Controllability, Observability, Pole Allocation, and State Reconstruction

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Abstract—In this paper we discuss the concepts of controllability, reachability, reconstructibility, and observability and attempt to show why these concepts are important in linear systems theory. We show how the above concepts allow us to solve the existence problem of closed-loop regulation of a linear time-invariant finite-dimensional system. The main results related to this are Theorems 4 and 5. Similar but less sharp results are also presented for time-varying systems. The discussion then proceeds to the precise relationships that exist between input–output and state descriptions of systems. Finally, the question of equivalence of internal and input–output stability is discussed.

I. Introduction

THE most innovative aspect of modern system theory is undoubtedly the prevalence of state-space models for dynamical systems. This has provided a framework which is at the same time extremely general, offers many advantages of a conceptual and philosophical nature, and yields concrete and specific practical results much more directly than other methods were able to provide.

In treating dynamical systems described by state-space models, it was recognized at a very early stage that certain regularity assumptions on the models were of essential importance for the validity of the various synthesis and analysis techniques which were being employed. These assumptions originally appeared as purely mathematical devices [1], [2]. However, it was soon recognized that these properties were of importance in their own right and related to the very possibility of achieving the desired degree of control and obtaining the desired information about the system. These notions were hence termed *controllability* and *observability*.

Reference [3] appears to be the first[1] fundamental study of controllability (in the context of finite-dimensional linear systems) and it is mainly the early work of Kalman and Bucy [4] and Kalman [5] which introduced these concepts in the now familiar synthesis techniques for linear systems. In fact, all of the results of this paper (if not of this issue) are directly or indirectly consequences of Kalman's pioneering work. Other authors who have made important contributions in this area are Gilbert [7] and Luenberger [8].

The concepts of controllability and observability are of particular importance in the design of linear feedback controllers and linear filters for linear stationary systems in the presence of white Gaussian disturbances. It is an important fact that if the system is controllable then the linear feedback system obtained by using the theory of optimal control with a quadratic cost is asymptotically stable. Similarly, if the system is observable the linear filter obtained by using Kalman filtering theory is asymptotically stable. The concepts of controllability and observability are also important in the context of mathematical model building. Indeed, although wanting to use a state-space model to carry out the analytic design task, one often starts with an input–output model which may have been obtained experimentally. In realizing a state-space model which produces the desired input–output relation, one may make an excellent case for requiring this realization to be minimal; that is, to be an accurate representation which does not introduce any phenomena which were not accounted for at least implicitly in the input–output description. It turns out more or less accidentally[2] that this minimality is intimately related to the circle of concepts including controllability and observability.

The purpose of this paper is to introduce the concepts of controllability and observability for linear systems formally. In doing this we will consider several related concepts, e.g., those of reachability and reconstructibility, which are of equal importance, but whose relevance is not as widely appreciated. We feel, moreover, that it is conceptually advantageous to start the discussion by considering these notions in their generality and then to specialize to finite-dimensional linear systems.

We then consider the regulator problem for finite-dimensional linear stationary systems. We first prove that for such systems it is possible to locate the closed-loop poles using state feedback if and only if the system is controllable. We then show that is is possible to build a

Manuscript received July 29, 1971. Paper recommended by R. W. Brockett, Associate Guest Editor. The work of J. C. Willems was supported by the United Kingdom Science Research Council. The work of S. K. Mitter was supported by the Air Force Office of Scientific Research under Grant AFOSR 70-1941.

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[1] For a detailed account of the historical development of this area, see [6, sec. 11].

[2] We are not implying that there is no simple intuitive explanation as to why this should be the case. What we mean is that there does not seem to be any *a priori* reason why the possibility of achieving control and observation is related to this aspect of model building.

state reconstructor (using input and output measurements) with arbitrary error dynamics provided the system is observable. By combining these two results we are able to demonstrate that a feedback compensator can be designed such that the closed-loop system has any preassigned poles provided the system is controllable and observable. The existence question for linear regulators is thus answered. It should be pointed out that the structure of the resulting feedback compensator that is obtained by using the above theory is exactly the same as that obtained by invoking the separation theorem of stochastic optimal control for the design of linear feedback systems with a quadratic performance criterion and in the presence of Gaussian disturbances.

The discussion then turns to time-varying systems, where we consider the questions of stabilizability and state reconstructibility. We consider qualitative aspects of input–output descriptions versus state-space descriptions and find that minimality of the state space is equivalent to controllability (reachability) and observability.

Finally we demonstrate the equivalence of internal Lyapunov stability and input–output stability for uniformly controllable and uniformly observable linear systems.

II. Dynamical Systems

We first introduce the notion of a dynamical system in state-space form. The formal definition attempts to capture the essential properties of finite-dimensional smooth linear systems. We have chosen our input and output functions to be continuous functions of time. This avoids various technical difficulties and simplifies questions related to the existence and uniqueness of solutions for finite-dimensional linear differential systems.

The following definition of a dynamical system is convenient for this discussion. More general concepts may be found in [9].

Let U, Y, and X be normed linear spaces, and let $\mathfrak{U}$, $\mathfrak{Y}$ denote the space of continuous functions defined on $R = (-\infty, \infty)$ with values in U, Y, respectively. $\mathfrak{U}$ is termed the *input space*, $\mathfrak{Y}$ the *output space*, U the *set of input values*, Y the *set of output values*, and X the *state space*.

Let $t_0 \in R$ and let $\mathfrak{J} = \{t \in R | t \geq t_0\}$. Consider the maps $\phi : \mathfrak{J} \times R \times X \times \mathfrak{U} \to X$ and $r : R \times X \times U \to Y$ termed the *state evolution map* and *read-out* map, respectively.

Definition 1: A *dynamical system* is a quintuple $\{\mathfrak{U}, \mathfrak{Y}, X, \phi, r\}$ satisfying the following axioms, for all $u_1, u_2 \in \mathfrak{U}$, $x_0 \in X$, $t_0, t_1, t_2 \in R$, $t_0 \leq t_1 \leq t_2$.

a) *Causality:* $\phi(t_1, t_0, x_0, u_1) = \phi(t_1, t_0, x_0, u_2)$ whenever $u_1(t) = u_2(t)$ for $t_0 \leq t \leq t_1$.

b) *Consistency:* $\phi(t_0, t_0, x_0, u) = x_0$.

c) *Semi-group property:* $\phi(t_2, t_0, x_0, u) = \phi(t_2, t_1, \phi(t_1, t_0, x_0, u), u)$.

d) *Smoothness:* The functions ϕ and r are continuous functions of t, $t \geq t_0$.

A dynamical system in state-space form thus views the generation of outputs from inputs and initial states as occurring via the mechanism of composition of the state evolution map and the output reading map. The state evolution map takes into consideration the memory of the system while the output reading map is memoryless and depends only on the current value of the time, the state, and the input.

Notation: The function $r(t, \phi(t, t_0, x_0, u), u(t))$ defined for $t \geq t_0$ will for convenience be unambiguously denoted by $y(t_0, x_0, u)$.

The most prominent example of a dynamical system in state-space form is the finite-dimensional linear system (FDLS) described by the ordinary differential equation

$$\dot{x} = A(t)x + B(t)u \qquad y = C(t)x \qquad \text{(FDLS)}$$

with $X = R^n$, $U = R^m$, and $Y = R^p$. The matrices $A(t)$, $B(t)$, and $C(t)$ are *throughout* assumed to have compatible dimensions and (again, mainly for technical reasons) to be *continuous and bounded* on $(-\infty, +\infty)$. It is well known that the above differential equation then defines a dynamical system in the sense of the above definition with the state evolution map given by the so-called *variation of constants formula:*

$$x(t) = \phi(t, t_0)x(t_0) + \int_{t_0}^{t} \phi(t, \tau)B(\tau)u(\tau)\, d\tau$$

where the *transition matrix* $\phi(t, \tau)$ is defined as the solution of the matrix differential equation $\dot{\phi}(t, \tau) = A(t)\phi(t, \tau)$, $\phi(\tau, \tau) = I$. The transition matrix satisfies the composition law $\phi(t_2, t_1)\phi(t_1, t_0) = \phi(t_2, t_0)$. See [10] for more details.

The above dynamical system has a very convenient additional property not expressed by the basic axioms of Definition 1; namely, the state evolution map is defined for all t and *not only* for $t \geq t_0$. Such dynamical systems are said to have the *group property*. Most systems described by partial differential equations and delay differential equations *do not* have the group property. Discrete systems and even time-varying systems for which the $A(t)$ matrix does not satisfy the smoothness properties stated above also need not have this property.

Of particular importance in practice are the so-called *stationary dynamical systems* in which ϕ and y commute with the shift operator, i.e., if S_T denotes the map from $\mathfrak{U}$ (respectively $\mathfrak{Y}$) into itself defined by $(S_T u)(t) = u(t + T)$, then $S_T \phi(t, t_0, x_0, u) = \phi(t + T, t_0 + T, x_0, S_T u)$ (respectively, $S_T y(t_0, x_0, u) = y(t_0 + T, x_0, S_T u)$).

Stationary linear finite-dimensional systems (SFDLS) are described by the differential equation

$$\dot{x} = Ax + Bu \qquad y = Cx \qquad \text{(SFDLS)}$$

which corresponds to a system with transfer function matrix $G(s) = C(Is - A)^{-1}B$, where s is a complex variable.

The state-space description of dynamical systems will be contrasted with the input–output description in Section VI.

III. Some Fundamental Properties of State-Space Models

In this section the basic concepts related to controllability, observability, and Lyapunov stability are introduced. The properties of controllability and observability of a dynamical system refer to the influence of the input on the state and of the state on the output, while Lyapunov stability refers to the asymptotic behavior of undriven systems. These notions will be introduced in the context of general dynamical systems as introduced in Section II and then, in the following section, specialized to finite-dimensional linear systems.

The first series of definitions refers to possible transfers in the state space which may result from applying inputs. The concept of controllability refers to transferring an arbitrary initial state to a desired trajectory. This desired trajectory is often an equilibrium point. We will assume this to be the case and take the zero element to represent this equilibrium.

Assumption: It will be assumed that for all $t_0 \in R$, $\phi(t, t_0, 0, 0) = 0$ for $t \geq t_0$ and $r(t_0, 0, 0) = 0$.

Definition 2: Let t_0 be an element of the real line. The state space of a dynamical system is said to be *reachable* at t_0 if, given any $x \in X$, there exists a $t_{-1} \leq t_0$ and a $u \in \mathfrak{U}$ such that $x = \phi(t_0, t_{-1}, 0, u)$. A dynamical system is said to be *controllable* at t_0 if, given any $x_0 \in X$, there exists a $t_1 \geq t_0$ and a $u \in \mathfrak{U}$ such that $\phi(t_1, t_0, x_0, u) = 0$. The state space of a dynamical system is said to be *connected* if, given any $x_0, x_1 \in X$, there exist $t_0 \leq t_1$ and a $u \in \mathfrak{U}$ such that $\phi(t_1, t_0, x_0, u) = x_1$.

Observability refers to the possibility of reconstructing the state from output measurements. As remarked by Kalman *et al.* in [9], there are however two separate state reconstruction problems which one should consider. One refers to deducing the present state from past output observations and the other refers to deducing the present state from future output observations. It is the first property which is essential in filtering. There is also the question of what happens to the inputs in this process. Fortunately this is of no consequence for linear systems, but for nonlinear systems one should distinguish three cases, depending on whether the input is *a priori* known, is arbitrarily assigned, or may be selected in the experiment. This leads us to consider the following series of definitions around the theme of deducing the state from output observations.

Definition 3: Let $t_0 \in R$. A dynamical system is said to be *zero input observable* at t_0 if knowledge of the output $y(t_0, x_0, 0)$ for $t \geq t_0$ uniquely determines x_0. A dynamical system is said to be *observable* at t_0 if for all $x_0 \in X$ and $u \in \mathfrak{U}$ knowledge of the observed output $y(t_0, x_0, u)$ for $t \geq t_0$ determines x_0 uniquely. The state space of a dynamical system is said to be *irreducible* at t_0 if for all $x_0 \in X$ there exists a $u \in \mathfrak{U}$ such that knowledge of the output $y(t_0, x_0, u)$ for $t \geq t_0$ uniquely determines x_0.

The difference between irreducibility and observability is that in the former case the u which yields the initial state x_0 may be a function of x_0 itself, whereas in the latter case any u will do. Any unobservable irreducible system is thus highly unsatisfactory from the viewpoint of state reconstruction. This seems to have been overlooked in the literature.

The analogous definitions referring to reconstructing the present state from past observations become as follow.

Definition 4: Let t_0 be an element of the real line. Then the state of a dynamical system is said to be *zero input reconstructible* at t_0 if knowledge of the output corresponding to an input $u = 0$ for $t \leq t_0$ uniquely determines $x_0 \in X$. Thus the output due to some initial state at "$t_{-1} = -\infty$" is observed and the present state x_0 is to be reconstructed. The state of a dynamical system is said to be *reconstructible* at t_0 if or all $u \in \mathfrak{U}$, knowledge of the output for $t \leq t_0$ uniquely determines $x_0 \in X$.

Note that connectedness implies reachability and controllability, and that observability implies zero-input observability, which in turn implies irreducibility. These notions are in general not equivalent unless, as will be shown in the next section, the system is linear and finite dimensional. The simplest example of a nonlinear system for which this equivalence does not hold is the system $\dot{x} = A\ x\ u;\ y = Cx$, where $x \in R^n$, $u \in R$, and $y \in R^p$, and the matrices A and C are compatible. It should also be noted [11] that reachability, controllability, connectedness, observability, irreducibility, and reconstructibility are preserved under (output) feedback, but that zero-input observability and zero-input reconstructibility are in general not unless the system is again linear and finite dimensional. If a system is not irreducible, then there exist two initial states such that the output to any input will be the same on the interval $[t_0, \infty)$. These two states are thus completely indistinguishable under experimentation and the state space may thus be reduced by eliminating one of these two states from the state space. Note that although the observability definitions ask for the reconstruction of the initial state, this is equivalent to reconstructing the state on the whole interval $[t_0, \infty)$. It may in fact be more logical to demand this in the very definitions.

It should be remarked that the above definitions might not be appropriate for certain applications. In particular, for distributed parameter systems it is sometimes more convenient to make definitions of controllability which require that every state can be driven arbitrarily close to the origin rather than exactly to it.

It is important for many applications to have somewhat stronger controllability and observability properties than merely those implied by the basic definitions. These are now introduced. With linear systems in mind we will norm the input and output spaces by means of an L_2-type norm.

Definition 5: A dynamical system is said to be *uniformly controllable* if there exists a $T > 0$ and a continuous function $\alpha: R \rightarrow R$ such that for all $x_0 \in X$ and $t_0 \in R$ there exists a $u \in \mathfrak{U}$ with $\int_{t_0}^{t_0+T} \|u(t)\|_U{}^2\, dt \leq \alpha(\|x_0\|)$ such that $\phi(t_0 + T, t_0, x_0, u) = 0$. The state space of a dynamical system is said to be *uniformly reachable* if there exists a

$T > 0$ and a continuous function $\alpha: R \to R$ such that for all $x_0 \in X$ and $t_0 \in R$ there exists a $u \in \mathcal{U}$ with $\int_{t_0-T}^{t_0} \|u(t)\|_U^2\, dt \leq \alpha(\|x_0\|)$ such that $\phi(t_0, t_0 - T, 0, u) = x_0$.

A dynamical system is said to be *uniformly zero-input observable* if there exists a $T > 0$ and a monotone increasing function $\beta: R \to R$ with $\beta(0) = 0$ such that for all x_0, $x_1 \in X$ and $t_0 \in R$

$$\int_{t_0}^{t_0+T} \|y(t, \phi(t, t_0, x_0, 0), 0) - y(t, \phi(t, t_0, x_1, 0), 0)\|^2\, dt \geq \beta(\|x_0 - x_1\|).$$

A dynamical system is said to be *uniformly zero-input reconstructible* if there exists a $T > 0$ and a monotone nonincreasing function $\beta: R \to R$ with $\beta(0) = 0$ such that for all x_0, $x_1 \in R$, and $t_0 \in R$

$$\int_{t_0-T}^{t_0} \|y(t, \phi(t, t_0 - T, x_0, 0), 0) - y(t, \phi(t, t_0 - T, x_1, 0), 0)\|^2\, dt \geq \beta(\|\varphi(t_0, t_0 - T, x_0, 0) - \varphi(t_0, t_0 - T, x, 0)\|).$$

The next notions refer to the zero-input stability of dynamical systems in state-space form. Since only the notion of exponential stability will be used in the sequel, attention will be limited to this concept. For a more detailed discussion of Lyapunov stability concepts see [12].

Definition 6: A dynamical system in state-space form is said to be *exponentially stable* if there exist constants M, $\lambda > 0$ such that

$$\|\phi(t_1, t_0, x_0, 0)\| \leq Me^{-\lambda(t_1 - t_0)} \cdot \|x_0\| \quad \text{for all } x_0 \in X \text{ and } t_1 \geq t_0.$$

IV. Finite-Dimensional Linear Systems: Algebraic Conditions for Controllability and Observability

It is usually quite difficult to obtain specific algebraic conditions for controllability and observability. The one class of systems for which such explicit tests may be obtained is the linear finite-dimensional system. The proof of the basic result which states these conditions in terms of invertibility of the W and M matrices given here is based on abstract vector space concepts. Although this approach is not standard and although the conditions may be obtained using much more modest means, it is felt that the results follow more naturally in this framework. In order to keep the discussion on an elementary level, topological notions will be avoided as much as possible.

Definition 7: Let V be an inner product space and let S be a subspace of V. Then the *orthogonal complement* of S, denoted by $S^\perp$, is defined as $S^\perp = \{v \in V | \langle v, s\rangle = 0$ for all $s \in S\}$. If S is a closed subspace (and thus in particular if S is finite dimensional) then $V = S \oplus S^\perp$; i.e., any element $v \in V$ has a unique decomposition into $v = x_1 + x_2$ with $x_1 \in S$ and $x_2 \in S^\perp$. Let L be a linear operator from V_1 into V_2 with V_1 and V_2 inner product spaces. Then the *null space* $\mathfrak{N}(L)$ and the *range space* $\mathfrak{R}(L)$ of L are the subspaces of V_1 and V_2, defined respectively by

$$\mathfrak{N}(L) = \{v_1 \in V_1 | Lv_1 = 0\}$$
$$\mathfrak{R}(L) = \{v_2 \in V_2 | v_2 = Lv_1, \text{ some } v_1 \in V_1\}.$$

The *adjoint* of L, denoted by L^*, is the operator from V_2 into V_1 which satisfies $\langle v_2, Lv_1\rangle_{V_2} = \langle L^*v_2, v_1\rangle_{V_1}$ for all $v_1 \in V_1$ and $v_2 \in V_2$. The adjoint is linear and uniquely defined whenever it exists. It exists when $\mathfrak{R}(L)$ is closed or whenever L is continuous and V_1 and V_2 are Hilbert spaces. In particular L^* exists whenever V_1 and/or V_2 are finite dimensional. It is this case which will be of interest in the sequel. If L^* exists, then so does $(L^*)^*$ and, in fact, $(L^*)^* = L$, and if L_1 and L_2 are linear operators from V_1 into V_2 and V_2 into V_3 which have an adjoint, then L_2L_1 has an adjoint and $(L_2L_1)^* = L_1^*L_2^*$. Consider now the following lemma which is proven in most texts on linear algebra [13].

Lemma 1: Let L be a linear operator from V_1 into V_2 with V_1 and/or V_2 finite dimensional. Then L^* exists, $\mathfrak{N}(L) = \mathfrak{R}(L^*)^\perp$, $\mathfrak{R}(L) = \mathfrak{N}(L^*)^\perp$, $\mathfrak{N}(L) = \mathfrak{N}(L^*L)$, and $\mathfrak{R}(L) = \mathfrak{R}(LL^*)$.

There are two types of linear operators which will be of particular interest in the sequel. Let R^n denote real Euclidean n-space with the usual Euclidean inner product $\langle x_1, x_2\rangle = x_1'x_2$ and let $C_2^n(t_0, t_1)$ denote the inner-product space of all continuous R^n-valued functions on $[t_0, t_1]$ with the inner product $\langle x_1, x_2\rangle = \int_{t_0}^{t_1} x_1'(t)x_2(t)\, dt$. Let $F(t)$ be a real $(m \times n)$ matrix-valued continuous function on $[t_0, t_1]$ and consider now the operators L_1 and L_2 from R^n into $C_2^m(t_0, t_1)$ and from $C_2^m(t_0, t_1)$ into R^n, respectively, defined by:

$$L_1x = F(t)x$$

and

$$L_2z = \int_{t_0}^{t_1} F'(t)z(t)\, dt.$$

It is easily verified that $L_1 = L_2^*$ and hence that $L_2 = L_1^*$. Notice that $L_2L_1 = L_1^*L_1 = L_2L_2^*$ is the linear transformation on R^n induced by the matrix $\int_{t_0}^{t_1} F'(t)F(t)\, dt$.

Turning now to the question of controllability and observability, observe that it follows from the variation of constants formula for the finite-dimensional linear system (FDLS) that the state $x_0 \in R^n$ at t_0 will be transferred to the state $x_1 \in R^n$ at t_1 by the continuous control $u(t)$ if and only if

$$\phi(t_0, t_1)x_1 - x_0 = \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)u(\tau)\, d\tau \triangleq Fu.$$

Similarly it follows that the output on $[t_0, t_1]$ due to the initial state x_0 at t_0 and the control $u(t)$ satisfies

$$y(t) - \int_{t_0}^{t} C(t)\phi(t, \tau)u(\tau)\, d\tau = C(t)\phi(t, t_0)x_0 \triangleq Hx_0$$

where $t_0 \leq t \leq t_1$.

It is immediately clear from the above that the finite-dimensional linear system (FDLS) is controllable at t_0 in some finite time interval $[t_0, t_1]$ *if and only if* the linear operator $F: C_2^m(t_0, t_1) \to R^n$ is onto R^n and is observable in the interval $[t_0, t_1]$ *if and only if* the linear operator $H:R^n \rightarrow C_2^p(t_0, t_1)$ is one–one on R^n. Consequently controllability on $[t_0, t_1]$ requires that $\mathcal{R}(F) = R^n$ and observability on $[t_0, t_1]$ requires that $\mathcal{N}(H) = \{0\}$. These characterizations are, however, somewhat unsatisfactory in the sense that they require knowledge of the range space and null space of operators defined from or into infinite-dimensional spaces. If one considers Lemma 1, it becomes apparent that these conditions can be stated as requiring, respectively, that $\mathcal{R}(FF^*) = R^n$ and $\mathcal{N}(H^*H) = \{0\}$. Since, however, FF^* and H^*H both map R^n into R^n, these linear operators will be characterized by matrices. It is thus entirely natural to rephrase the conditions in terms of FF^* and HH^*. This leads to the following theorem.

Theorem 1: The finite-dimensional linear system (FDLS) is controllable at t_0 if and only if det $W(t_0, t_1) \neq 0$ for some $t_1 \geq t_0$, its state space is reachable at t_0 if and only if det $W(t_{-1}, t_0) \neq 0$ for some $t_{-1} \leq t_0$, and its state space is connected if and only if det $W(t_{-1}, t_1) \neq 0$ for some t_{-1}, t_1. It is observable (zero-input observable, irreducible) at t_0 if and only if det $M(t_0, t_1) \neq 0$ for some $t_1 \geq t_0$ and its state is reconstructible (zero-input reconstructible) at t_0 if and only if det $M(t_{-1}, t_0) \neq 0$ for some $t_{-1} \leq t_0$. The $(n \times n)$ matrices W and M are defined by

$$W(t_0, t_1) \triangleq \int_{t_0}^{t_1} \phi(t_0, \tau)B(\tau)B'(\tau)\phi'(t_0, \tau)\, d\tau$$

$$M(t_0, t_1) \triangleq \int_{t_0}^{t_1} \phi'(\tau, t_0)C'(\tau)C(\tau)\phi(\tau, t_0)\, d\tau.$$

Proof: The operator $FF^*:R^n \rightarrow R^n$ corresponds to the matrix $W(t_0, t_1)$ and the operator $H^*H:R^n \rightarrow R^n$ corresponds to the matrix $M(t_0, t_1)$. Since a linear operator mapping a finite-dimensional space into a space of the same finite dimension is one–one if and only if it is onto, it follows that invertibility of these matrices is equivalent to, respectively, controllability and observability in the interval $[t_0, t_1]$. Since however $\mathcal{R}(W(t_0, t_1))$ is monotone nondecreasing (in the set theoretic sense) with t_1 and since $(M(t_0, t_1))$ is monotone nonincreasing with t_1, the controllability and observability claims follow. The other cases are proven in a similar way.

It remains to determine a control which makes the desired transfer in the case of controllability and to give an algorithm to compute the initial state in the case of observability or reconstructibility. In fact, $u^*(t) = B'(t)\phi'(t_0, t)W^{-1}(t_0, t_1)(\phi(t_0, t_1)x_1 - x_0)$ transfers the system from x_0 at t_0 to x_1 at t_1, while minimizing $\int_{t_0}^{t_1}\|u(t)\|^2\, dt$, and $x_0 = M^{-1}(t_0, t_1)\int_{t_0}^{t_1}\phi'(\tau, t_0)\cdot C'(\tau)v(\tau)\, d\tau$, where $v(t) = y(t) - \int_{t_0}^{t} C(t)\phi(t, \tau)u(\tau)\, d\tau$, is the unique state at t_0 which will yield the output $y(t)$ on $t_0 \leq t \leq t_1$ (or $t_1 \leq t \leq t_0$) when the input is $u(t)$.

The minimizing property of the above control leads immediately to the following conditions for uniform controllability and uniform observability.

Theorem 2: The finite-dimensional linear system (FDLS) is uniformly controllable (uniformly reachable) if and only if for some $T > 0$ there exists an $\epsilon_1 > 0$ such that $W(t_0, t_0 + T) \geq \epsilon_1 I$ for all $t_0 \in R$; it is uniformly observable (uniformly reconstructible) if and only if for some $T > 0$ there exists an $\epsilon_2 > 0$ such that $M(t_0, t_0 + T) \geq \epsilon_2 I$ for all $t_0 \in R$.

Notice that, in view of the boundedness assumption on $A(t)$, uniform controllability is equivalent to the existence for some $T > 0$ of constants $\epsilon_1, \epsilon_2 > 0$ and M_1, M_2 such that

$$0 \leq \epsilon_1 I \leq W(t_0, t_0 + T) \leq M_1 I$$

and

$$0 \leq \epsilon_2 I \leq \phi(t_0 + T, t_0)W(t_0, t_0 + T)\phi'(t_0 + T, t_0) \leq M_2 I$$

for all $t_0 \in R$. Thus for the systems under consideration the present definition is equivalent to the one originally proposed by Kalman [5]. The same holds for observability.

Theorem 1 suffers from the drawback that it does not give controllability and observability conditions in terms of the original model which involves the matrix $A(t)$, but instead in terms of the associated transition matrix $\phi(t, \tau)$. It is, however, possible to remedy this situation, at least for sufficiently smooth systems. In the case of stationary systems, controllability and observability turn out to be determined by the following well-known conditions.

Theorem 3: The stationary finite-dimensional linear system (SFDLS) is controllable (reachable, connected) at t_0 if and only if rank $\{B, AB, A^2B, \cdots, A^{n-1}B\} = n$. It is observable (reconstructible, irreducible) at t_0 if and only if rank $\{C', A'C', \cdots, (A')^{n-1}C'\} = n$.

Proof: See, for instance, [10, p. 79].

Silverman and Meadows [14] and Chang [15] have developed algebraic conditions which are applicable to linear time-varying dynamical systems. These results require that the matrices $A(t)$, $B(t)$, and $C(t)$ be sufficiently smooth.

Remark: For finite-dimensional stationery linear systems, if the system is controllable in finite time, then it is controllable in an arbitrarily small time. This result is in general no longer true for infinite-dimensional stationery linear systems.

A very useful concept is that of *dual* systems. Dual dynamical systems have the intriguing property that control problems of one become observation problems for the other, and vice versa. Optimal control problems thus lead to optimal estimation problems. It is fair to say that the concept is so far not understood in any degree of generality. It is analogous, but not identical, to the concept of reciprocal systems: one strongly suspects these concepts to be relevant for general nonlinear systems.

We begin with a simple identification of two adjoint operators. It is easy to show that if $L_1:C_2^m(t_0, t_1) \rightarrow R^n$ is defined by $x(t_1) = L_1 u$ with

$$\dot{x} = A(t)x + B(t)u, \qquad x(t_0) = 0$$

then $L_1^*:R^n \rightarrow C_2^m(t_0, t_1)$ is given by $y(t) = L_1^*p(t_1)$, $t_0 \leq t \leq t_1$ with

$$\dot{p} = -A'(t)p, \qquad y = B'(t)p.$$

Similarly, if $L_2: R^n \to C_2^p(t_0, t_1)$ is defined by $y(t) = L_2 x(t_0)$, $t_0 \leq t \leq t_1$, with

$$\dot{x} = A(t)x, \qquad y = C(t)x,$$

then $L_2^*: C_2^p(t_0, t_1) \to R^n$ is given by $p(t_0) = L_2^* u$ with

$$\dot{p} = -A'(t)p - C'(t)u, \qquad p(T_1) = 0.$$

These formulas are easy to verify after noticing that the transition matrices of $\dot{x} = A(t)x$ and $\dot{p} = -A'(t)p$ are related by $\psi(t, \tau) = \phi'(\tau, t)$.

By Lemma 1, it follows that there exists a very simple relation between the null spaces and the range spaces of L_1, L_2 and their adjoints. These associations give a relationship between controllability and observability concepts. The difficulty, however, is that the operators L_1^* and L_2^* are defined backwards in time (the appearance of $-A'(t)$ strongly suggests that this better be the case if these are to be concepts of any degree of generality). Thus, in order to associate controllability properties with observability properties of the mathematical adjoint, one would have to introduce the concept of a dynamical system which runs backwards in time or reverse the time direction of the adjoint and consider the dynamical system thus obtained. Thus one associates with the system

$$\dot{x} = A(t)x + B(t)u, \qquad y = C(t)x$$

its so-called *dual* defined by

$$\dot{z} = A_1(t)z + B_1(t)v, \qquad w = C_1(t)z$$

where

$$A_1(t_0 + t) = A'(t_0 - t)$$
$$B_1(t_0 + t) = C'(t_0 - t)$$
$$C_1(t_0 + t) = B'(t_0 - t).$$

In view of the above remarks it is now very simple to prove the following correspondences between a dynamical system and its dual: controllability at $t_0 \leftrightarrow$ reconstructibility at t_0, and reachability at $t_0 \leftrightarrow$ observability at t_0.

Remark: Any finite-dimensional linear dynamical system may be decomposed into four subsystems: (1) a controllable and observable subsystem; (2) a subsystem which is controllable but not observable; (3) a subsystem which is observable but not controllable; and finally, 4) a subsystem which is neither controllable nor observable.This is *Kalman's canonical structure theorem* (see the paper by Silverman, this issue).

V. Pole Allocation, State Reconstruction, and Closed-Loop Regulation

The following questions about the control of dynamical systems are of fundamental importance in mathematical system theory—how to: 1) design a state feedback law which regulates the closed-loop response; 2) design a state reconstructor, i.e., a system which deduces the current state from past observations of the output; and 3) design an output feedback compensator which regulates the closed-loop response.

In this section we will treat these questions for the stationary finite-dimensional linear dynamical system introduced in Section II.

$$\dot{x} = Ax + Bu, \qquad y = Cx. \qquad \text{(SFDLS)}$$

We assume as before that the input $u(t)$ is a continuous function of t. The problems of regulation and state reconstruction discussed above will be "solved" in the same class of dynamical systems, i.e., we will only use stationary finite-dimensional linear systems to achieve a possible design procedure.

Consider first the question of regulation under state feedback and assume that the feedback $-Kx$ is being applied to the system. The closed-loop response is then governed by the dynamical equations:

$$\dot{x} = (A - BK)x + Bu, \qquad y = Cx.$$

A representative feature of the response of this closed-loop system is given by the location of its poles, i.e., by the zeros of det $(Is - A + BK)$. The question thus arises of under what conditions is it possible to assign the poles of a dynamical system arbitrarily by suitably choosing the feedback gain matrix K. More precisely, given an arbitrary polynomial

$$r(s) = s^n + r_{n-1}s^{n-1} + \cdots + r_0$$

with real coefficients, when does there exist, for given matrices A and B, at least one $(m \times n)$ matrix K such that det $(Is - A + BK) = r(s)$? The possibility of pole assignment turns out to be equivalent to controllability. This property of state feedback appears to have been known for a long time in the single-input case (see [6] for historical comments), but has only recently come to the foreground for the multiple-input case.

In the multi-input case the first results in this direction are due to Langenhop [16] and Popov [17]. They proved that, given an arbitrary polynomial $r(s)$ with coefficients in the field of real or complex numbers, there exists a matrix K (possibly complex) such that det $(Is - A + BK) = r(s)$ if and only if the system is controllable. The proof given by Popov [17] makes interesting use of Kalman's canonical structure theorem and is different from Langenhop's. Wonham [18] gave a different proof and further showed that, if the polynomial has coefficients in the field of real numbers, then the matrix K can be chosen to be real. In our context, Wonham's appears to be the first complete proof. A different proof and algorithms for pole assignment was presented by Simon and Mitter [19] and Simon [34]. The proof given in this paper uses a lemma due to Heymann [20]. The lemma also appears in Popov [31, p. 261, proposition 2, appendix A].

Theorem 4: There exists a real $m \times n$ matrix L such that det $(Is - A - BL) = s^n + r_{n-1}s^{n-1} + \cdots + r_0$ for arbitrary real coefficients $\{r_0, r_1, \cdots, r_{n-1}\}$ if and only if the system $\dot{x} = Ax + Bu$ is controllable; i.e., if and only if the $(nm \times n)$ matrix $[B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ is of rank n.

The proof of the theorem will proceed via several propositions.

Proposition 1—Sufficiency for the Case where B is a Column Vector: Let A be an $n \times n$ matrix and let b be an $n \times 1$ matrix. If the system is controllable, then there is a k' $(1 \times n)$ such that the characteristic polynomial $A - bk'$ is an arbitrary preassigned polynomial (of degree n).

Sketch of Proof: Since the system is controllable, the controllability matrix $H = (b, Ab, \cdots, A^{n-1}b)$ has rank n. Therefore the n columns of H span R^n. It is then well known that there exists a basis of the state-space $X = R^n$ such that the system $\dot{x} = Ax + bu$ may be brought in the standard controllable form [10] $\dot{z} = A_1 z + b_1 u$ where

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \cdots & 0 \\ 0 & 0 & 1 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots 0 & 1 \\ a_0 & a_1 & \cdots & a_{n-1} \end{pmatrix}.$$

It is then easy to see that the characteristic polynomial of $A + b_1k'$ may be chosen to have any preassigned form.

Remark: The proof of the above proposition consists of first putting the system in standard controllable form, and once this is done, the pole assignment becomes transparent. The two parts of this algorithm may be combined to give the following direct algorithm for pole assignment [16]. Let

$$p(s) = s^n + p_{n-1}s^{n-1} + \cdots + p_0 = \det(Is - A)$$

and

$$r(s) = s^n + r_{n-1}s^{n-1} + \cdots + r_0 = \det(Is - A + bk')$$

be the open-loop and desired closed-loop characteristic polynomials, respectively. Form the matrix

$$P = \begin{pmatrix} 1 & 0 \cdots & 0 \\ p_{n-1} & & \vdots \\ \vdots & & \vdots \\ p_1 & & 1 \\ p_0 & p_1 \cdots & p_{n-1} \end{pmatrix}$$

then the vector k is given by

$$k = \begin{bmatrix} b' \\ b'A' \\ \vdots \\ b'(A')^{n-1} \end{bmatrix} P^{-1} \begin{bmatrix} r_{n-1} - p_{n-1} \\ r_{n-2} - p_{n-2} \\ \vdots \\ r_0 - p_0 \end{bmatrix}.$$

We will now show that the case when B is $n \times m$ may be reduced to the case that $m = 1$ by first applying state feedback.

Proposition 2: If the system $\dot{x} = Ax + Bu$ is controllable, then there exist matrices $L(m \times n)$ and $b_1(m \times 1)$ such that the system $\dot{x} = (A - BL)x + b_1 v$ is controllable and b_1[3] is in the column space of B.

Proof: Let b_i denote the ith column of B and let E_i be the subspace spanned by the vectors $b_i, Ab_i, \cdots, A^{n-1}b_i$. Since $[B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ is of full rank n, there are n vectors of the form $A^j b_i$ which form a basis for R^n. Let $\{e_1, e_2, \ldots, e_n\}$ denote this basis. Thus any $x \in R^n$ may be written as $\sum_{i=1}^n a_i e_i$. But $e_i \in E_i$ and thus the linear combination $\sum_{i=1}^n a_i e_i$ may be rearranged as $x_1 + x_2 + \cdots + x_m$ with $x_i \in E_i$.

In general the subspace E_i will not be independent in the sense that $E_i \cap E_j \neq \{0\}$ for $i \neq j$. However there are subspaces S_i of E_i such that $S_i \cap S_j = \{0\}$ for $i \neq j$ and such that $R^n = S_1 + S_2 + \cdots + S_m$. To see this, let $S_1 = E_1$. Then S_1 has a basis $b_1, Ab_1, \cdots, A^{k_1-1}b_1$ where K_1 is an integer ≥ 1.[4] Let $b_2{}^*$ be the first column of B not in S_1 and let T_1 be a subspace of R^n containing $b_2{}^*$ such that $R^n = S_1 \oplus T_1$.[5] This may be done by extending the basis for S_1 to a basis for R^n containing $b_2{}^*$. Let S_2 be the subspace spanned by $b_2{}^*$. $Ab_2{}^*, \cdots, A^{k_2-1}b_2{}^*$ with k_2 the largest integer such that these vectors are linearly independent in T_1. Then $S_1 \oplus S_2$ is a subspace of R^n with basis $b_1, Ab_1, \cdots, A^{k_1-1}b_1, b_2{}^*, Ab_2{}^*, \cdots, A^{k_2-1}b_2{}^*$; moreover, $S_1 \cap S_2 = \{0\}$.

Now let $b_3{}^*$ be the first column of B not in $S_1 \oplus S_2$. By the same process as above, there is a subspace T_2 containing $b_3{}^*$ such that $E = S_1 \oplus S_2 \oplus T_2$. Also, there is a number k_3 so that $b_3{}^*, \cdots, A^{k_3-1}b_3{}^*$ are linearly independent in T_2. Let S_3 be the subspace spanned by $b_3{}^*, \cdots, A^{k_3-1}b_3{}^*$. Then $S_1 \oplus S_2 \oplus S_3$ is a subspace of R^n with basis $b_1, \cdots, A^{k_3-1}b_3{}^*$; moreover, $(S_1 \oplus S_2) \cap S_3 = 0$.

Since R^n is finite dimensional and equal to $E_1 \oplus E_2 \cdots \oplus E_m$, this process terminates at some stage. Hence $R^n = S_1 \oplus S_2 + \cdots \oplus S_m$ as indicated, and a basis for E is obtained by combining the bases for the subspaces. By rearranging the columns of B (hence, the coordinates of the control) it can be assumed that the first r columns of B were in this process. Hence, the basis is $b_1, \cdots, A^{k_1-1}b_1, \cdots, b_r, \cdots, A^{k_r-1}b_r$, and $\sum_{i=1}^r k_i = n$. Let $Q = [b_1, \cdots, A^{k_1-1}b_1, \cdots, b_r, \cdots, A^{k_r-1}b_r]$ be the matrix whose columns are these basis vectors. Hence Q is invertible.

Define an $m \times n$ matrix $S = [s_1, \cdots, s_n]$ where each column is an m-vector defined as follows: $s_{r_j} = \epsilon_{j+1}{}^{(m)}$ if $r_j = \sum_{i=1}^j k_i$ and $j = 1, \cdots, r-1$; $s_j = 0$ otherwise, where $\epsilon_i{}^{(m)}$ is the ith standard basis vector of R^m. Finally, let $L = SQ^{-1}$; then $LQ = S$. Consider $LA^{k_j-1}b_j$ since $A^{k_j-1}b_j$ is the r_jth column of Q, $LA^{k_j-1}b_j = \epsilon_{j+1}{}^{(m)}$ for each $j = 1, \cdots, r-1$. Similarly, $LA^i b_j = 0$ for all other powers of A.

Let $A_1 = A - BL$, then $H_1 = [b_1, \cdots, A_1{}^{n-1}b_1]$ is the

[3] Any nonzero vector in the column space of B will do for b_1.

[4] The reader may find it useful to look up the notion of a cyclic subspace of a vector space V with respect to a linear map A in a book on matrix theory. See, for example, [33, p. 185].

[5] $\oplus$ indicates direct sum; i.e., $R^n = S_1 \oplus T_1$, and $S_1 \cap T_1 = \{0\}$.

controllability matrix of the pair (A_1, b_1). Consider the following columns of H_1.

$$
\begin{aligned}
b_1 &= b_1.\\
A_1 b_1 &= Ab_1 - BLb_1 = Ab_1.\\
A_1^2 b_1 &= A^2 b_1 - BLAb_1 = A^2 b_1.\\
&\vdots\\
A_1^{k_1-1} b_1 &= A^{k_1-1} b_1.\\
A_1^{k_1} b_1 &= A^{k_1} b_1 + BLA^{k_1-1} b_1 = b_2 + \text{linear combination of previous vectors.}\\
A_1^{k_1+1} b_1 &= Ab_2 + \text{linear combination of previous vectors.}\\
&\vdots\\
A_1^{n-1} b_1 &= A^{k_r-1} b_r + \text{linear combination of previous vectors.}
\end{aligned}
$$

Thus the jth column of H_1 is the jth column of Q plus a linear combination of the previous $j-1$ columns. Hence the columns of H_1 are linearly independent so rank of H_1 is n. Clearly b_1 is in the column space of B. Proposition 1 can now be applied to the controllable system $\dot{x} = (A - BL)x + b_1 v$ to complete the proof of the theorem.

Proof of Theorem 3—Necessity: Let $a_1, \cdots, a_n$ be distinct scalars such that $\det(A - a_i I) \neq 0$ for $i = 1, 2, \cdots, n$. By hypothesis, there is an L such that $\det(A + BL - sI) = (a_1 - s) \cdots (a_n - s)$, that is, $a_1, \cdots, a_n$ are the eigenvalues of $A + BL$. Then for each a_i there is a $v_i \in R^n$ such that $(A + BL)v_i = a_i v_i$ and $v_i \neq 0$. Since $(a_i I - A)$ is invertible, this can be written as

$$(a_i I - A)^{-1} BLv_i = v_i, \qquad i = 1, 2, \cdots, n.$$

For each a_i, there are scalars $b_j(a_i)$ such that

$$(a_i I - A)^{-1} = \sum_{j=1}^{n} b_j(a_i) A^{j-1}.$$

To see this, let $d = \sum_{i=0}^{n} d_i s^i$ be the characteristic polynomial of $a_i I - A$. Then since $d_0 = \det(a_i I - A)$, $I = (a_i I - A)(-(d_1/d_0)I -, \cdots, -(d_n/d_0)(a_i I - A)^{n-1})$. This shows the above, noting that $(a_i I - A)^k = \sum_{j=0}^{k} \binom{k}{j} (a_i I)^j (-1)^{k-j} A^{k-j}$.

Combining the above,

$$\sum_{j=1}^{n} A^{j-1} B (b_j(a_i) L v_i) = v_i \tag{1}$$

for $i = 1, \cdots, n$.

Let $H = (B, AB, \cdots, A^{n-1}B)$ and consider Hy for some $y \in R^{nm}$. If y is written as n blocks, each of length m, i.e., $y = (y_1', \cdots, y_n')$ where each y_i is an m-tuple, then

$$Hy = (B, AB, \cdots, A^{n-1}B) \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \sum_{j=1}^{n} A^{j-1} B y_j.$$

Letting $y_j = b_j(a_i) L v_i$, (1) becomes

$$Hy_i^* = v_i \tag{2}$$

where $y_i^* = (b_1(a_i)Lv_i, \cdots, b_n(a_i)Lv_i)'$.

Since the eigenvalues of $A + BL$ are distinct, the eigenvectors $v_1, \cdots, v_n$ are linearly independent and form a basis for R^n. Then any $v \in R^n$ can be expressed as $v = \sum_{i=1}^{n} k_i v_i$; so, using (2), $v = \sum_i k_i v_i = \sum_i k_i H y_1^* = H(\sum_i k_i y_1^*)$. Then the range of H is R^n, so H has rank n. Therefore, the system is controllable.

Theorem 3 provides a method for approaching a large class of control problems. Other design procedures based on optimality criteria are treated elsewhere in this issue.

Remarks: Theorem 4 has been generalized in several directions. A generalization to periodic systems may be found in [35]. For a restricted class of time-varying systems a pole-allocation result has been proved in [29]. The theorem is also true for discrete-time finite-dimensional linear systems defined over an arbitrary field [36].

The pole-allocation result of Theorem 4 states that the characteristic polynomial of the closed-loop system matrix may be chosen at will by the use of state feedback. More generally, one would like to answer the question as to what Jordan forms of the system matrix can be realized using state feedback (in fact it is clear from the single-input case that this Jordan form cannot be chosen at will since $A - bk'$ will always have the same characteristic as minimal polynomial). This aspect of the state feedback problem has been studied by Rosenbrock [32]. It turns out that for a Jordan form to be possible certain *a priori* inequalities have to be satisfied. From an algebraic point of view, the results of Rosenbrock appear to present pretty much the definitive story as to what can be achieved using state feedback.

We will consider the second question raised in the introduction to this section, namely, the design of a state reconstructor. A natural approach for such a design is to attempt to discover a dynamical system whose state will be an estimate of the state to be reconstructed. The system to be designed has knowledge of the input and output of the dynamical system for which we are designing a state reconstructor.

Consider as a possible choice for such a system the n-dimensional linear system

$$\dot{\hat{x}} = F\hat{x} + Lu + Hy$$

where u and y represent the input and output of the systems (SFDLS) and where $\hat{x}$ represents the estimated state. Since one would like the dynamics of the original system and the state reconstructor to be compatible, it is natural to choose $L = B$ and $F = A - HC$ such that the estimator dynamics become

$$\dot{\hat{x}} = A\hat{x} + Bu - H(\hat{y} - y)$$

with $\hat{y} = C\hat{x}$. The estimator is thus driven by the error of the estimated output and the observed output through the feedback gain H. The error $e = \hat{x} - x$ is governed by the differential equation

$$\dot{e} = (A - HC)e.$$

A possible criterion for the quality of the state reconstructor are the eigenvalues of the matrix governing the error equation, i.e., the zeros det $(Is - A + HC)$. The question thus arises of under what conditions on A and C can one make the zeros of det $(Is - A + HC)$ arbitrary by appropriately choosing H. This question is precisely the dual of the pole-allocation question, and this observation immediately leads to the following theorem.

Theorem 5: There exists an $(n \times p)$ matrix H such that the error dynamics of the state reconstructor are governed by $\dot{e} = (A - HC)e$ where $e = \hat{x} - x$ with det $(Is - A + HC) = s^n + r_{n-1}s^{n-1} + \cdots + r_0$ for arbitrary real coefficients $\{r_0, r_1, \cdots, r_{n-1}\}$ if and only if the system $\dot{x} = Ax + Bu$, $y = Cx$ is reconstructible (observable); i.e., if and only if the $(np \times n)$ matrix $[C' \vdots A'C' \vdots \cdots \vdots (A')^{n-1}C']$ is of full rank n.

Proof: Since $\dot{x} = Ax + Bu, y = Cx$ is reconstructible, $\dot{x} = A'x + C'v$ is controllable. Hence H' may be chosen such that det $(Is - A' + C'H')$ is preassigned. Since det $(Is - A' + C'H')$ = det $(Is - A + HC)$, the result follows.

Remark: The above state reconstructor is sometimes called an *observer*. It suffers from one fundamental drawback; namely, that in reconstructing the state from the outputs y we ignore the fact that we know $y = Cx$ exactly and thus that it would be logical to choose $\hat{x}$ such that $C\hat{x} = y$. In other words, rather than estimating the whole state x, it suffices to estimate the components of x in the null space of C. This problem has been studied by Luenberger [8] who showed that there exists a state reconstructor of order $(n - p)$ whose state in combination with the observed output results in an error vector which has p components identically zero and whose $(n - p)$ remaining components are governed by a stationary linear dynamical system of order $(n - p)$ with preassigned eigenvalues of its system matrix. For a more complete account of this see Wonham [22].

The regulator design based on pole allocation explained in the first part of this section is based on exact knowledge of all the states. For technological reasons it is often very difficult and inefficient to measure the complete state vector, and one only has access to the output for measurements. The question thus arises whether it is possible to use the above ideas to design a compensator which has as its input the output of the system to be controlled. A logical and intuitive procedure in obtaining such an output compensator is to separate the task of state reconstruction and feedback regulation by first designing a state reconstructor and then using the estimated value of the state (instead of the actual value of the state) in the feedback controller. This approach is a prelude to the separation theorem for stochastic optimal control. In the present context it should be considered as a reasonable first approach to the design of an output feedback controller.

Using this idea we obtain the following closed-loop dynamical system:

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx\\ \dot{\hat{x}} &= A\hat{x} + Bu - H(\hat{y} - y)\\ \hat{y} &= C\hat{x}\\ u &= -K\hat{x}.\end{aligned}$$

Written in terms of x and $e = \hat{x} - x$, this closed-loop dynamical system may be written as

$$\begin{aligned}\dot{x} &= (A - BK)x - BKe\\ \dot{e} &= (A - HC)e\end{aligned}$$

or

$$\frac{d}{dt}\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} A - BK & -BK \\ 0 & A - HC \end{pmatrix}\begin{pmatrix} x \\ e \end{pmatrix}.$$

The above representation shows that the poles of the closed-loop system are the zeros of det $(Is - A + BK)$ det $(Is - A + HC)$ and may hence be allocated at will by choosing K and H *if and only if* the system (SFDLS) is both controllable and reconstructible (observable); i.e., if and only if the $(nm \times n)$ and $(np \times n)$ matrices $[B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ *and* $[C' \vdots A'C' \vdots \cdots \vdots (A')^{n-1}C']$ have full rank n.

One may of course replace the state reconstructor in this compensator by a Luenberger observer. This in fact yields a simpler design which specializes to the state feedback case.

The state feedback regulator, the state reconstructor, and the output feedback regulator are shown in Fig. 1.

Remark: The structure of the feedback compensator shown in Fig. 1 is precisely the same as that obtained by invoking the separation theorem of stochastic control and designing the feedback compensator for a linear system with Gaussian disturbances on the basis of deterministic optimal control theory for a quadratic cost and Kalman filtering theory.

VI. Stabilizability and State Reconstructibility for Time-Varying Systems

It was shown in Section III that the control

$$u(t) = -B'(t)\phi'(t_0, t)W^{-1}(t_0, t_1)x_0, \qquad t_0 \le t \le t_1$$

will transfer the state of a controllable linear finite-dimensional system from state x_0 at time t_0 to state 0 at time t_1. Implemented in a feedback form, this would lead to the feedback control law

$$u = -B'(t)W^{-1}(t, t_1)x$$

which, since $\lim_{t \to t_1} \|W^{-1}(t, t_1)\| = \infty$, calls for unbounded feedback gains. This is not surprising since it amounts to driving a linear system to zero in finite time. The problem thus arises as to whether it is possible to design a continuous bounded feedback control law which makes the closed-loop system exponentially stable. It will be shown that under suitable controllability assumptions this is indeed possible.

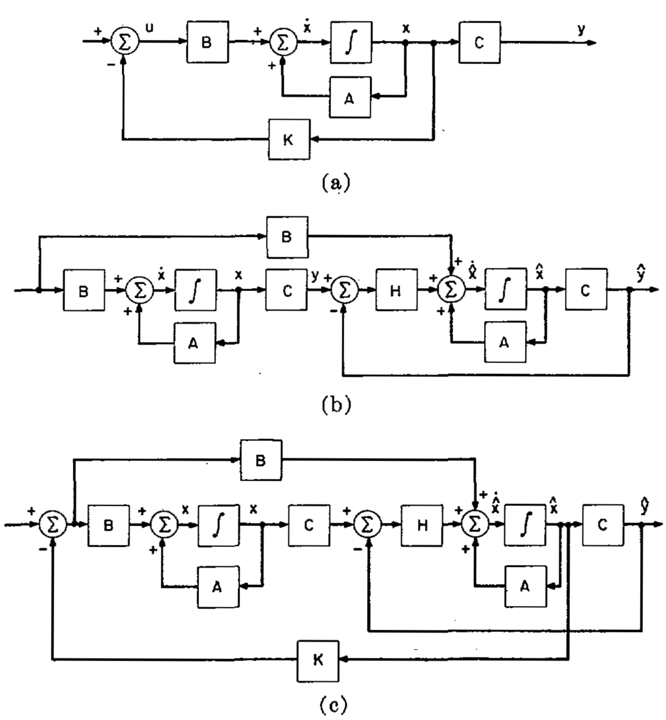

Fig. 1. (a) State feedback controller. (b) State reconstructor. (c) Output feedback controller.

Theorem 6: Assume that the system (FDLS) is uniformly controllable. Then there exists a bounded continuous matrix $G(t)$ such that the feedback system $\dot{x} = A(t)x + B(t)u$; $u = -G(t)x$ is exponentially stable.

The proof is based on the optimal control results of [5]. The details of the proof may be found there and only an outline will be included.

Outline of the Proof: Consider the Riccati differential equation $\dot{K}_T = -A'(t)K_T - K_TA(t) + K_TB(t)B'(t)K_T - I$ with $K_T(T) = 0$. Then $\lim_{T\to\infty} K_T(t) = K_\infty(t)$ exists for all t. This limit is approached monotonically from below and also satisfies the above Riccati differential equation [$K_\infty(t)$ is thus differentiable and hence continuous]. Moreover $K_\infty(t)$ is bounded on any half line $[t_0, \infty)$ by uniform controllability. The result follows if it is proved that the system $\dot{x} = (A(t) - B(t)B'(t)K_\infty(t))x$ is exponentially stable. This, however, follows by considering $x'K_\infty(t)x$ as a Lyapunov function. For details see [5].

It was shown in Section III that one may reconstruct the present state from past observations of the input and the output. This problem is the dual of the control problem. The control nature of the state reconstruction problem may be brought out as follows.

Suppose we consider a state reconstructor governed by the equations

$$\dot{\hat{x}} = A(t)\hat{x} + B(t)u + r \qquad \hat{y} = C(t)\hat{x}$$

with $\hat{x}$ the state of the state reconstructor, u the input to the plant, and r a correction signal (which is to be designed). The error $e = \hat{x} - x$ is then governed by the equation

$$\dot{e} = A(t)e + r.$$

In choosing the correction input r in a feedback form it should be realized that one has incomplete information of e in that only x and Cx are known. Assume, however, that r is chosen to be the dual of the feedback control law considered in the beginning of this section, i.e.,

$$r = -M^{-1}(t, t_0)C(t)e = M^{-1}(t, t_0)(y - \hat{y}).$$

This correction signal is admissible since it only depends on y and leads to the error equation

$$\dot{e} = (A(t) - M^{-1}(t, t_0)C(t))e$$

which is such that $e(t_0) = 0$. This state reconstructor, however, asks for unbounded gains and will be unacceptable in most situations. By dualizing Theorem 6 one obtains a procedure for designing a state reconstructor for which the error approaches zero at an exponential rate.

Theorem 7: Assume that the system (FDLS) is uniformly observable. Then there exists a bounded continuous matrix $H(t)$ such that the system

$$\dot{\hat{x}} = A(t)\hat{x} + B(t)u - H(t)(\hat{y} - y)$$

with $\hat{y} = C(t)\hat{x}$ is a state reconstructor such that the error equation $\dot{e} = (A(t) - H(t)C(t))e$ is exponentially stable.

The combination of the stabilizing control law with the above state reconstructor in a loop for which the state reconstruction and control function are separated leads to the following result

Theorem 8: Assume that the system (FDLS) is uniformly controllable and uniformly observable. Then there exist bounded continuous matrices $G(t)$ and $H(t)$ such that the closed-loop system

$$\begin{aligned}
\dot{x} &= A(t)x + B(t)u \\
y &= C(t)x \\
\dot{\hat{x}} &= A(t)\hat{x} - B(t)u + H(t)(y - \hat{y}) \\
\hat{y} &= C(t)\hat{x} \\
u &= -G(t)\hat{x}
\end{aligned}$$

is exponentially stable.

VII. Input–Output Descriptions of Dynamical Systems

There are two (fundamentally different but essentially equivalent) possible descriptions of dynamical systems. One is the so-called state-space description. An appropriate set of axioms for a large class of such systems has been introduced in Section II. The other is the input–output description. An axiomatic framework for the study of a class of systems described in terms of input–output data and which possesses all of the essential properties of finite-dimensional linear systems will now be introduced.

As before, let $\mathcal{U}$ denote the collection of all continuous U-valued functions on $(-\infty, +\infty)$. The spaces Y and $\mathcal{Y}$ are similarly defined. We assume again that U and Y are normed vector spaces.

Consider[6] now the subset $\mathcal{U}^+$ of $\mathcal{U}$ defined as

$$\mathcal{U}^+ = \{u \in \mathcal{U} | u(t) = 0 \text{ for } t \leq t_0, \text{ some } t_0 \in R\}$$

and assume that $\mathcal{Y}^+$ is similarly defined. Consider now a mapping F from $\mathcal{U}^+$ into $\mathcal{Y}^+$. Then F is said to be *causal* on $\mathcal{U}^+$ if $u_1, u_2 \in \mathcal{U}^+$ with $u_1(t) = u_2(t)$ for $t \leq t_0$ implies that $(Fu_1)(t) = (Fu_2)(t)$ for $t \leq t_0$.

Definition 8: A dynamical system in input–output form is a map from $\mathcal{U}^+$ into $\mathcal{Y}^+$ which is causal on $\mathcal{U}^+$.

The problem of realization is to construct a dynamical system in state-space form such that it generates the same input–output pairs as the dynamical system in input–output form. The question thus reduces to constructing an appropriate state-space X and suitable maps ϕ and r.

The problem of realization is trivially solved if one requires no additional properties of the state-space model. Indeed let X denote the set of all continuous U-valued functions on $[0, \infty)$ which vanish for sufficiently large values of their argument. Consider now a dynamical system in input–output form and let us take the state space at time t to be the element of X which satisfies $x(s) = u(t - s)$ for $t \geq 0$. This element of X will indeed qualify for a state since, in trying to satisfy the requirement that the state should summarize the essential features about the past input, we in fact decided to store the complete past input. It is in fact a simple matter [11] to induce the appropriate state transition map and readout map to go along with this choice of the state. This shows that every input–output dynamical system has a state-space realization and yields a rather interesting decomposition of a dynamical system into a linear stationary reachable dynamical part and a memoryless part. Of course one can essentially never expect this system to be observable. This realization is indeed extremely inefficient.

A much more interesting realization is the minimal realization for which the state-space X has as few elements as possible. This realization is always reachable and irreducible. It is logical to consider as the state at time t the equivalence class of inputs up to time t which yield the same output after time t, no matter how these inputs are continued after time t. In other words, the inputs $u_1 \in \mathcal{U}$ and $u_2 \in \mathcal{U}$ will result in the same state at time t if any $v_1, v_2 \in \mathcal{U}$ with $v_1(s) = u_1(s)$ and $v_2(s) = u_2(s)$ for $s \leq t$ and $v_1(s) = v_2(s)$ for $s \geq t$ yield outputs $(Fv_1)(t) = (Fv_2)(t)$ for $t > s$. One may then proceed to induce the maps ϕ and r from there. This realization procedure works well for stationary systems. The difficulty with time-varying systems arises from the fact that usually the above equivalence class idea will result in a state space which is itself time varying. This difficulty is basically a consequence of a deficiency in the axiomatic framework for dynamical systems in state-space form in that these axioms do not allow for systems with a time-varying state space. For most applications this inconsistency is of no consequence, but in order to obtain in a simple manner an abstract solution to the minimal realization problem for time-varying systems, this problem proves to be a stumbling block. This difficulty may be overcome by a suitable modification of the axioms of dynamical systems.

The input–output dynamical system which will be studied in this paper is described by a Volterra integral equation. Let $w(t, \tau)$ be a $(p \times m)$ matrix defined and continuous for $t \geq \tau$, and consider the input–output system defined with $U = R^m$ and $Y = R^p$ as follows. Consider $u \in \mathcal{U}^+$ and let t_0 be such that $u(t) = 0$ for $t \leq t_0$. Then

$$y(t) \triangleq \begin{cases} 0, & \text{for } t \leq t_0 \\ \displaystyle\int_{t_0}^{t} w(t, \tau)u(\tau)\, d\tau, & \text{for } t \geq t_0 \end{cases}. \qquad \text{(LS)}$$

This system will in short be denoted by $y = Wu$.

The next section is concerned with the realization of a system (LS) by means of a system (FDLS).

VIII. Minimal Realizations of Linear Systems

The question of state-space realizations of the system (LS) has been actively investigated in recent times, particularly as a result of the work of Kalman, Youla, and Silverman. The present section is devoted to one special aspect of this problem, namely, the relationship of minimality and controllability and observability. The full implications and the algorithmic questions related to this realization theory may be found in the paper by Silverman in this issue.

It is clear that the input–output system (LS) is realized by the state-space model (FDLS) *if and only if* $w(t, \tau) = C(t)\phi(t, \tau)B(\tau)$ for all $t \geq \tau$. Furthermore the system (LS) has a finite-dimensional linear realization (FDLS) if and only if there exist continuous matrices G and H such that $w(t, \tau) = H(t)G(\tau)$ for $t \geq \tau$.

Definition 9: Assume that the state-space model (FDLS) is a realization of the input–output system (LS). Then it is said to be a *minimal realization* of (LS) if every other realization of the type (FDLS) has a state space of greater or equal dimension.

It is important to note that minimality of the realization does not preclude the existence of *nonlinear* state-space realizations with a lower dimensional state space. In fact, such lower dimensional (indeed, *one-dimensional*) nonlinear realizations will always exist. It suffices, therefore, to consider space-filling curves which map R^n one–one and onto R. In order to ensure that every finite-dimensional realization requires a state space of a dimension at least that of the minimal (linear) realization, one has to require some smoothness on the maps ϕ and r.

One may expect from the discussion in Section VIII that the problem of constructing a minimal realization will (at least in principle) be simple for stationary systems, but

[6] There is no real need to restrict attention to $\mathcal{U}^+$. The difficulty with inputs which extend to $-\infty$ is that is is usually difficult to prove well-posedness of typical mathematical models. This problem is completely avoided by considering $\mathcal{U}^+$ as the class of admissible inputs.

will cause some difficulties for time-varying systems. We will thus consider the stationary case first.

Theorem 9: The linear system (LS) is realizable by a state-space model (SFDLS) if and only if: 1) the kernel $w(t, \tau)$ is separable, i.e., there exist matrices H and G such that $H(t)G(\tau) = w(t, \tau)$ for $t \geq \tau$; and 2) $w(t, \tau) = w(t - \tau, 0)$.

Proof: The "only if" part of the theorem is obvious. To prove the "if" part we need to identify a triplet of matrices A, B, C such that $Ce^{At}B = w(t, 0)$ for $t \geq 0$.

We will first identify the state space. Let $\mathfrak{U}_-$ be the functions in $\mathfrak{U}$ restricted to $(-\infty, 0]$ and let $\mathcal{Y}_+$ be the functions in $\mathcal{Y}$ restricted to $[0, \infty)$. Consider now the mapping from $\mathfrak{U}_-$ to $\mathcal{Y}_+$ defined by

$$y(t) = \int_{-\infty}^{0} w(t, \tau)u(\tau)\, d\tau, \qquad t \geq 0$$

and denote this mapping by $y_+ = W^+u$. Since $w(t, \tau) = H(t)G(\tau)$, W^+ may be viewed as the composition $W^+ = HG$ with $G: \mathfrak{U} \to R^q$ and $H: R^q \to \mathcal{Y}_+$ defined by $Gu = \int_{-\infty}^{0} G(\tau)u(\tau)\, d\tau$ and $Hz = H(t)z$, $t \geq 0$, where q denotes the number of rows of G (= the number of columns of H). Consider now the quotient space $X = \mathcal{R}(G)/\mathfrak{N}(H)$. This is clearly a finite-dimensional (say n-dimensional) vector space.

We will now identify the A matrix. To do this, observe that the space X qualifies as a state space by viewing the state at $t = T$ corresponding to the input $u \in \mathfrak{U}$ as the element in $R(G)/N(H)$ corresponding to $u_T \in \mathfrak{U}_+$ with $u_T(t) = u(T + t)$ for $t \leq 0$. It is an easy matter to formalize the linear maps ϕ and r associated with this choice of the state space. Let $\phi(t)x_0$, $t \geq 0$, be the zero-input response resulting from $x(0) = x_0$. Then $\phi(t)$ is an $(n \times n)$ matrix which satisfies $\phi(0) = I$ and $\phi(t_1)\phi(t_2) = \phi(t_1 + t_2)$ for $t_1, t_2 \geq 0$. Thus $\phi(t) = e^{At}$ for some matrix A.

The matrix C is then the linear mapping form R^n into R^p which takes x_0 into $y(0)$. To identify the B matrix, notice that by properly redefining G and H we may assume that $H(t) = Ce^{At}$ and $x_0 = \int_{-\infty}^{0} G(\sigma)u(\sigma)\, d\sigma$. Then $B = G(0)$ satisfies $w(t, 0) = H(t)G(0) = Ce^{At}B$ for $t \geq 0$, which shows that $w(t, \sigma) = Ce^{A(t-\sigma)}B$ for $t \geq \sigma$ and that the triplet $\{A, B, C\}$ is indeed a realization. Notice that this realization is in fact a minimal one.

Extensions and more details of the ideas invoked in the proof of the above theorem may be found in [6] and in the paper by Silverman in this issue.

The following theorem is an immediate consequence of the proof of Theorem 9 and yields the desired relationship between minimality and reachability–observability.

Theorem 10: A realization A, B, C is minimal if and only if the system $\dot{x} = Ax + Bu$; $y = Cx$ is observable and has a reachable state space. Moreover a minimal realization always exists and any two minimal realizations A_1, B_1, C_1 and A_2, B_2, C_2 are related via the similarity transformation $A_2 = TAT^{-1}$, $B_2 = TB_1$, $C_2 = C_1T^{-1}$ for some invertible matrix T.

Proof: Let $G(\sigma) = e^{-A\sigma}B$, $\sigma \leq 0$, $H(t) = Ce^{At}$, $t \geq 0$, and let the operators G and H be as defined in the proof of Theorem 9. Since minimality is equivalent to $R(G) = R^n$ (= reachability) and $N(H) = \{0\}$ (= observability) the first part of the theorem follows. Existence of a minimal realization follows from the proof of Theorem 9 and the similarity transformation follows since in every minimal realization the state space must be isomorphic (in the vector space sense) to the quotient space $\mathcal{R}(G)/\mathfrak{N}(H)$.

The realization theory for stationary linear systems thus presents in principle no difficulties. The time-varying case is much more involved, however. Indeed assume that $w(t, \tau) = H(t)G(\tau)$ [which is a necessary condition for a realization (FDLS) to exist], and let G_T and H_T be defined as $G_Tu = \int_{-\infty}^{T} G(\sigma)u(\sigma)\, d\sigma$ and $H_Tz = H(t)z$ for $t \geq T$. It is then natural to consider $X_T = R(G_T)/N(H_T)$ as the state space at time T and to define the maps ϕ and r from there. There is, however, one basic difficulty with this idea, namely X_T is in general an explicit function of T, and in order to account for this it becomes necessary to start from a new axiomatic framework for treating dynamical systems. A second difficulty is that even if X_T is always a subspace of, say, R^n, it may not be possible to construct an n-dimensional realization exhibiting the required smoothness for the parameters $A(t)$, $B(t)$, $C(t)$. This situation is illustrated by trying to obtain a one-dimensional realization of the system $\dot{x}_1 = b_1(t)u$, $\dot{x}_2 = b_2(t)u$, $y = c_1(t)x_1 + c_2(t)x_2$ with $b_1(t) = c_1(t) = 0$ for $t \geq 0$ and $b_2(t) = c_2(t) = 0$ for $t \leq 0$. Thus, although reachability and observability at some time are certainly sufficient conditions for minimality, they are not necessary.

There is, however, another (somewhat artificial) procedure for achieving a certain amount of structural invariance for time-varying systems. This procedure is based on considering anti-causal systems in association with the causal systems considered so far. This requires the state transition map to possess the group property and the input–output maps to be defined as a causal map when time runs in the usual forward direction and an anticausal map when time runs backward. The state of a state-space realization of such an input–output dynamical system is this alternatively required to summarize past *and* future inputs. This then yields the required invariance properties.

This device has been successfully applied to dynamical systems described by

$$\int_{t_0}^{t} w(t, \tau)u(\tau)\, d\tau$$

for $t \geq t_0$ when t runs forward and for $t \leq t_0$ when t runs backwards. Finite-dimensional realizations are then required to satisfy the equality $w(t, \tau) = C(t)\phi(t, \tau)B(\tau)$ for all $t, \tau \in R$. In this context one may in fact prove that a realization is minimal if and only if the resulting system is reachable and observable at some time. These notions are of course to be extended to dynamical systems which possess the group property. For details see Weiss and Kalman [23], Youla [24], and Desoer and Varaiya [25].

IX. Stability

Another interesting application of controllability and observability in connection with deducing internal from external properties of systems and vice versa is the equivalence of internal stability and input–output stability. This is the subject of the following theorem. First, however, we will define input–output stability.

Definition 10: Let G be a dynamical system in input–output form and assume that $Go = 0$. Then it is said to be[7] *L_p-input-output stable*, $1 \leq p \leq \infty$, if there exists a constant $K < \infty$ such that all $u \in \mathfrak{U}$, $u \in L_p$, yield $Gu \in L_p$ and $\|Gu\|_{L_p} \leq K\|u\|_{L_p}$.

It may be shown [26, sec. 2] that for systems of the class (FDLS) and for any system of the class (LS) when $p = \infty$ [27] the existence of the K follows from the fact that $y \in L_p$ whenever $u \in L_p$, i.e., if G is a map from $L_p \cap \mathfrak{U}$ into L_p then it is automatically bounded. It is interesting to note [18] that L_p-stability, $1 \leq p \leq \infty$, is equivalent to the existence of a constant K such that for all $u \in \mathfrak{U}$ the inequality

$$\left(\int_{-\infty}^{T} \|(Gu)(t)\|_Y{}^p \, dt\right)^{1/p} \leq K\left(\int_{-\infty}^{T} \|u(t)\|_Y{}^p \, dt\right)^{1/p}$$

is satisfied.

Theorem 11: Assume that the system (FDLS) is uniformly reachable and uniformly observable. Then exponential stability and L_p-input–output stability, $1 \leq p \leq \infty$, are equivalent.

Proof: If the system is exponentially stable, then there exists M, $\alpha > 0$, such that $\|w(t, \tau)\| \leq Me^{-\alpha(t-\tau)}$ for $t \geq \tau$ (recall that we assumed $B(t)$ and $C(t)$ to be bounded). Hence $\|y(t)\| \leq M \int_{t_0}^{t} e^{-\alpha(t-\tau)}\|u(\tau)\| \, d\tau$. The convolution of the L_p-function $\|u(t)\|$ against the L_1-kernel $e^{-\alpha t}$, $t \geq 0$, maps L_p into itself by Minkowski's inequality and defines a bounded linear transformation with bound by $M \int_0^\infty e^{-\alpha t} \, dt = M/\alpha$. This establishes that exponential stability implies L_p-stability. The converse will now be shown in the case $p = 2$. The method of proof works equally well when $1 \leq p < \infty$ and the case $p = \infty$ is well documented in the literature (see e.g., [8] and [10, sec. 30, theorem 3]. Assume that the system (LS) is L_2-input–output stable and that (FDLS) is a uniformly reachable and uniformly observable realization. Consider the real-valued function defined on $R^n \times R$ by $V(x_0, t_0) = \int_{t_0}^{\infty}\|y(t, \phi(t, t_0, x_0, 0), 0)\|^2 \, dt$. It will now first be shown that V is well defined. Let T be as in the definition of uniform reachability and uniform observability, and let u be the control which minimizes $\int_{t_0-T}^{T} \|u(t)\|^2 \, dt$ subject to $\dot{x} = A(t)x + B(t)u$, $y = C(t)x$, and $x(t_0 - T) = 0$, $x(t_0) = x_0$. By L_p-input–output stability there exists a constant $K < \infty$ such that $\int_{t_0-T}^{\infty} \|y(t, \phi(t, t_0, x_0, 0), 0)\|^2 \, dt \leq K \int_{t_0-T}^{T}\|u(t)\|^2 \, dt$. Thus $V(x, t)$ is well defined and by uniform reachability and uniform observability there exist constants $\epsilon > 0$ and R such that $\epsilon\|x\|^2 \leq V(x, t) \leq R\|x\|^2$. By uniform observability $V(x_0, t_0) - V(\phi(t_0 + T, t_0, x_0, 0), t_0 + T) = \int_{t_0}^{t_0+T} \|y(t, \phi(t, t_0, x_0, 0), 0)\|^2 \, dt \geq \epsilon\|x\|^2$. Since $V(\phi(t_0 + T, t_0, x_0, 0), t_0 + T) \leq (1 - \epsilon/R)V(x_0, t_0)$, exponential stability is established as claimed.

Note that the above theorem states as a side result the equivalence of L_p input–output stability for all $1 \leq p \leq \infty$ for systems (LS) with a uniformly controllable and uniformly observable state-space realization. Theorems along the line of Theorem 10 for nonlinear systems may be found in [10].

X. Conclusions

In this paper we have attempted to give a broad-based review of what we consider the most important applications of the controllability–observability circle of ideas. We have tried to emphasize concepts, but also discussed some specific results in the context of finite-dimensional linear systems. One of the things which may have come out of this treatment is that (for linear systems) one should properly be discussing *four* concepts, namely, reachability and controllability (input $\leftrightarrow$ state) on one hand and reconstructibility and observability (state $\leftrightarrow$ output) on the other. Problems of control face the question of controllability, problems of state reconstruction and filtering face the question of reconstructibility and problems of output–feedback control will face both the questions of reconstructibility and controllability. On the other hand, questions related to deducing internal properties from input–output properties, such as minimality and stability of the state-space realization, face the questions of reachability and observability. (The question then is: What did the input do and what will the output be?) By considering dynamical systems defined backwards in time, one may also drag the questions of controllability and reconstructibility in the minimality question in a somewhat artificial way.

Many problems in this area remain to be investigated. First among these is to find conditions and structure theorems for classes of nonlinear systems and systems defined via algebraic structures which do not involve the usual vector space assumptions. Some results in this vein are presented in [30].

[7] The space L_p is the collection of those (U- or Y-valued) functions for which $\|u\|_{L_p} = \left(\int_{-\infty}^{+\infty} \|u(t)\|^p \, dt\right)^{1/p} < \infty$ when $1 \leq p < \infty$ and with $\|u\|_{L_\infty} =$ essential $\sup_{t \in R}\|u(t)\| < \infty$ when $p = \infty$.

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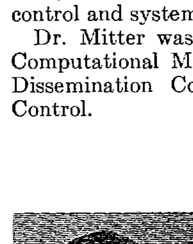
Dr. Mitter was the Chairman of the Technical Committee on Computational Methods, Distributed Systems of the Information Dissemination Committee of the IEEE Group on Automatic Control.



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