol. AC-13, No. 1, February 1968.

11. D. L. Kleinman, "An Easy Way to Stabilize a Linear Constant System", IEEE Trans A.C., vol. AC-15, No. 6, December 1970.

12. D. E. Knuth, The Art of Computer Programming Vol. 2: Seminumerical Algorithms, Addison-Wesley, Reading, Mass., 1969.

13. MACSYMA Reference Manual, R. Bogen et al., Project MAC, M.I.T., Cambridge, Mass., 1975.

14. L. Mirsky, An Introduction To Linear Algebra, Oxford University Press, 1961.