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CONTROLLABILITY, OBSERVABILITY AND OPTIMAL FEEDBACK CONTROL OF AFFINE HEREDITARY DIFFERENTIAL SYSTEMS*

M. C. DELFOUR† AND S. K. MITTER‡

Abstract. This paper is concerned with two aspects of the control of affine hereditary differential systems. They are (i) the theory of various types of controllability and observability for such systems and (ii) the problem of optimal feedback control with a quadratic cost. The study is undertaken within the framework of hereditary differential systems with initial data in the space M^2 (cf. Delfour and Mitter [6], [7]). The main result of this paper is the existence and characterization of the optimal feedback operator for the system.

1. Introduction. Perhaps the most useful part of optimal control theory for ordinary differential equations is the theory of optimal control of linear differential systems with a quadratic cost criterion. This theory is also the most complete, both for systems evolving in a finite-time interval as well as over an infinite-time interval. It is well known that in the finite-time case the optimal control can be expressed in linear feedback form, where the "feedback gains" satisfy a matrix differential equation of Riccati type. In the infinite-time case by using the theory of controllability and observability, the asymptotic behavior of the controlled system can be studied and a rather complete solution to the problem is available.

The present paper is concerned with (i) generalization of the theory of controllability and observability to affine hereditary differential systems and (ii) a study of the optimal feedback control problem for affine hereditary differential systems with a quadratic cost. The theory is currently being completed in order to show the relation of the theory of controllability and observability to the infinite-time quadratic cost problem.

The optimal control problem studied in this paper was first formulated and studied by Krasovskii [23], [24] using the space of continuous functions as the space of initial data and using dynamic programming arguments. This problem has also been studied by Ross and Flügge-Lotz [30], Eller, Aggarwal and Banks [13], Kushner and Barnea [25] and Alekal, Brunovsky, Chyung and Lee [1], in each case using Carathéodory–Hamilton–Jacobi type arguments. The basic disadvantage of the method used by these authors is that it necessitates a direct study of a complicated set of coupled ordinary and first order partial differential equations before the existence of a feedback control can be asserted.

In Delfour and Mitter [6], [7] we have developed a theory of hereditary differential systems where the initial datum is chosen to lie in the space

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† Centre de Recherches Mathématiques, Université de Montréal, Montréal 101, Québec, Canada.

‡ Electronic Systems Laboratory and Electrical Engineering Department, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139. The work of this author was supported in part by the National Science Foundation under Grant GK-25781 and by the Air Force Office of Scientific Research under Grant AFOSR-70-1941.

$M^p(-b, 0; H)$, $1 \leqq p < \infty$. In particular, $M^2(-b, 0; H)$ is a Hilbert space. By choosing the initial datum to lie in $M^2(-b, 0; H)$, the techniques developed by J. L. Lions [26] for the control of parabolic partial differential equations can be appropriately modified to solve the optimal feedback control problem for affine hereditary differential systems. It should be emphasized that in contrast to the Hamilton–Jacobi method this is a direct method where the existence of the "feedback operator" is first demonstrated and it is then shown to satisfy an operational differential equation of Riccati type. Part of the results on the feedback control problem were announced in Delfour and Mitter [10].

The concepts of controllability and observability for hereditary differential systems are also studied within the framework developed in Delfour and Mitter [6], [7]. This is accomplished by using certain results on controllability and observability of abstract linear control systems (cf. Delfour and Mitter [11]). We present necessary and sufficient conditions for various types of controllability and we examine the dual system. We also show how various existing results on controllability fit into the framework adopted in this paper (cf. A. F. Buckalo [2], Chyung and Lee [3], D. R. Haley [19], Kirillova and Churakova [22], G. S. Tahim [32] and L. Weiss [33]–[36]).

1.1. Notation and terminology. Given two real linear spaces X and Y and a linear map $T: X \to Y$, the image of T in Y will be denoted by $\mathrm{Im}(T)$ and the kernel of T in X by $\mathrm{Ker}(T)$. Let H and K be two Hilbert spaces and $T: H \to K$ be a continuous linear map. The adjoint of T will be denoted T^* $(\in \mathscr{L}(K^*, H^*))$. When $H = K$ we shall say that T is self-adjoint if $T^* = T$ and we shall write $T \geqq 0$ for a positive self-adjoint operator $((x|Tx) \geqq 0$ for all $x)$ and $T > 0$ for a positive definite self-adjoint operator $((x|Tx) > 0$ for all $x \neq 0)$. The identity map in $\mathscr{L}(H)$ is written I. The restriction of the map $x: [0, \infty[\to X$ to the interval $[0, t]$ is denoted $\pi_t x$ for all $t \in]0, \infty[$. The set of real numbers is denoted by R.

In the sequel we shall abbreviate hereditary differential system as HDS.

2. Basic properties of affine HDS. Let H and U be Hilbert spaces. Let $N \geqq 1$ be an integer, let $a > 0$, $0 = \theta_0 > \theta_1 > \cdots > \theta_N = -a$ be real numbers and $b \in [a, \infty]$. Let $I(\alpha, \beta) = R \cap [\alpha, \beta]$ for any $\alpha < \beta$ in $[-\infty, \infty]$. Let $|\cdot|_H$ (resp. $|\cdot|_U$) and $(\cdot|\cdot)_H$ (resp. $(\cdot|\cdot)_U$) denote the norm and inner products on H (resp. U).

2.1. Space of initial data and space of solutions. Our first task consists of choosing an appropriate space of initial data.

Consider the space $\mathscr{L}^2(-b, 0; H)$ (not to be confused with $L^2(-b, 0; H)$) of all maps $I(-b, 0) \to H$ which are square integrable in $I(-b, 0)$ endowed with the seminorm

$$\|y\|_{M^2} = \left[|y(0)|_H^2 + \int_{-b}^{0} |y(\theta)|_H^2 \, d\theta \right]^{1/2}.$$

The quotient space of $\mathscr{L}^2(-b, 0; H)$ by the linear subspace of all y such that $\|y\|_{M^2} = 0$ is a Hilbert space which is isometrically isomorphic to the product space $H \times L^2(-b, 0; H)$. It will be denoted by $M^2(-b, 0; H)$ and its norm by $\|\cdot\|_{M^2}$. The isomorphism between $H \times L^2(-b, 0; H)$ and $M^2(-b, 0; H)$ is denoted by κ.

In order to discuss the Cauchy problem we must also describe the space in which solutions will be sought. Let $1 \leqq p < \infty$, $t_0 \in R$. For all $t \in]t_0, \infty[$ we denote by $AC^p(t_0, t; H)$ the vector space of all absolutely continuous maps $[t_0, t] \to H$ with a derivative in $L^p(t_0, t; H)$. When $AC^p(t_0, t; H)$ is endowed with the norm

$$\|x\|_{AC^p} = \left[|x(t_0)|_H^p + \int_{t_0}^t \left| \frac{dx}{ds}(s) \right|_H^p ds \right]^{1/p},$$

it is a Banach space isometrically isomorphic to $H \times L^p(t_0, t; H)$. In particular, $AC^2(t_0, t; H)$ is a Hilbert space. We shall also need $C(t_0, t; H)$, the Banach space of all continuous maps $[t_0, t] \to H$ endowed with the sup norm $\| \cdot \|_C$.

When we consider the evolution of a system in an infinite-time interval it is useful and quite natural to introduce the following spaces. Let $\pi_t(x)$ be the restriction of the map $x: [t_0, \infty[\to H$ to the interval $[t_0, t]$, $t \in]t_0, \infty[$. Denote by $L^p_{\text{loc}}(t_0, \infty; H)$, $AC^p_{\text{loc}}(t_0, \infty; H)$ and $C_{\text{loc}}(t_0, \infty; H)$ the vector space of all maps $x: [t_0, \infty[\to H$ such that for all $t \in]t_0, \infty[$, $\pi_t(x)$ is in $L^p(t_0, t; H)$, $AC^p(t_0, t; H)$ and $C(t_0, t; H)$, respectively. They are Fréchet spaces (cf. Delfour [5]) when their respective topologies are defined by the saturated family of seminorms $q_t(x) = \|\pi_t(x)\|_F$, $t \in]t_0, \infty[$, where F is either L^p, AC^p or C.

2.2. System description. Consider the affine hereditary differential system $\mathscr{A}$ defined on $[0, \infty[$:

$$(2.1)\qquad \begin{aligned} \frac{dx}{dt}(t) &= A_{00}(t)x(t) + \sum_{i=1}^N A_i(t) \begin{cases} x(t + \theta_i), & t + \theta_i \geqq 0 \\ h(t + \theta_i), & t + \theta_i < 0 \end{cases} \\ &\quad + \int_{-b}^0 A_{01}(t, \theta) \begin{cases} x(t + \theta), & t + \theta \geqq 0 \\ h(t + \theta), & t + \theta < 0 \end{cases} d\theta \\ &\quad + B(t)v(t) + f(t) \quad \text{a.e. in } [0, \infty), \\ x(0) &= h(0), \quad h \in M^2(-b, 0; H), \end{aligned}$$

where A_{00} and $A_i\,(i = 1, 2, \cdots, N)$ are in $L^\infty_{\text{loc}}(0, \infty; \mathscr{L}(H))$, $A_{01} \in L^\infty_{\text{loc}}(0, \infty; -b, 0; \mathscr{L}(H))$, $B \in L^\infty_{\text{loc}}(0, \infty; \mathscr{L}(U, H))$, $v \in L^2_{\text{loc}}(0, \infty; U)$ and $f \in L^2_{\text{loc}}(0, \infty; H)$.

v is to be thought of as the control to be applied to the system and f is a known external input to the system. Under the above hypotheses, (2.1) has a unique solution $\phi(\cdot\,; h, v)$ in $AC^2_{\text{loc}}(0, \infty; H)$ and the map

$$(2.2)\qquad (h, v) \mapsto \phi(\cdot\,; h, v): M^2(-b, 0; H) \times L^2_{\text{loc}}(0, \infty; U) \to AC^2_{\text{loc}}(0, \infty; H)$$

is affine and continuous (cf. Delfour and Mitter [6], [7] and Delfour [5]). We also have the variation of constants formula

$$(2.3)\qquad \phi(t; h, v) = \Phi(t, 0)h + \int_0^t \Phi^0(t, s)B(s)v(s)\, ds + \int_0^t \Phi^0(t, s)f(s)\, ds,$$

where

$$\Phi(t, s)h = \Phi^0(t, s)h(0) + \int_{-b}^0 \Phi^1(t, s, \alpha)h(\alpha)\, d\alpha,$$

and $\Phi^0(t,s) \in \mathscr{L}(H)$ is the unique solution in $AC^2_{\mathrm{loc}}(s,\infty;\mathscr{L}(H))$ of the system

$$\frac{\partial \Phi^0}{\partial t}(t,s) = A_{00}(t)\Phi^0(t,s) + \sum_{i=1}^{N} A_i(t)\begin{cases}\Phi^0(t+\theta_i,s), & t+\theta_i \geqq s\\ 0, & \text{otherwise}\end{cases}$$
$$+ \int_{-b}^{0} A_{01}(t,\theta)\begin{cases}\Phi^0(t+\theta,s), & t+\theta \geqq s\\ 0, & \text{otherwise}\end{cases} d\theta \quad \text{a.e. in } [s,\infty[\tag{2.4}$$

and

$$\Phi^1(t,s,\alpha) = \sum_{i=1}^{N}\begin{cases}\Phi^0(t,s+\alpha-\theta_i)A_i(s+\alpha-\theta_i), & \alpha+s-t<\theta_i \leqq \alpha\\ 0, & \text{otherwise}\end{cases}$$
$$+ \begin{cases}\int_{-b}^{\alpha} \Phi^0(t,s+\alpha-\theta)A_{01}(s+\alpha-\theta,\theta)\,d\theta, & s+\alpha \leqq t-b\\ \int_{\alpha-t+s}^{\alpha} \Phi^0(t,s+\alpha-\theta)A_{01}(s+\alpha-\theta,\theta)\,d\theta, & s+\alpha > t-b\end{cases}. \tag{2.5}$$

2.3. State equation of the system.

DEFINITION 2.1. Let $f = 0$, $v = 0$ in (2.1). The *evolution of the state* of the homogeneous system is given by the map

$$t \mapsto \tilde{\phi}(t;h):[0,\infty[\to M^2(-b,0;H) \tag{2.6}$$

defined as

$$\tilde{\phi}(t;h)(\theta) = \begin{cases}\phi(t+\theta;h), & t+\theta \geqq 0,\\ h(t+\theta), & t+\theta < 0.\end{cases} \tag{2.7}$$

It is easy to verify the following theorem.

THEOREM 2.2. *Consider* (2.1) *with* $f = 0, v = 0$ *on* $[s,\infty[$ *with initial datum* h *at time* s. *Let* $\tilde{\phi}_s(\cdot\,;h)$ *denote the solution of this system in* $AC^2_{\mathrm{loc}}(s,\infty;H)$. *The map* $(t,s) \mapsto \tilde{\phi}_s(t;h)$ *generates a two-parameter semigroup* $\tilde{\Phi}(t,s)$ *satisfying the following properties*:

(i) $\tilde{\Phi}(t,s) \in \mathscr{L}(M^2)$, $t \geqq s \geqq 0$;

(ii) $\tilde{\Phi}(t,r) = \tilde{\Phi}(t,s)\tilde{\Phi}(s,r)$, $t \geqq s \geqq r \geqq 0$;

(iii) $t \mapsto \tilde{\Phi}(t,s)h:[s,\infty[\to M^2$ *is continuous for all* $h \in M^2$ *and* $s \in [0,\infty[$;

(iv) $\tilde{\Phi}(s,s) = I$, *where* I *is the identity operator in* $\mathscr{L}(M^2)$;

(v) *for* $t - s \geqq b$, $\tilde{\Phi}(t,s):M^2 \to M^2$ *is compact* (*i.e., maps bounded sets into relatively compact sets*);

(vi) *Let* $\mathscr{D} = AC^2(-b,0;H) \cap M^2(-b,0;H)$. *Then for all* $h \in \mathscr{D}$, $\tilde{\Phi}(t,s)h \in \mathscr{D}$. ▧

Since M^2 is isomorphic to $H \times L^2(-b,0;H)$, $\tilde{\Phi}(t,s)$ can be decomposed into two operators $\tilde{\Phi}^0(t,s) \in \mathscr{L}(H,M^2)$ and $\tilde{\Phi}^1(t,s) \in \mathscr{L}(L^2(-b,0;H),M^2)$ such that

$$\tilde{\Phi}(t,s)h = \tilde{\Phi}^0(t,s)h^0 + \tilde{\Phi}^1(t,s)h^1,$$

where

$$[\tilde{\Phi}^0(t,s)h^0](\alpha) = \begin{cases}\Phi^0(t+\alpha,s)h^0, & t+\alpha \geqq s,\\ 0, & t+\alpha < s,\end{cases} \tag{2.8}$$

and

$$[\tilde{\Phi}^1(t,s)h^1](\alpha) = \begin{cases} \int_{-b}^{0} \Phi^1(t+\alpha, s, \eta)h^1(\eta)\,d\eta, & t+\alpha \geqq s, \\ h^1(t+\alpha-s), & t+\alpha < s. \end{cases} \tag{2.9}$$

Finally corresponding to (2.1) we have the *state equation* in *integral form*

$$\tilde{\phi}(t;h,v) = \tilde{\Phi}(t,0)h + \int_0^t \tilde{\Phi}^0(t,s)B(s)v(s)\,ds + \int_0^t \tilde{\Phi}^0(t,s)f(s)\,ds. \tag{2.10}$$

We now wish to obtain the state equation in differential form. We first construct an unbounded operator $\tilde{A}(t)$ whose domain is

$$\mathscr{D} = AC^2(-b,0;H) \cap M^2(-b,0;H).$$

For this purpose define the linear maps

$$\tilde{A}^0(t):\mathscr{D} \to H \quad \text{and} \quad \tilde{A}^1:\mathscr{D} \to L^2(-b,0;H)$$

as follows:

$$\tilde{A}^0(t)h = A_{00}(t)h(0) + \sum_{i=1}^{N} A_i(t)h(\theta_i) + \int_{-b}^{0} A_{01}(t,\theta)h(\theta)\,d\theta \tag{2.11}$$

and

$$(\tilde{A}^1h)(\theta) = \frac{dh(\theta)}{d\theta}. \tag{2.12}$$

From the operators $\tilde{A}^0(t)$ and $\tilde{A}^1$ we construct the unbounded operator $\tilde{A}(t):\mathscr{D} \to M^2(-b,0;H)$ as

$$[\tilde{A}(t)h](\alpha) = \begin{cases} \tilde{A}^0(t)h, & \alpha = 0, \\ [\tilde{A}^1h](\alpha), & \alpha \neq 0. \end{cases} \tag{2.13}$$

Define also the operator $\tilde{B}(t):U \to M^2(-b,0;H)$ as

$$[\tilde{B}(t)u](\alpha) = \begin{cases} B(t)u, & \alpha = 0, \\ 0, & \text{otherwise}, \end{cases} \tag{2.14}$$

and $\tilde{f}(t) \in M^2(-b,0;H)$ as

$$[\tilde{f}(t)](\alpha) = \begin{cases} f(t), & \alpha = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{2.15}$$

We then have the following theorem.

THEOREM 2.3. (i) *For all $h \in \mathscr{D}$ and all $u \in L^2_{\text{loc}}(0,\infty;U)$, the system*

$$\begin{aligned} \frac{dy}{dt}(t) &= \tilde{A}(t)y(t) + \tilde{B}(t)u(t) + \tilde{f}(t) \quad \text{a.e. in } [0,\infty[, \\ y(0) &= h \end{aligned} \tag{2.16}$$

has a unique solution in $AC^2_{\text{loc}}(0,\infty;M^2)$ which coincides with $\tilde{\phi}(\cdot\,;h,u)$.

(ii) *The map $(h,u) \mapsto \Lambda(h,u) = \tilde{\phi}(\cdot\,;h,u):\mathscr{D} \times L^2_{\text{loc}}(0,\infty;U) \to AC^2_{\text{loc}}(0,\infty;M^2)$ can be lifted to a unique continuous affine map $\bar{\Lambda}:M^2 \times L^2_{\text{loc}}(0,\infty;U) \to C_{\text{loc}}(0,\infty;M^2)$ and for all pairs (h,u), $\bar{\Lambda}(h,u)$ coincides with $\tilde{\phi}(\cdot\,;h,u)$.*

Proof. Cf. Delfour and Mitter [6], [7]. ▧

Remark 2.4. In the autonomous case ($\tilde{A}(t) = \tilde{A} = \text{const.}$) the semigroup $\{\tilde{\Phi}(t,s)\}$ becomes a one-parameter semigroup $\{\tilde{\Phi}(t)\}$ and its infinitesimal generator is precisely $\tilde{A}$ and the domain of $\tilde{A}$ is $\mathscr{D}$ (cf. Hale [16] for analogous considerations).

2.4. Adjoint systems. One of the truly fascinating aspects of linear HDS is the existence of two types of adjoint systems: a *hereditary adjoint system* and a *topological adjoint system.*

2.4.1. Hereditary adjoint system. The *hereditary adjoint system* is defined in the interval $[0, T]$ for some $T > 0$:

$$\begin{aligned}\frac{dp}{dt}(t) + A_{00}(t)^*p(t) + \sum_{i=1}^{N} \begin{cases} A_i(t-\theta_i)^*p(t-\theta_i), & t-\theta_i \leqq T \\ 0, & t-\theta_i > T \end{cases} \\ + \int_{-b}^{0} \begin{cases} A_{01}(t-\theta)^*p(t-\theta), & t-\theta \leqq T \\ 0, & t-\theta > T \end{cases} d\theta + g(t) = 0 \quad \text{a.e. in } [0, T],\end{aligned} \tag{2.17}$$

$$p(T) = k^0, \quad k^0 \in H, \tag{2.18}$$

for some $g \in L^2(0, T; H)$. Under the hypotheses of § 2.2, (2.17)–(2.18) has a unique solution $\psi(\cdot\,; T, k^0)$ in $AC^2(0, T; H)$ and the map

$$k^0 \mapsto \psi(\cdot\,; T, k^0): H \to AC^2(0, T; H) \tag{2.19}$$

is affine and continuous. We also have the *variation of constants* formula

$$\psi(t; T, k^0) = \Phi^0(T, t)^*k^0 + \int_t^T \Phi^0(r, t)^*g(r)\,dr, \tag{2.20}$$

where Φ^0 is defined in (2.4) (cf. Delfour and Mitter [7], [9]).

It will be convenient to construct the following unbounded operator $\tilde{A}_T^0(t)$: $\mathscr{D}^* \to H$ with appropriate domain $\mathscr{D}^*$:

$$\begin{aligned}\tilde{A}_T^0(t)h = A_{00}(t)^*h(0) + \sum_{i=1}^{N} \begin{cases} A_i(t-\theta_i)^*h(\theta_i), & t-\theta_i \leqq T \\ 0, & t-\theta_i > T \end{cases} \\ + \int_{-b}^{0} \begin{cases} A_{01}(t-\theta,\theta)^*h(\theta), & t-\theta \leqq T \\ 0, & t-\theta > T \end{cases} d\theta.\end{aligned} \tag{2.21}$$

Equation (2.19) can now be rewritten in a condensed form:

$$\begin{aligned}&\frac{dp}{dt}(t) + \tilde{A}_T^0(t)p_t + g(t) = 0 \quad \text{a.e. in } [0, T], \\ &p(T) = k^0 \in H,\end{aligned} \tag{2.22}$$

where $p_t \in M^2$ is defined as

$$p_t(\theta) = \begin{cases} p(t-\theta), & t-\theta \leqq T, \\ 0, & \text{otherwise.} \end{cases} \tag{2.23}$$

Remark 2.5. Even when $A_{00}, A_1, \cdots, A_N$ and A_{01} are time invariant, the hereditary adjoint system depends on both T and the time t.

2.4.2. Topological adjoint system. Owing to some delicate technical considerations, we restrict our attention to the autonomous case ($\tilde{A}(t) = \tilde{A} = \text{const.}$). $\tilde{A}^*$ denotes the M^2-adjoint of $\tilde{A}$.

THEOREM 2.6. *Given $T > 0$, the densely defined closed operator $-\tilde{A}^*$ generates the one-parameter semigroup $\{\Psi(T-t)^*\}$ and for all $k \in \mathscr{D}(\tilde{A}^*)$, $z(t) = \Psi(T-t)^*k$*

is the unique solution in $AC^2(0, T; M^2)$ *of the equation*

$$\begin{aligned} &\frac{dz}{dt}(t) + \tilde{A}^* z(t) = 0 \quad \text{in } [0, T], \\ &z(T) = k \in \mathscr{D}(A^*). \end{aligned} \tag{2.24}$$

Proof. Cf. Delfour and Mitter [7]. ▧

For obvious reasons system (2.24) will be referred to as the *topological adjoint system.*

3. Controllability and observability. In §§ 3.1 and 3.2 we successively look at various notions of controllability and observability, discuss their relative merits and prove various results on controllability and observability. In the last section we construct a system which is dual to the original system. The relationship between controllability and observability and the feedback problem will be considered in a forthcoming paper.

3.1. Controllability. The notions of controllability for hereditary differential systems have been explored by several authors since 1965 and precise conditions have been presented for controllability (cf. G. S. Tahim [32], Chyung and Lee [3], Kirillova and Churakova [22], L. Weiss [33]–[36], A. F. Buckalo [2] and A. Halanay [17]) of different types. In this section we look at two types of controllability in the framework of the space M^2, derive necessary and sufficient conditions and discuss the relationship of earlier results in the literature with ours.

DEFINITION 3.1. (i) The data $h \in M^2$ is *controllable* (resp. M^2-*controllable*) *at time* T to $x \in H$ (resp. $k \in M^2$) if there exists a sequence $\{u_n\}$ in $L^2(0, T; U)$ such that $\phi(T; h, u_n)$ (resp. $\tilde{\phi}(T; h, u_n)$) converges to x (resp. k). System $\mathscr{A}$ is *controllable* (resp. M^2-*controllable*) *at time* T if all $h \in M^2$ are controllable (resp. M^2-controllable) at time T to all $x \in H$ (resp. $k \in M^2$).

(ii) The data $h \in M^2$ is *controllable to the origin* (resp. *to the zero function*) if there exists a finite time $T > 0$ for which h is controllable to $0 \in H$ (resp. $0 \in M^2$) at time T. If every h in M^2 is *controllable to the origin* (resp. *to the zero function*), system $\mathscr{A}$ is said to be *controllable to the origin* (resp. *to the zero function*).

DEFINITION 3.2. (i) The data $h \in M^2$ is *strictly controllable* (resp. M^2-*controllable*) *at time* T to $x \in X$ (resp. $k \in M^2$) if there exists u in $L^2(0, T; U)$ such that $\phi(T; h, u) = x$ (resp. $\tilde{\phi}(T; h, u) = k$). System $\mathscr{A}$ is said to be *strictly controllable* (resp. M^2-*controllable*) *at time* T if every $h \in M^2$ is strictly controllable (resp. M^2-controllable) at time T to all $x \in H$ (resp. $k \in M^2$).

(ii) The data $h \in M^2$ is *strictly controllable to the origin* (resp. *to the zero function*) if there exists a finite time $T > 0$ for which h is strictly controllable to $0 \in H$ (resp. $0 \in M^2$) at time T. If all data h in M^2 are strictly controllable to the origin (resp. to the zero function), system $\mathscr{A}$ is said to be *controllable to the origin* (resp. *to the zero function*).

For completeness we have included this last definition.

DEFINITION 3.3. Let $\mathscr{H}$ be a linear subspace of M^2. System $\mathscr{A}$ is *strictly controllable to a function* ψ *in* $\mathscr{H}$ if for each $h \in M^2$, there exist a finite time $T > 0$ and a control map $u \in L^2(0, T; U)$ such that $\tilde{\phi}(T; h, u) = \psi$. System $\mathscr{A}$ is said to be *strictly controllable to the space* $\mathscr{H}$ if it is strictly controllable to all functions ψ of $\mathscr{H}$.

PROPOSITION 3.4. *When H is finite-dimensional the notion of strict controllability at time T (resp. strict controllability to the origin) is equivalent to the notion of controllability at time T (resp. controllability to the origin).*

In this paper we shall not consider the "strict" notions unless we are in the situation of Proposition 3.4. The following results are obtained directly from the definitions.

PROPOSITION 3.5. (i) *$\mathscr{A}$ is never strictly M^2-controllable at time T.*

(ii) *For all $T < b$, $\mathscr{A}$ is never M^2-controllable at time T or controllable to the zero function.*

(iii) *The controllability of $\mathscr{A}$ at time T is a necessary condition for the M^2-controllability of $\mathscr{A}$ at time T.* ▧

Remark. (i) When $b = \infty$, $\mathscr{A}$ is never controllable to the zero function or M^2-controllable at any finite $T \geqq 0$.

(ii) Proposition 3.5 (i) implies that when $\mathscr{A}$ is M^2-controllable at time T, the initial states in M^2 are only strictly controllable to points in a dense subspace of M^2 which is different from M^2.

PROPOSITION 3.6. *$\mathscr{A}$ is controllable to the origin (to the zero function) if there exists a finite time $T > 0$ such that $\mathscr{A}$ is controllable (M^2-controllable) at time T.* ▧

Remark. A similar statement is true for the "strict" notions.

All the above definitions were originally given in the literature for initial data in $C(-b, 0; H)$ rather than $M^2(-b, 0; H)$. As for the space of control maps, it is safer and technically more advantageous to use the larger $L^2(0, T; U)$ rather than $L^\infty(0, T; U)$ or the space of piecewise continuous maps. Whether the control map can be picked in a smaller subspace of $L^2(0, T; U)$ will depend on the nature of the operators A_{00}, A_i $(i = 1, \cdots, N)$ and A_{01}. If they are "sufficiently nice" so will the control maps be. Table 1 summarizes some of the details concerning previous research. In all cases, $H = R^n$ and the controllability is strict (s.c.).

TABLE 1

	Types of Controllability	Control Maps
Chyung and Lee, 1966 [3]	• $A_{01} = 0$, continuous matrices A_{00}, A_i $(i = 1, \cdots, N)$ and B. • s.c. at time T.	$L^\infty(0, T; H)$
Kirillova and Churakova, 1967 [22]	• $N = 1$, $A_{01} = 0$, constant matrices A_{00}, A_1 and B. • s.c. to the origin and to the zero function.	piecewise continuous maps $[0, \infty[\to U$
L. Weiss, 1967 [33] and 1970 [34]	• $N = 1$, $A_{01} = 0$, continuous matrices A_{00}, A_1 and B. • s.c. to the origin, to the zero function and to a function.	$L^\infty_{\text{loc}}(0, \infty; U)$
A. F. Buckalo, 1968 [2]	• $N = 1$, $A_{01} = 0$, $n - 1$ differentiable matrices A_{00}, A_1 and B. • s.c. to the zero function.	$L^\infty_{\text{loc}}(0, \infty; U)$
A. Halanay, 1970 [18]	• $N = 1$, $A_{01} = 0$, special constant matrices A_{00}, A_1 and B. • "complete controllability" (= s.c. to the zero function for all $T > a$).	piecewise continuous maps $[0, \infty[\to U$

Definitions 3.1 (ii), (3.2) (ii) and 3.3 are conceptually interesting but technically difficult to deal with since the final time T is not fixed. Even from the engineering standpoint it is desirable to have a uniform bound on T independent of the initial data h in M^2. In fact, most conditions for "controllability of Definitions 3.1 (ii), 3.2 (ii) and 3.3" are only sufficient. They make use of Proposition 3.6, the converse of which is obviously not true. The notion of M^2-controllability is new in the context of hereditary differential systems, though the idea of density has often been used in partial differential equations where it naturally arises. It is clear that at time $t > 0$ the state $\tilde{\phi}(t; h, u)$ will be absolutely continuous in $[-t, 0]$ (see (2.7)). Thus it will be impossible to synthesize an M^2-map or even a continuous map defined in $I(-b, 0)$ which is not at least differentiable in the interval $[-t, 0]$. For all the above reasons we shall limit the scope of our investigation to the notions of controllability of Definitions 3.1 (i) and 3.2 (i).

THEOREM 3.7. *The following statements are equivalent*:

(i) $\mathscr{A}$ *is controllable* (*resp.* M^2-*controllable*) *at time* T;

(ii) *the map* $u \mapsto S(T)u = \int_0^T \Phi^0(T, s)B(s)u(s)\,ds : L^2(0, T; U) \mapsto H$ (*resp.* $u \mapsto \tilde{S}(T)u = \int_0^T \tilde{\Phi}^0(T, s)B(s)u(s)\,ds : L^2(0, T; U) \mapsto M^2(-b, 0; H)$) *has a dense image in* H (*resp.* $M^2(-b, 0; H)$);

(iii) *the map* $x \mapsto S(T)^*x : H \to L^2(0, T; U)$ (*resp.* $k \mapsto \tilde{S}(T)^*k : M^2(-b, 0; H) \to L^2(0, T; U)$), *where* $(S(T)^*x)(t) = B(t)^*\Phi^0(T, t)^*x$ (*resp.* $(\tilde{S}(T)^*k)(t) = B(t)^*\tilde{\Phi}^0(T, t)^*k$), *is injective*;

(iv) *the symmetric operator*

$$W_c(T) = \int_0^T \Phi^0(T, s)B(s)B(s)^*\Phi^0(T, s)^*\,ds$$

$$(\textit{resp. } \tilde{W}_c(T) = \int_0^T \tilde{\Phi}^0(T, s)B(s)B(s)^*\tilde{\Phi}^0(T, s)^*\,ds)$$

is positive definite.

Proof. This is a corollary to Delfour and Mitter [11, Thm. 9 and Cor. 10]. ▧

COROLLARY 3.8. *Let* $H = R^n$. (i) *The condition*

$$\operatorname{rank}(W_c(T)) = n \tag{3.1}$$

is necessary and sufficient for the strict controllability of $\mathscr{A}$ *at time T.*

(ii) *Condition* (3.1) *is necessary for the* M^2-*controllability of* $\mathscr{A}$ *at time T.*

(iii) *If there exists a time* T, $0 \leqq T < \infty$, *for which condition* (3.1) *holds, then system* $\mathscr{A}$ *is strictly controllable to the origin.* ▧

Remark. Part (i) of the corollary is due to Chyung and Lee [3] and part (iii) to L. Weiss [33, Lemma 1].

PROPOSITION 3.9. *Assume there exists* T, $0 < T < \infty$, *such that* $\operatorname{Im}(\Phi(T, 0)) = H$. *If all initial states* $h \in M^2$ *are controllable to the origin at time T, system* $\mathscr{A}$ *is controllable at time T.*

Proof. For all $h \in M^2$, $x \in H$ there exists $k \in M^2$ such that

$$-x + \Phi(T, 0)h = \Phi(T, 0)k. \tag{3.2}$$

Since $\mathscr{A}$ is controllable to the origin at time T there exists $\{u_n\}$ in $L^2(0, T; U)$ such that

$$\Phi(T, 0)k + \int_0^T \Phi^0(T, s)B(s)u_n(s)\,ds + \int_0^T \Phi^0(T, s)f(s)\,ds \to 0. \tag{3.3}$$

Hence $\mathscr{A}$ is controllable at time T by combining (3.2) and (3.3). ▧

Remark. The condition $\operatorname{Im}(\Phi(T,0)) = H$ is equivalent to have the "force free attainable set"

$$\{\Phi(T,0)h \mid h \in M^2\}$$

equal to H. When the property is true for all $T \geqq 0$, then system $\mathscr{A}$ is said to be *pointwise complete*. This definition is due to L. Weiss [33] who conjectured that for $H = R^n$ all systems of the form

$$\frac{dx}{dt}(t) = A_0 x(t) + A_1 x(t-a) + Bu(t), \quad t \geqq 0,$$
$$x(s) = h(s), \quad s \in [-a,0],$$

are pointwise complete. This point has been investigated by V. M. Popov [29] who has shown that the conjecture is false for $n > 2$. Popov has further found necessary and sufficient conditions for the system to be pointwise complete.

Finally we restrict our attention to $H = R^n$ and systems of the form

$$\text{(3.4)}\qquad \begin{aligned} &\frac{dx}{dt}(t) = A_0(t)x(t) + A_1(t)x(t-a) + B(t)u(t) + f(t), \quad t \geqq 0,\\ &x(s) = h(s), \quad s \in [-a,0], \end{aligned}$$

where A_0 and A_1 are in $L^\infty_{\mathrm{loc}}(0,\infty;\mathscr{L}(R^n))$, $B \in L^\infty_{\mathrm{loc}}(0,\infty;\mathscr{L}(R^m,R^n))$ and $f \in L^2(0,\infty;R^n)$. Notice that

$$\text{(3.5)}\qquad \begin{aligned} &\frac{\partial \Phi^0}{\partial t}(t,0) = A_0(t)\Phi^0(t,0), \quad t \in [0,a],\\ &\Phi^0(0,0) = I \end{aligned}$$

and that for $T \geqq a$,

$$\text{(3.6)}\qquad \begin{aligned} &\frac{\partial \Phi^0}{\partial s}(T,s) + \Phi^0(T,s)A_0(s) = 0, \quad s \in [T-a,T],\\ &\Phi^0(T,T) = I. \end{aligned}$$

This means that the force free attainable set is equal to R^n in the interval $[0,a]$ since

$$R^n = \operatorname{Im}(\Phi^0(t,0)) \subset \operatorname{Im}(\Phi(t,0)) \subset R^n.$$

Also, we have the following proposition.

PROPOSITION 3.10. (i) *If there exists $T_0 \in [T-a, T[\cap [0,T]$ for which the system*

$$\frac{dx}{dt}(t) = A_0(t)x(t) + B(t)u(t), \quad t \in [T_0,T],$$

is strictly controllable at time T, then system $\mathscr{A}$ is strictly controllable at time T.

(ii) *If in addition A_0 and B are respectively $n-2$ and $n-1$ times continuously differentiable in $[T_0,T]$, we can construct the* controllability matrix *of Silverman* *and Meadows* [31]

$$Q_c(t) = [P_0(t) \vdots P_1(t) \vdots \cdots \vdots P_{n-1}(t)],$$

where

$$P_{k+1}(t) = -A_0(t)P_k(t) + \dot{P}_k(t),$$
$$P_0(t) = B(t),$$

and the condition of part (i) *is equivalent to the existence of some* $t \in [T_0, T]$ *for which* rank $Q_c(t) = n$. ▧

Remark. A. F. Buckalo [2] used the condition of L. Weiss and incorporated the ideas of Silverman and Meadows [31] to essentially obtain part (ii) of the above proposition. The classical rank condition is obtained when the matrices A_0 and B are not time dependent.

In addition to the above results one should mention the work of Kirillova and Churakova [22] and L. Weiss [34]. It is the first attempt to obtain direct conditions on the various matrices in contrast to the above results where the (strict) controllability of a nonhereditary system serves as a sufficient condition for the (strict) controllability of $\mathscr{A}$.

3.2. Observability. To our knowledge the notion of observability for HDS has not been studied in the published literature. We have seen that there are several notions of controllability. Likewise, there are more than one way to observe system $\mathscr{A}$ and different things to observe.

Let Y be a Hilbert space which might be thought of as the *observation space*. We can observe the map $\phi(\cdot\,; h, u)$ with an *observer* $Z \in L^\infty(0, T; \mathscr{L}(H, Y))$; the *observation* at time t is defined by

$$z(t; h, u) = Z(t)\phi(t; h, u). \tag{3.7}$$

We can also observe the map $\tilde{\phi}(\cdot\,; h, u)$ with an M^2-*observer* $\tilde{Z} \in L^\infty(0, T; \mathscr{L}(M^2, Y))$; the M^2-*observation* at time t is defined by

$$\tilde{z}(t; h, u) = \tilde{Z}(t)\tilde{\phi}(t; h, u). \tag{3.8}$$

Since $M^2(-b, 0; H)$ is isomorphic to $H \times L^2(-b, 0; H)$, there exist $\tilde{Z}^0(t) \in \mathscr{L}(H, Y)$ and $\tilde{Z}^1(t) \in \mathscr{L}(L^2(-b, 0; H), Y)$ such that

$$\tilde{Z}(t)(\kappa^{-1}(h^0, 0)) = \tilde{Z}^0(t)h^0 \tag{3.9}$$

and

$$\tilde{Z}(t)(\kappa^{-1}(0, h^1)) = \tilde{Z}^1(t)h^1. \tag{3.10}$$

Notice that our observer satisfies hypothesis (ii) in Definition 12 (cf. Delfour and Mitter [11]). Now starting from either of the above two types of observations, we can either determine the state $h \in M^2(-b, 0; H)$ or simply $h^0 \in H$.

DEFINITION 3.11. (i) System $\mathscr{A}$ is *observable* in $[0, T]$ if for all $h \in M^2(-b, 0; H)$ and $u \in L^2(0, T; U)$ the point $h^0 \in H$ can be uniquely determined from a knowledge of u, h^1 and the *observation map* $z(\cdot\,; h, u)$, where $\kappa(h) = (h^0, h^1)$ and κ is the isometric isomorphism between $M^2(-b, 0; H)$ and $H \times L^2(-b, 0; H)$.

(ii) System $\mathscr{A}$ is *strongly observable* in $[0, T]$ if for all $h \in M^2(-b, 0; H)$ and $u \in L^2(0, T; U)$, the state h can be uniquely determined from a knowledge of u and the *observation map* $z(\cdot\,; h, u)$.

(iii) System $\mathscr{A}$ is *M^2-observable* in $[0, T]$ if for all $h \in M^2(-b, 0; H)$ and $u \in L^2(0, T; U)$ the state h can be uniquely determined from u and the *observation map* $\tilde{z}(\cdot\,; h, u)$.

PROPOSITION 3.12. *Let* $\tilde{Z}^0(t) = Z(t)$.

(i) *$\mathscr{A}$ strongly observable $\Rightarrow$ $\mathscr{A}$ M^2-observable and observable.*

(ii) *For all $T < b$, $\mathscr{A}$ is not strongly observable in* $[0, T]$.

Proof. The proof follows from the definitions. ▧

Remarks. (i) When $b = \infty$, system $\mathscr{A}$ is never strongly observable in $[0, T]$ for all finite T.

(ii) When $\tilde{Z}^1(t) = 0$, strong observability and M^2-observability are equivalent.

PROPOSITION 3.13. *The following statements are equivalent*:

(i) *$\mathscr{A}$ is observable in* $[0, T]$;

(ii) *the map* $F^0 : H \to L^2(0, T; Y)$, *where* $((F^0)h^0)(t) = Z(t)\Phi^0(t, 0)h^0$, *is injective*;

(iii) *the map* $y \mapsto (F^0)^* y = \int_0^T \Phi^0(t, 0)^* Z(t)^* y(t)\, dt : L^2(0, T; Y) \to H$ *has a dense image in* H;

(iv) *the symmetric operator*

$$W_0^0(T) = \int_0^T \Phi^0(t, 0)^* Z(t)^* Z(t) \Phi^0(t, 0)\, dt \tag{3.11}$$

is positive definite.

Proof. The proof is similar to the proof of Theorem 3.7. ▧

COROLLARY 3.14. *Let* $H = R^n$. (i) *The condition*

$$\operatorname{rank}(W_0^0(T)) = n \tag{3.12}$$

is necessary and sufficient for the observability of $\mathscr{A}$ in $[0, T]$.

(ii) *Condition* (3.12) *is necessary for strong observability of system* $\mathscr{A}$.

Proof. The proof is similar to the proof of Corollary 3.8. ▧

COROLLARY 3.15. *Consider the system of equations* (3.4) *with observer* $Z \in L^\infty(0, T; \mathscr{L}(R^n, Y))$.

(i) *If there exists $T_f \in [0, T] \cap [0, a]$ for which the system*

$$\frac{dx}{dt}(t) + A_0(t)^* x(t) + Z(t)^* y(t) = 0 \quad \text{in } [0, T_f],$$

$$x(T_f) = h^0$$

is controllable at time 0, *then system $\mathscr{A}$ is observable in* $[0, T]$.

(ii) *If in addition A_0 and Z are respectively $n - 2$ and $n - 1$ times continuously differentiable in $[0, T_f]$, we can construct the* observability matrix *of Silverman and Meadows* [31],

$$Q_0(t) = [S_0(t) \vdots S_1(t) \vdots \cdots \vdots S_{n-1}(t)], \tag{3.13}$$

where

$$S_{k+1}(t) = A_0(t)^* S_k(t) + \dot{S}_k(t), \quad S_0(t) = Z(t)^*, \tag{3.14}$$

and the condition of part (i) *is equivalent to the existence of some* $t \in [0, T_f]$ *for which* rank $(Q_0(t)) = n$. ▧

Remark. The classical rank condition is obtained when the matrices A_0 and Z are not time dependent.

PROPOSITION 3.16. *The following statements are equivalent:*

(i) $\mathscr{A}$ *is strongly observable* (*resp.* M^2*-observable*) *in* $[0, T]$;

(ii) *the map* F (*resp.* $\tilde{F}$)$: M^2(-b, 0; H) \to L^2(0, T; Y)$ *defined by* $(Fh)(t) = Z(t)\Phi(t, 0)h$ (*resp.* $(\tilde{F}h)(t) = \tilde{Z}(t)\tilde{\Phi}(t, 0)h$) *is injective*;

(iii) *the map* F^* (*resp.* $\tilde{F}^*$)$: L^2(0, T; Y) \to M^2(-b, 0; H)$ *defined by*

$$F^*y = \int_0^T \Phi(t,0)^*Z(t)^*y(t)\,dt \quad (\textit{resp. } \tilde{F}^*y = \int_0^T \tilde{\Phi}(t,0)^*\tilde{Z}(t)^*y(t)\,dt)$$

has a dense image in $M^2(-b, 0; H)$;

(iv) *the symmetric operator*

$$W_0(T) = \int_0^T \Phi(t,0)^*Z(t)^*Z(t)\Phi(t,0)\,dt$$

$$(\textit{resp. } \tilde{W}_0(T) = \int_0^T \tilde{\Phi}(t,0)^*\tilde{Z}(t)^*\tilde{Z}(t)\tilde{\Phi}(t,0)\,dt)$$

is positive definite.

Proof. The proof is similar to the proof of Theorem 3.7. ▧

3.3. Duality. In general, it is difficult to find a differential system which synthesizes the dual system $\mathscr{A}^*$ (cf. Delfour and Mitter [11, Def. 12 and Thm. 13]). However, it is not too difficult to construct the dual system corresponding to the notions of "controllability at time T" and "observability at time T." The simultaneously controlled and observed dual of $\mathscr{A}$ is defined as follows:

$$\begin{aligned}(3.15)\quad &\frac{dx}{dt}(t) + A_{00}(t)^*x(t) + \sum_{i=1}^{N} \begin{cases} A_i(t-\theta_i)^*x(t-\theta_i), & t-\theta_i \leqq T \\ 0, & t-\theta_i > T \end{cases} \\ &+ \int_{-b}^{0} \begin{cases} A_{01}(t-\theta,\theta)^*x(t-\theta), & t-\theta \leqq T \\ 0, & t-\theta > T \end{cases} d\theta + Z(t)^*y(t) = 0 \\ &\hspace{20em}\text{a.e. in } [0, T], \\ &x(T) = x_T \in H \text{ (evolution equation)}, \\ &\chi(t; x_T, y) = B(t)^*\phi^*(t; x_T, y) \text{ (observation map)},\end{aligned}$$

where $\phi^*(\cdot\,; x_T, y)$ is the unique solution of (3.15) in $AC^2(0, T; H)$.

PROPOSITION 3.17. *System* $\mathscr{A}$ *is controllable at time* T (*resp. observable in* $[0, T]$) *if and only if system* $\mathscr{A}^*$ *is observable in* $[0, T]$ (*resp. controllable at* 0). ▧

It is extremely important to notice that "controllability at time T" is a dual notion of "observability in $[0, T]$." It would have been extremely unpleasant to have "strong observability" in lieu of "observability."

4. The optimal control problem with a quadratic cost.

4.1. Formulation of the problem. Consider the controlled system (2.1). We fix the final time $T \in]0, \infty[$ and consider the solution of (2.1) in the interval

$[0, T]$. We also consider f to be given. The solution in $[0, T]$ corresponding to $h \in M^2(-b, 0; H)$ and $v \in L^2(0, T; U)$ is denoted by $x(\cdot\,; h, v)$. We associate with v and h the cost function $J(v, h)$ given by

$$
\begin{aligned}
J(v, h) &= (x(T; h, v)|Fx(T; h, v)) \\
&\quad + \int_0^T [(x(s; h, v)|Q(s)x(s; h, v)) + (v(s)|N(s)v(s))]\, ds \\
&\quad + 2\int_0^T (v(s)|m(s))\, ds + 2\int_0^T (x(s; h, v)|g(s))\, ds,
\end{aligned} \tag{4.1}
$$

where $g \in L^2(0, T; H)$, $m \in L^2(0, T; U)$, $F \in \mathscr{L}(H)$, $Q \in L^\infty(0, T; \mathscr{L}(H))$, $N \in L^\infty(0, T;$ $\mathscr{L}(U))$, F, $Q(s)$ and $N(s)$ are positive symmetric transformations and there exists a constant $c > 0$ such that $(y|N(s)y) \geqq c\|y\|_U^2$ for all s in $[0, T]$.

For each h we shall show that there exists a unique control u which minimizes the cost function $J(v, h)$ over all v in $L^2(0, T; U)$. The minimizing control u will be completely characterized in terms of the adjoint system. We shall also show that the control u can be synthesized using a linear feedback law and that the minimum of the cost function can be expressed in terms of the initial datum h.

4.2. Existence of the optimal control; Necessary and sufficient conditions for optimality. The existence and uniqueness of the optimal control u minimizing the cost $J(v, h)$ is a direct consequence of the hypotheses of §§ 2 and 4.1 and two theorems of Lions (cf. [26, Thm. 1.1, p. 4, and Thm. 1.2, p. 7]). In summary, given a continuous bilinear form $\tilde{\pi}$ defined in a Hilbert space $\mathscr{U}$ (with norm $\|\cdot\|$) satisfying the properties

$$\tilde{\pi}(v, w) = \tilde{\pi}(w, v) \quad \text{for all } w, v \in \mathscr{U}, \tag{4.2a}$$

$$\tilde{\pi}(v, v) \geqq c\|v\|^2 \quad \text{for all } v \in \mathscr{U}, \quad c > 0, \tag{4.2b}$$

and a continuous linear form $\tilde{L}$ also defined in $\mathscr{U}$, we define the cost

$$\tilde{J}(v) = \tilde{\pi}(v, v) - 2\tilde{L}(v) \tag{4.3}$$

which is to be minimized over the closed convex subset $\mathscr{U}_{ad}$ of $\mathscr{U}$. For such a cost there exists a unique u in $\mathscr{U}_{ad}$ minimizing $\tilde{J}(v)$ and this element can be uniquely characterized by

$$\tilde{\pi}(u, v - u) \geqq \tilde{L}(v - u) \quad \text{for all } v \in \mathscr{U}_{ad}. \tag{4.4}$$

For fixed f the cost function $J(v, h)$ given by (4.1) is of the form

$$J(v, h) = \pi(v, v) - 2L_h(v) + c(h), \tag{4.5}$$

where π and L_h satisfy the same hypotheses as $\tilde{\pi}$ and $\tilde{L}$ and $c(h)$ is a constant which solely depends on h. If $y(\cdot\,; w) = x(\cdot\,; 0, u + w) - x(\cdot\,; 0, u)$ is the solution of system $\mathscr{A}$ with $f = 0$, $h = 0$ and control w, a straightforward computation will show that inequality (4.4) becomes

$$
\begin{aligned}
&\int_0^T [(Q(s)x(s; h, u) + g(s)|y(s; w)) + (N(s)u(s) + m(s)|w(s))]\, ds \\
&\quad + (Fx(T; h, u)|y(T; w)) \geqq 0 \quad \text{for all } w \in L^2(0, T; U).
\end{aligned} \tag{4.6}
$$

In order to improve the above characterization, we introduce the adjoint system corresponding to $x(\cdot\,; h, v)$:

$$\begin{aligned}&\frac{dp}{ds}(s;h,v) + A_{00}(s)^*p(s;h,v) + \sum_{i=1}^{N}\begin{cases}A_i(s-\theta_i)^*p(s-\theta_i;h,v), & s-\theta_i \leqq T\\ 0, & s-\theta_i > T\end{cases}\\ &\qquad + \int_{-b}^{0}\begin{cases}A_{01}(s-\theta,\theta)^*p(s-\theta;h,v), & s-\theta \leqq T\\ 0, & s-\theta > T\end{cases}d\theta \\ &\qquad + Q(s)x(s;h,v) + g(s) = 0 \quad \text{a.e. in } [0,T],\\ &p(T;h,v) = Fx(T;h,v).\end{aligned} \tag{4.7}$$

The notation $p(s; h, v)$ emphasizes the dependence of the adjoint solution on the control v and the initial datum h. From Lemma 3.3 in Delfour and Mitter [7] we know that by letting $y(s; w) = x(s; h, w) - x(s; h, 0)$,

$$\begin{aligned}&\mathscr{H}(T; T, (0 \circ y(\cdot\,;w))_T, p(\cdot\,;h,u)) - \mathscr{H}(0; T, (0\circ y(\cdot\,;w))_0, p(\cdot\,;h,u))\\ &= \int_0^T \Bigg(p(s;h,u)\Big|\frac{dy}{ds}(s;w) - A_{00}(s)y(s;w) - \sum_{i=1}^{N}\begin{cases}A_i(s)y(s+\theta_i;w), & s+\theta_i \geqq 0\\ 0, & s+\theta_i<0\end{cases}\\ &\quad - \int_{-b}^{0}\begin{cases}A_{01}(s,\theta)(s+\theta;w), & s+\theta\geqq 0\\ 0, & s+\theta<0\end{cases}d\theta\Big)\\ &\quad + \int_0^T\Bigg(\frac{dp}{ds}(s;h,u) + A_{00}(s)^*p(s;h,u)\\ &\quad + \sum_{i=1}^{N}\begin{cases}A_i(s-\theta_i)^*p(s-\theta_i;h,u), & s-\theta_i\leqq T\\ 0, & s-\theta_i>T\end{cases}\\ &\quad + \int_{-b}^{0}\begin{cases}A_{01}(s-\theta,\theta)^*p(s-\theta;h,u), & s-\theta\leqq T\\ 0, & s-\theta>T\end{cases}d\theta\Big|y(s;w)\Bigg)ds.\end{aligned} \tag{4.8}$$

Computing $Q(s)x(s; h, u) + g(s)$ in (4.6) by using (4.7) and (4.8), we can rewrite inequality (4.6) explicitly as

$$\int_0^T (B(s)^*p(s;h,u) + N(s)u(s) + m(s)|w(s))\,ds \geqq 0 \quad \text{for all } w\in L^2(0,T;U). \tag{4.9}$$

Hence the optimal control u is uniquely characterized by

$$u(s) = -N(s)^{-1}[B(s)^*p(s;h,u) + m(s)] \quad \text{a.e. in } [0,T].$$

Thus we have established the following result.

THEOREM 4.1 (Necessary and sufficient conditions of optimality). *Given h, there exists a unique control u which minimizes $J(v, h)$ in $L^2(0, T; U)$. The optimal control u is completely characterized by the identity*

$$u(s) = -N(s)^{-1}[B(s)^*p(s;h) + m(s)] \quad \textit{a.e. in } [0,T], \tag{4.10}$$

where $(p(\cdot\,; h), x(\cdot\,; h))$ is the unique pair of maps in $AC^2(0, T; H)$ which satisfies *the following system of equations*:

$$\frac{dx}{ds}(s;h) = A_{00}(s)x(s;h) + \sum_{i=1}^{N} A_i(s)\begin{Bmatrix} x(s+\theta_i;h), & s+\theta_i \geqq 0 \\ h(s+\theta_i), & s+\theta_i < 0 \end{Bmatrix}$$

$$(4.11) \qquad + \int_{-b}^{0} A_{01}(s,\theta)\begin{Bmatrix} x(s+\theta;h), & s+\theta \geqq 0 \\ h(s+\theta), & s+\theta < 0 \end{Bmatrix} d\theta$$

$$- B(s)N(s)^{-1}[B(s)^*p(s;h) + m(s)] + f(s) \qquad \text{a.e. in } [0,T],$$

$$x(0;h) = h(0);$$

$$\frac{dp}{ds}(s;h) + A_{00}(s)^*p(s;h) + \sum_{i=1}^{N} \begin{Bmatrix} A_i(s-\theta_i)^*p(s-\theta_i;h), & s-\theta_i \leqq T \\ 0, & s-\theta_i > T \end{Bmatrix}$$

$$(4.12) \qquad + \int_{-b}^{0} \begin{Bmatrix} A_{01}(s-\theta,\theta)^*p(s-\theta;h), & s-\theta \leqq T \\ 0, & s-\theta > T \end{Bmatrix} d\theta + Q(s)x(s;h) + g(s) = 0 \qquad \text{a.e. in } [0,T],$$

$$p(T;h) = Fx(T;h).$$

Proof. The proof of this theorem is an immediate consequence of the existence and uniqueness of the pair $(x(\cdot : h, v), p(\cdot\, ; h, v))$ as solutions of (2.1) and (4.7) and the characterization given by (4.10). ▧

COROLLARY 4.2. *Let* $m = 0$ *in* (4.11). *The two-point boundary value problem* (4.11)–(4.12) *has a unique solution* $(x(\cdot\,;h), p(\cdot\,;h))$ *in* $AC^2(0,T;H) \times AC^2(0,T;H)$. ▧

Remark. A different type of boundary value problem for HDE can be found in a paper by A. Halanay [17].

4.3. "Decoupling" of the equations of Theorem 4.1; the operators *D* and *P*. In this section we consider the initial datum h to be fixed. Let $f'(r) = f(r) - B(r)N(r)^{-1}m(r)$ and $R(r) = B(r)N(r)^{-1}B(r)^*$. We shall write $x(r)$, $p(r)$ and $J(v)$ in place of $x(r;h)$, $p(r;h)$ and $J(v,h)$ as in Theorem 4.1. In order to "decouple" the system of equations (4.11)–(4.12) we consider the problem of § 4.2 in the interval $[s,T]$, $s \in [0,T[$ instead of $[0,T]$. In this case the solution of (2.1) in the interval $[s,T]$ is denoted by $\phi(\cdot\,; s, v)$ and the cost is defined by

$$J_s(v) = (\phi(T;s,v)|F\phi(T;s,v))$$

$$(4.13) \qquad + \int_s^T [(\phi(r;s,v)|Q(r)\phi(r;s,v)) + (v(r)|N(r)v(r))]\, dr$$

$$+ 2\int_s^T [(\phi(r;s,v)|g(r)) + (v(r)|m(r))]\, dr.$$

Corresponding to $\phi(\cdot\,; s, v)$, the solution of

$$\frac{d\phi}{dr}(r;s,v) = A_{00}(r)\phi(r;s,v) + \sum_{i=1}^{N} A_i(r)\begin{Bmatrix} \phi(r+\theta_i;s,v), & r+\theta_i \geqq s \\ h(r+\theta_i-s), & r+\theta_i < s \end{Bmatrix}$$

$$(4.14) \qquad + \int_{-b}^{0} A_{01}(r,\theta)\begin{Bmatrix} \phi(r+\theta;s,v), & r+\theta \geqq s \\ h(r+\theta-s), & r+\theta < s \end{Bmatrix} d\theta$$

$$+ B(r)v(r) + f(r) \quad \text{a.e. in } [s,T],$$

$$\phi(s;s,v) = h(0),$$

we introduce the adjoint solution $\psi(\cdot\,;s,v)$ as the solution of

$$
\begin{aligned}
&\frac{d\psi}{dr}(r;s,v) + A_{00}(r)^*\psi(r;s,v) + \sum_{i=1}^{N} \begin{cases} A_i(r-\theta_i)^*\psi(r-\theta_i;s,v), & r-\theta_i \leqq T \\ 0, & r-\theta_i > T \end{cases} \\
&\qquad + \int_{-b}^{0} \begin{cases} A_{01}(r-\theta,\theta)^*\psi(r-\theta;s,v), & r-\theta \leqq T \\ 0, & r-\theta > T \end{cases} d\theta \qquad (4.15)\\
&\qquad + Q(r)\phi(r;s,v) + g(r) = 0 \quad \text{a.e. in } [s,T], \\
&\psi(T;s,v) = F\phi(T;s,v).
\end{aligned}
$$

We now obtain the analogue of Theorem 4.1 for the optimal control $u \in L^2(s,T;U)$ which is characterized by

$$u(r) = -N(r)^{-1}[B(r)^*\psi(r;s) + m(r)] \quad \text{a.e. in } [s,T], \tag{4.16}$$

where the pair $(\phi(\cdot\,;s), \psi(\cdot\,;s)) = (\phi(\cdot\,;s,u), \psi(\cdot\,;s,u))$ is the solution in $AC^2(s,T;H) \times AC^2(s,T;H)$ of the coupled system

$$
\begin{aligned}
&\frac{d\phi}{dr}(r;s) = A_{00}(r)\phi(r;s) + \sum_{i=1}^{N} A_i(r) \begin{cases} \phi(r+\theta_i;s), & r+\theta_i \geqq s \\ h(r+\theta_i-s), & r+\theta_i < s \end{cases} \\
&\qquad + \int_{-b}^{0} A_{01}(r,\theta) \begin{cases} \phi(r+\theta;s), & r+\theta \geqq s \\ h(r+\theta-s), & r+\theta < s \end{cases} d\theta \qquad (4.17)\\
&\qquad - R(r)\psi(r;s) + f'(r) \quad \text{a.e. in } [s,T], \\
&\phi(s,s) = h(0);
\end{aligned}
$$

$$
\begin{aligned}
&\frac{d\psi}{dr}(r;s) + A_{00}(r)^*\psi(r;s) + \sum_{i=1}^{N} \begin{cases} A_i(r-\theta_i)^*\psi(r-\theta_i;s), & r-\theta_i \leqq T \\ 0, & r-\theta_i > T \end{cases} \\
&\qquad + \int_{-b}^{0} A_{01}(r-\theta,\theta)^* \begin{cases} \psi(r-\theta;s), & r-\theta \leqq T \\ 0, & r-\theta > T \end{cases} d\theta \qquad (4.18)\\
&\qquad + Q(r)\phi(r;s) + g(r) = 0 \quad \text{a.e. in } [s,T], \\
&\psi(T;s) = F\phi(T;s).
\end{aligned}
$$

LEMMA 4.3. *The map*

$$
\begin{aligned}
&(h,f,g,m) \mapsto (\phi(\cdot\,;s), \psi(\cdot\,;s)) \\
&\quad : M^2(-b,0;H) \times L^2(s,T;H) \times L^2(s,T;H) \times L^2(s,T;U) \qquad (4.19)\\
&\qquad \to AC^2(s,T;H) \times AC^2(s,T;H)
\end{aligned}
$$

is linear and continuous.

Proof. The map (4.19) is clearly linear. To show it is continuous, choose an arbitrary sequence $\{(h_n, f_n, g_n, m_n)\}$ in $M^2 \times L^2 \times L^2 \times L^2$ which converges to (h,f,g,m). Let $(\phi_n(\cdot\,;s,v), \psi_n(\cdot\,;s,v))$ (resp. $(\phi(\cdot\,;s,v), \psi(\cdot\,;s,v))$) be the solution of the system (4.14)–(4.15) for some $v \in L^2(s,T;U)$ and the data (h_n, f_n, g_n, m_n) (resp. (h,f,g,m)). For fixed v,

$$(h_n, f_n, g_n, m_n) \to (h,f,g,m) \Rightarrow \phi_n(\cdot\,;s,v) \to \phi(\cdot\,;s,v) \quad \text{in } AC^2(s,T;H) \tag{4.20}$$

by [7, Cor. 2.7]. Let u_n (resp. u) be the optimal control for the cost $J_s^n(v)$ (resp. $J_s(v)$) defined in terms of (h_n, f_n, g_n, m_n) (resp. (h, f, g, m)). We clearly obtain

$$J_s^n(u_n) = \inf_{v \in L^2} J_s^n(v) \leqq J_s^n(u) \tag{4.21}$$

and

$$J_s^n(u) \to J_s(u) \tag{4.22}$$

by (4.20) with $v = u$ and the very construction of J_s^n and J_s.

Hence

$$\limsup J_s^n(u_n) \leqq \limsup J_s^n(u) = J_s(u) = \inf_{v \in L^2} J_s(v). \tag{4.23}$$

Because of the hypothesis on $N(r)$, there exist $c_1 > 0$ and $c_2 > 0$ for which

$$J_s^n(u_n) \geqq c_1 |u_n|^2 - c_2 |u_n|. \tag{4.24}$$

From the last inequality and (4.23) it is necessary that

$$u_n \in \text{bounded subset of } L^2 \quad \text{as } (h_n, f_n, g_n, m_n) \to (h, f, g, m). \tag{4.25}$$

By weak compactness there is a subsequence u_μ such that

$$u_\mu \to w \quad \text{in } L^2 \text{ weakly.} \tag{4.26}$$

Thus

$$\phi_\mu(\cdot\,; s, u_\mu) \to \phi(\cdot\,; s, w) \quad \text{in } AC^2(s, T; H) \text{ weakly} \tag{4.27}$$

and finally (by convexity of $J_s(v)$ in v),

$$\liminf J_s^\mu(u_\mu) \geqq J_s(w). \tag{4.28}$$

If we combine inequalities (4.23) and (4.28), we necessarily have $w = u$. The results are summarized below:

$$\begin{aligned} &u_n \to u \quad \text{in } L^2(s, T; H) \text{ weakly,} \\ &J_s^n(u_n) \to J_s(u); \end{aligned} \tag{4.29}$$

$$\begin{aligned} &\phi_n(\cdot\,; s, u_n) \to \phi(\cdot\,; s, u) \quad \text{in } AC^2(s, T; H) \text{ weakly,} \\ &\psi_n(\cdot\,; s, u_n) \to \psi(\cdot\,; s, u) \quad \text{in } AC^2(s, T; H) \text{ weakly.} \end{aligned} \tag{4.30}$$

This shows the continuity on $M^2 \times L^2 \times L^2 \times L^2$ (in the strong topology) into $AC^2(s, T; H) \times AC^2(s; T; H)$ (in the weak topology) of the map (4.11). Since all the spaces in presence are Hilbert this is sufficient to prove the theorem. ▧

Remark. The above is essentially Lions' proof [26, Lemma 4.2, pp. 148–150].

COROLLARY 4.4. *The map*

$$(h, f, g, m) \mapsto \psi(r; s) : M^2 \times L^2 \times L^2 \times L^2 \to H \tag{4.31}$$

is linear and continuous for $r \in [s, T]$. Hence it has the representation

$$\psi(r; s) = D(r, s)h + F(r, s)f + G(r, s)g + M(r, s)m \tag{4.32}$$

for

$$D(r, s) \in \mathscr{L}(M^2(-b, 0; H), H), \qquad F(r, s) \in \mathscr{L}(L^2(s, T; H), H),$$
$$G(r, s) \in \mathscr{L}(L^2(s, T; U), H) \quad \text{and} \quad M(r, s) \in \mathscr{L}(L^2(s, T; K), H). \quad ▧$$

In the remainder of this section we assume that f, g and m are fixed. In this case we may write

$$\psi(r;s) = D(r,s)h + d(r,s) \tag{4.33}$$

instead of (4.32).

LEMMA 4.5. *Let* $(x,p) = (x(\cdot\,;h), p(\cdot\,;h))$ *be the solution of the system* (4.11)–(4.12). *Then*

$$p(t) = D(t,s)\tilde{x}(s;h) + d(t,s) \tag{4.34}$$

for all pairs $s \leqq t$ *in* $[0,T]$, *where* $D(t,s)$ *and* $d(t,s)$ *are defined by the following rules*:

(i) *We solve the system*

$$\begin{aligned}\frac{d\beta}{dr}(r) = A_{00}(r)\beta(r) &+ \sum_{i=1}^{N} A_i(r)\begin{Bmatrix}\beta(r+\theta_i), & r+\theta_i \geqq s\\ h(r+\theta_i - s), & r+\theta_i < s\end{Bmatrix}\\ &+ \int_{-b}^{0} A_{01}(r,\theta)\begin{Bmatrix}\beta(r+\theta), & r+\theta \geqq s\\ h(r+\theta-s), & r+\theta < s\end{Bmatrix} d\theta - R(r)\gamma(r)\end{aligned} \tag{4.35}$$

a.e. in $[s,T]$ *and* $\beta(s) = h(0)$,

$$\begin{aligned}\frac{d\gamma}{dr}(r) + A_{00}(r)^*\gamma(r) &+ \sum_{i=1}^{N}\begin{Bmatrix}A_i(r-\theta_i)^*\gamma(r-\theta_i), & r-\theta_i \leqq T\\ 0, & r-\theta_i > T\end{Bmatrix}\\ &+ \int_{-b}^{0}\begin{Bmatrix}A_{01}(r-\theta,\theta)^*\gamma(r-\theta), & r-\theta \leqq T\\ 0, & r-\theta > T\end{Bmatrix} d\theta + Q(r)\beta(r) = 0\end{aligned} \tag{4.36}$$

a.e. in $[s,T]$ *and* $\gamma(T) = F\beta(T)$;

then

$$D(t,s)h = \gamma(t), \quad t \in [s,T].$$

(ii) *We solve the system*

$$\begin{aligned}\frac{d\eta}{dr}(r) = A_{00}(r)\eta(r) &+ \sum_{i=1}^{N} A_i(r)\begin{Bmatrix}\eta(r+\theta_i), & r+\theta_i \geqq 0\\ 0, & r+\theta_i < 0\end{Bmatrix}\\ &+ \int_{-b}^{0} A_{01}(r,\theta)\begin{Bmatrix}\eta(r+\theta), & r+\theta \geqq 0\\ 0, & r+\theta < 0\end{Bmatrix} d\theta - R(r)\xi(r) + f'(r)\end{aligned} \tag{4.37}$$

a.e. in $[s,T]$ *and* $\eta(s) = 0$,

$$\begin{aligned}\frac{d\xi}{dr}(r) + A_{00}(r)^*\xi(r) &+ \sum_{i=0}^{N}\begin{Bmatrix}A_i(r-\theta_i)^*\xi(r-\theta_i), & r-\theta_i \leqq T\\ 0, & r-\theta_i > T\end{Bmatrix}\\ &+ \int_{-b}^{0}\begin{Bmatrix}A_{01}(r-\theta,\theta)^*\xi(r-\theta), & r-\theta \leqq T\\ 0, & r-\theta > T\end{Bmatrix} d\theta + Q(r)\eta(r) + g(r) = 0\end{aligned} \tag{4.38}$$

a.e. in $[s,T]$ *and* $\xi(T) = F\eta(T)$;

then

$$d(t,s) = \xi(t), \quad t \in [s,T].$$

Proof. $D(t, s)$ and $d(t, s)$ are clearly obtained from the rules (i) and (ii) of the lemma: it suffices to decompose the map $h \to \psi(r)$ into its linear part and its constant part. We only need to establish identity (4.34). Consider the system (4.14)–(4.15) with initial datum $\tilde{x}(s; h)$ (see Definition 2.1) at time s, where x is the solution of the system (4.11)–(4.12) with initial datum h. The solution is denoted by (ϕ, ψ).

We also define

$$\bar{\phi} \text{ (resp. } \bar{\psi}) = \text{restriction of } x \text{ (resp. } p) \text{ to } [s, T],$$

where $\bar{\phi}$ and $\bar{\psi}$ are the solutions of the system (4.14)–(4.15) with initial datum $\tilde{x}(s; h)$. By uniqueness, $(\phi, \psi) = (\bar{\phi}, \bar{\psi})$ and $p(t) = \bar{\psi}(t) = D(t, s)\tilde{x}(s; h) + d(t, s)$. This proves the lemma. ▧

Remark. The above is essentially Lions' original proof (cf. J. L. Lions [26, Lemma 4.3]).

COROLLARY 4.6. *Given* $s \in [0, T[$ *and* $h \in M^2(-b, 0; H)$, *the maps* $t \mapsto D(t, s)h$ *and* $t \mapsto d(t, s)$ *are in* $AC^2(s, T; H)$. ▧

DEFINITION 4.7. For all $s \in [0, T[$ and $\eta \in I(-b, 0)$, $s - \eta \leqq T$, let

$$P(s, \eta) = \begin{cases} D(s - \eta, s), & \eta \in [s - T, 0] \cap I(-b, 0), \\ 0, & \text{otherwise}. \end{cases} \tag{4.39}$$

This defines the operator $P(s) \in \mathscr{L}(M^2(-b, 0; H))$ in the natural way $(P(s)h)(\eta) = P(s, \eta)h$. Similarly, let

$$r(s, \eta) = \begin{cases} d(s - \eta, s), & \eta \in [s - T, 0] \cap I(-b, 0), \\ 0, & \text{otherwise}; \end{cases} \tag{4.40}$$

and this defines $r(s) \in M^2(-b, 0; H)$, where $(r(s))(\eta) = r(s, \eta)$.

Remark. The conclusions of Lemma 4.5 can also be written in state form. Let $\tilde{x}, \tilde{p}, \tilde{\gamma}$ and $\tilde{\xi}$ denote the state variables associated with the variables x, p, γ and ξ, respectively. We can write $\tilde{p}(s) = P(s)\tilde{x}(s) + r(s)$, where $P(s)h = \tilde{\gamma}(s)$ and $r(s) = \tilde{\xi}(s)$. Here $P(s)$ and $r(s)$ are defined directly.

4.4. The operator $\Pi(s)$ and the optimal cost; relations between Π and P. In this section we introduce the operator $\Pi(s)$ which characterizes the optimal cost. It is constructed from $D(s - \eta, s)$, $\eta \in I(-b, 0)$, $s - \eta \leqq T$, or simply from $P(s, \eta)$ (Definition 4.5). Since there is an isometric isomorphism κ between $M^2(-b, 0; H)$ and $H \times L^2(-b, 0; H)$ the operator $P(s, \eta)$ can be decomposed in the following way:

$$P(s, \eta)h = P^0(s, \eta)h^0 + P^1(s, \eta)h^1, \tag{4.41}$$

where $h \in M^2(-b, 0; H)$, $\kappa(h) = (h^0, h^1)$ (see [6, Prop. 2.1]), $P^0(s, \eta) \in \mathscr{L}(H)$ and $P^1(s, \eta) \in \mathscr{L}(L^2(-b, 0; H), H)$.

PROPOSITION 4.8. *Let* $f = g = 0$ *and* $m = 0$ *in system* (4.17)–(4.18). *We denote by* $(\phi(\cdot\,; s), \psi(\cdot\,; s))$ (*resp.* $(\bar{\phi}(\cdot\,; s), \bar{\psi}(\cdot\,; s))$) *its solution for the initial datum* h (*resp.* $\bar{h}$). *Then*

$$\begin{aligned} \mathscr{H}(s; T, h, D(\cdot, s)\bar{h}) = (\phi(T; s)|F\bar{\phi}(T; s)) + \int_s^T [(\bar{\psi}(r; s)|R(r)\psi(r; s)) \\ + (Q(r)\bar{\phi}(r; s)|\phi(r; s))]\, dr. \end{aligned} \tag{4.42}$$

The map

$$(4.43)\quad (h,\bar{h}) \mapsto \mathscr{H}(s;T,h,D(\cdot,s)\bar{h}) : M^2(-b,0;H) \times M^2(-b,0;H) \to R$$

is a continuous bilinear form which is symmetric and positive.

Proof. From Lemma 3.3 of [7] and equations (4.17)–(4.18),

$$(4.44)\quad \begin{aligned} &\mathscr{H}(T;T,(h\circ\phi(\cdot,s))_T,\bar{\psi}(\cdot\,;s)) - \mathscr{H}(s;T,h,\bar{\psi}(\cdot\,;s)) \\ &\quad = -\int_s^T [(\bar{\psi}(r;s)|R(r)\psi(r;s)) + (Q(r)\bar{\phi}(r;s)|\phi(r;s))]\,dr, \end{aligned}$$

where $\mathscr{H}$ is given by Definition 3.2 in [7]. But

$$(4.45)\quad \mathscr{H}(T;T,(h\circ\phi(\cdot\,;s))_T,\bar{\psi}(\cdot\,;s)) = (\phi(T;s)|\bar{\psi}(T;s)) = (\phi(T;s)|F\bar{\phi}(T;s))$$

and we obtain (4.42) from identity (4.44). The map (4.43) is clearly bilinear and continuous since the map $h \mapsto (\phi(\cdot\,;s),\psi(\cdot\,;s))$ (resp. $\bar{h} \mapsto (\bar{\phi}(\cdot\,;s),\bar{\psi}(\cdot\,;s))$ is linear and continuous by Lemma 4.1. The symmetry and positivity of the map (4.43) follow from the symmetry and positivity of the operators F, $R(r)$ and $Q(r)$. ▧

COROLLARY 4.9. (i) *The map* (4.43) *can be written in a unique way in terms of the transformation* $\Pi(s) \in \mathscr{L}(M^2(-b,0;H))$:

$$(4.46)\quad \mathscr{H}(s;T,h,D(\cdot,s)\bar{h}) = (\Pi(s)\bar{h}|h)_{M^2}.$$

(ii) $\Pi(s)$ *is equivalent to the matrix of operators*

$$(4.47)\quad \begin{pmatrix} \Pi^{00}(s) & \Pi^{01}(s) \\ \Pi^{10}(s) & \Pi^{11}(s) \end{pmatrix},$$

where $\Pi^{00}(s) \in \mathscr{L}(H)$, $\Pi^{01}(s) \in \mathscr{L}(L^2,H)$, $\Pi^{11}(s) \in \mathscr{L}(L^2)$ *and* $\Pi^{10}(s) \in \mathscr{L}(H,L^2)$.

(iii) *Moreover,*

$$(4.48)\quad \Pi^{00}(s)^* = \Pi^{00}(s) \geqq 0,$$

$$(4.49)\quad \Pi^{01}(s) = \Pi^{10}(s)^*,$$

$$(4.50)\quad \Pi^{11}(s)^* = \Pi^{11}(s) \geqq 0.$$

(iv) *Let u be the optimal control obtained under the hypotheses of Proposition* 4.6. *The optimal cost can be written as*

$$(4.51)\quad J_s^h(u) = (\Pi^{00}(s)h^0|h^0) + 2(\Pi^{01}(s)h^1|h^0) + (\Pi^{11}(s)h^1|h^1)_{L^2}.$$

Moreover, there exists a constant $c > 0$ such that

$$(4.52)\quad \|\Pi(s)h\|_{M^2} \leqq c\|h\|_{M^2}.$$

Proof. (i)–(iv) are obvious. The inequality in (iv) follows from the positivity and symmetry of the operator $\Pi(s)$ and

$$|(\Pi(s)h|h)_{M^2}| = J_s^h(u) \leqq J_s^h(0) \leqq c\|h\|_{M^2}^2. \quad ▧$$

The operator $\Pi(s)$ can now be expressed in terms of $P(s,\eta)$, $\eta \in I(-b,0)$, $s-\eta \leqq T$. In doing this we obtain further information on $\Pi(s)$.

COROLLARY 4.10. (i) $\Pi^{00}(s) = P^0(s,0), \Pi^{01}(s) = P^1(s,0)$ *and* $(\Pi^{10}(s)h^0)(\alpha) = \Pi^{10}(s,\alpha)h^0$, *where*

$$\Pi^{01}(s,\alpha) = \sum_{i=1}^{N} \begin{Bmatrix} A_i(s+\alpha-\theta_i)^* P^0(s,\theta_i-\alpha), & \alpha+s-T<\theta_i \leqq \alpha \\ 0, & \textit{otherwise} \end{Bmatrix} + \int_{\max\{-b,\alpha+s-T\}}^{\alpha} A_{01}(s+\alpha-\theta,\theta)^* P^0(s,\theta-\alpha)\,d\theta. \tag{4.53}$$

(ii) *The kernel* $\Pi^{01}(s,\alpha)$ *of* $\Pi^{01}(s)$ *is equal to* $\Pi^{10}(s,\alpha)^*$.

(iii)

$$(\Pi^{11}(s)h^1)(\alpha) = \sum_{i=1}^{N} \begin{Bmatrix} A_i(s+\alpha-\theta_i)^* P^1(s,\theta_i-\alpha)h^1, & \alpha+s-T<\theta_i \leqq \alpha \\ 0, & \textit{otherwise} \end{Bmatrix} + \int_{\max\{-b,\alpha+s-T\}}^{\alpha} A_{01}(s+\alpha-\theta,\theta)^* P^1(s,\theta-\alpha)h^1\,d\theta. \tag{4.54}$$

If we now go back to the system (2.1) *defined in* $[0, T]$, *the minimizing control* $u(s)$ *at time* s *is given by*

$$u(s) = -N(s)^{-1}\left[B(s)^*\left[\Pi^{00}(s)x(s) + \int_{-b}^{0} \Pi^{10}(s,\alpha)^*\tilde{x}(s;h,u)(\alpha)\,d\alpha + r(s,0)\right] + m(s)\right]. \tag{4.55}$$ ▧

5. Operational differential equation of Riccati type for the operators $P(t)$ and $\Pi(t)$. So far we have established the existence of operators $P(t)$ and $\Pi(t)$ in $\mathscr{L}(M^2)$ and studied their properties. We have also shown how $P(t)$ and $\Pi(t)$ can be indirectly computed. In this section we show that $P(t)$ and $\Pi(t)$ satisfy operational differential equations of Riccati type. In order to study $P(t)$ and $\Pi(t)$ we assume that in (2.1), $f = 0$, and in (4.1) that $m = 0$ and $g = 0$ (Proposition 4.6).

5.1. Formal derivation of an operational differential equation for $\Pi(t)$. We use the fact that there is an isometric isomorphism κ between $M^2(-b,0;H)$ and $H \times L^2(-b,0;H)$, where $\kappa(h) = (h^0,h^1)$. Since $\Pi(s) \in \mathscr{L}(M^2(-b,0;H))$, we use the above isomorphism and write

$$\begin{aligned} [\Pi(s)h]^0 &= \Pi^0(s)h, \quad \Pi^0(s) \in \mathscr{L}(M^2,H), \\ [\Pi(s)h]^1 &= \Pi^1(s)h, \quad \Pi^1(s) \in \mathscr{L}(M^2,L^2). \end{aligned} \tag{5.1}$$

We denote by (p,x) (resp. $(\bar{p},\bar{x})$) the solution of system (4.11)–(4.12) with initial datum h (resp. $\bar{h}$). We use the notation (see Definition 2.1)

$$x_s = \tilde{x}(s;h) \quad (\text{resp. } \bar{x}_s = \tilde{\bar{x}}(s;h)). \tag{5.2}$$

From (4.34), Definition 4.7, Corollary 4.10 and (5.1),

$$p(s) = \Pi^0(s)x_s \quad (\text{resp. } \bar{p}_s = \Pi^0(s)\bar{x}_s). \tag{5.3}$$

Define operators $\tilde{R}(t)$, $\tilde{Q}(t)$ and $\tilde{F}$ in $\mathscr{L}(M^2)$ as

$$[\tilde{R}(t)h](\alpha) = \begin{cases} R(t)h(0), & \alpha = 0, \\ 0, & \text{otherwise}; \end{cases} \tag{5.4}$$

$$[\tilde{Q}(t)h](\alpha) = \begin{cases} Q(t)h(0), & \alpha = 0, \\ 0, & \text{otherwise}; \end{cases} \tag{5.5}$$

$$[\tilde{F}h](\alpha) = \begin{cases} Fh(0), & \alpha = 0, \\ 0, & \text{otherwise}. \end{cases} \tag{5.6}$$

It is easy to verify the following:

$$[\tilde{R}(t)\Pi(t)x_t](0) = R(t)\Pi^0(t)x_t = R(t)p(t), \tag{5.7}$$

and

$$(p(t)|R(t)\bar{p}(t))_H = (\Pi(t)x_t|\tilde{R}(t)\Pi(t)\bar{x}_t), \tag{5.8}$$

$$(x(T)|F\bar{x}(T))_H = (x_T|\tilde{F}\bar{x}_T)_{M^2}, \tag{5.9}$$

$$(x(t)|Q(t)\bar{x}(t))_H = (x_t|\tilde{Q}(t)\bar{x}_t)_{M^2}. \tag{5.10}$$

Then from (2.16) in Theorem 2.3, (4.34) and (5.4),

$$\begin{aligned} \frac{dx_t}{dt} &= \tilde{A}(t)x_t - \tilde{R}(t)\Pi(t)x_t, \quad t \in [0, T], \\ x_0 &= h. \end{aligned} \tag{5.11}$$

Equations (5.4) through (5.11) can also be written with $\bar{h}$ and $\bar{x}$ in place of h and x. From (4.42), (4.46) and (5.8),

$$(x_s|\Pi(s)\bar{x}_s) = (x_T|\tilde{F}\bar{x}_T) + \int_s^T [(x_r|\tilde{Q}(r)\bar{x}_r) + (\Pi(r)x_r|\tilde{R}(r)\Pi(r)\bar{x}_r)]\, dr \tag{5.12}$$

and

$$(x_T|\Pi(T)\bar{x}_T) = (x_T|\tilde{F}\bar{x}_T). \tag{5.13}$$

Formal differentiation of both sides of (5.12) and use of (5.11) yields

$$([\dot{\Pi}(s) + \Pi(s)\tilde{A}(s) + \tilde{A}(s)^*\Pi(s) - \Pi(s)\tilde{R}(s)\Pi(s) + \tilde{Q}(s)]x_s|\bar{x}_s)_{M^2} = 0.$$

Since this has to be true for all x_s and $\bar{x}_s$, we get

$$\begin{aligned} &\dot{\Pi}(s) + \Pi(s)\tilde{A}(s) + \tilde{A}(s)^*\Pi(s) - \Pi(s)\tilde{R}(s)\Pi(s) + \tilde{Q}(s) = 0, \\ &\Pi(T) = \tilde{F}, \end{aligned} \tag{5.14}$$

where $\tilde{A}(s)^*$ is the M^2-adjoint of $\tilde{A}(s)$.

5.2. Interpretation of equation (5.14). The first question is to determine in what sense equation (5.14) has a solution. There are two ways to proceed: either to study (5.14) directly or to study an equivalent integral equation. In the first situation we can apply certain results of Da Prato [4], but we need further assumptions.

In the second situation we study an equivalent integral equation rather than the differential equation directly.

5.3. Direct study of equation (5.14). In order to apply Da Prato's results to equation (5.14) we need further hypotheses:

(i) A_{00}, A_i, A_{01} and B in (2.1) and Q and R in (4.1) do not depend on t;
(ii) there exist $\omega \geqq 0$ and $K > 0$ such that

$$\|\tilde{\Phi}(t)\|_{\mathscr{L}(M^2)} \leqq K\, e^{-\omega t} \quad \text{for all } t \geqq 0;$$

(iii) $\tilde{F} = 0$.

5.3.1. Results of Da Prato. We now state the results of Da Prato (cf. [4, Thms. 7.5 and 7.6]) which are of interest to us.

Let X be a Banach space. Let $\mathscr{L}(X, X)$ denote the algebra of bounded linear operators on X, $\mathscr{L}_s(X, X)$ the space $\mathscr{L}(X, X)$ endowed with the topology of simple convergence in X. Let M and N be two unbounded operators in X which are infinitesimal generators of strongly continuous semigroups $t \to e^{tM}$ and $t \to e^{tN}$ respectively. Moreover, we assume that there exist positive constants K_M, K_N and $\omega_N \in R_+$ such that

$$\|e^{tM}\| \leqq K_M \|e^{tN}\| \leqq K_N\, e^{-\omega_N t} \quad \text{for all } t \in R_+.$$

We consider the equation

$$(5.15) \qquad \begin{aligned} &\frac{dU}{dt} - MU(t) - U(t)N - f(U(t)) = V(t), \\ &U(0) = 0, \quad t \in [0, T], \quad T \in R_+. \end{aligned}$$

In the above, f is holomorphic in an open set Ω of the complex plane containing the origin and $V \in C(0, T; \mathscr{L}_s(X, X))$.

DEFINITION 5.1. $U \in C(0, T; \mathscr{L}_s(X, X))$ is said to be a *weak solution of* (5.15) if there exists a sequence $\{U_n\}$ in $C^1(0, T; \mathscr{L}_s(X, X))$ such that[1]

(i) $U_n(0) = 0$ for all $n \in N$;
(ii) $U_n(t)x \in \mathscr{D}(M)$ (domain of M) for all $x \in X$, $t \in R_+$ and $t \mapsto MU_n(t) \in C(0, T; \mathscr{L}_s(X, X))$;
(iii) $U_n(t)N$ can be extended to a bounded operator $\overline{U_n(t)N}$, $t \in R_+$ and $t \mapsto \overline{U_n(t)N} \in C(0, T; \mathscr{L}_s(X, X))$;
(iv) $U_n \to U$ and $dU_n/dt - MU_n - \overline{U_nN} - f(U_n) \to V$ in $C(0, T; \mathscr{L}_s(X, X))$.

THEOREM 5.2 (Da Prato). *Let* $T_0 \in R_+$, $V \in C(0, T_0; \mathscr{L}_s(X, X))$. *Then there exists a* $T \leqq T_0$ *such that* (5.15) *has a unique weak solution in* $[0, T]$. ▧

Suppose further that there exists a constant $C \in R_+$ such that if $T \in [0, T_0]$ and U is a solution of (5.15) in $[0, T]$, we have

$$\|U(t)\| \leqq C \quad \text{for all } t \in [0, T].$$

THEOREM 5.3 (Da Prato). *Equation* (5.15) *has a unique global weak solution.* ▧

[1] $C^1(0, T; \mathscr{L}_s(X, X))$ = space of functions with values in $\mathscr{L}_s(X, X)$ which are strongly differentiable.

5.3.2. Existence of a global solution for (5.14). In view of the fact that $\tilde{A}$ and $\tilde{A}^*$ are infinitesimal generators of strongly continuous semigroups which are adjoint to each other and $f(\Pi) = \Pi\tilde{R}\Pi$, $\tilde{R}$ being a bounded positive operator, all the assumptions of Theorem 5.2 are satisfied and we can conclude that when $F = 0$ there exists a unique weak solution of (5.15) locally. Finally, in view of the a priori bound (4.52) we can conclude that the local solution is also global.

5.4. Study of equation (5.14) via an equivalent integral equation. We now derive an integral equation equivalent to (5.14).

PROPOSITION 5.4. *The operator* Π *defined in Corollary* 4.9 *is the unique minimal*[2] *solution of the following system*:

$$(5.16)\qquad \begin{aligned} &\frac{\partial}{\partial r}\tilde{\Phi}(r,s) = [\tilde{A}(r) - \tilde{R}(r)\Pi(r)]\tilde{\Phi}(r,s) \quad a.e.\ in\ [s,T],\\ &\tilde{\Phi}(s,s) = I, \end{aligned}$$

$$(5.17)\qquad \Pi(s) = \tilde{\Phi}(T,s)^*\tilde{F}\tilde{\Phi}(T,s) + \int_s^T \tilde{\Phi}(r,s)^*[\tilde{Q}(r) + \Pi(r)\tilde{R}(r)\Pi(r)]\tilde{\Phi}(r,s)\,dr.$$

Proof. (i) We start with Proposition 4.8. We know that for $f = g = 0$ and $m = 0$ system (4.17)–(4.18) has a unique solution and that

$$\psi(r,s) = P^0(r)\tilde{\phi}(r,s) = \Pi^0(r)\tilde{\phi}(r,s)$$

(cf. Lemma 4.5 and Definition 4.7).

Equation (4.17) can now be written in state form:

$$\begin{aligned} &\frac{\partial}{\partial r}\tilde{\phi}(r,s) = [\hat{A}(r) - \tilde{R}(r)\Pi(r)]\tilde{\phi}(r,s) \quad \text{in } [s,T],\\ &\tilde{\phi}(s,s) = h. \end{aligned}$$

This clearly generates the semigroup $\tilde{\Phi}(r,s)$. Now using (4.42) in Proposition 4.8 and (4.46) in Corollary 4.9, we obtain

$$\begin{aligned} (h|\Pi(s)\bar{h}) = {} & (\tilde{\Phi}(T,s)h|\tilde{F}\tilde{\Phi}(T,s)\bar{h})\\ & + \int_s^T [(\Pi(r)\tilde{\Phi}(r,s)h|\tilde{R}(r)\Pi(r)\tilde{\Phi}(r,s)\bar{h})\\ & + (\tilde{\Phi}(r,s)h|\tilde{Q}(r)\tilde{\Phi}(r,s)\bar{h})]\,dr. \end{aligned}$$

This is sufficient to show that Π is a solution of system (5.16)–(5.17).

(ii) Consider another solution $\tilde{\Pi}$ of (5.16)–(5.17). Fix a time $s \in [0, T[$. This corresponds to the feedback control

$$\tilde{u}(t) = -N^{-1}B^*\tilde{\Pi}^0(t), \quad t \in [s,T],$$

in the time interval $[s, T]$. However, by definition,

$$(\Pi(s)h|h) = \min_{v \in L^2(s,T;U)} J_s(v) \leqq J_s(\tilde{u}) = (\tilde{\Pi}(s)h|h).$$

This proves the minimality property. ▧

[2] That is, the corresponding control gives us the minimal cost.

COROLLARY 5.5. *For all h and $\bar{h}$ in $\mathscr{D}$ the operator Π defined in Corollary 4.9 is a positive self-adjoint solution of the following system*:

$$\begin{aligned}(5.18)\qquad &\frac{d}{dt}(\bar{h}|\Pi(t)h) + (\tilde{A}(t)\bar{h}|\Pi(t)h) + (\bar{h}|[\Pi(t)\tilde{A}(t) - \Pi(t)\tilde{R}(t)\Pi(t)\\ &\qquad + \tilde{Q}(t)]h) = 0 \quad a.e.\ in\ [0, T],\\ &\Pi(T) = \tilde{F}.\end{aligned}$$

Proof. Let $\tilde{\bar{\phi}}(t, s) = \tilde{\Phi}(t, s)\bar{h}$ and $\tilde{\phi}(t, s) = \tilde{\Phi}(t, s)h$. Using the fact that $\tilde{\phi}(r, s) = \tilde{\Phi}(r, t)\tilde{\phi}(t, s)$, we compute $(\tilde{\bar{\phi}}(t, s)|\Pi(t)\tilde{\phi}(t, s))$ from (5.17):

$$\begin{aligned}(\tilde{\bar{\phi}}(t, s)|\Pi(t)\tilde{\phi}(t, s)) = (\tilde{\bar{\phi}}(T, s)|\tilde{F}\tilde{\phi}(T, s)) + \int_t^T (\tilde{\bar{\phi}}(r, s)|[\tilde{Q}(r)\\ + \Pi(r)\tilde{R}(r)\Pi(r)]\tilde{\phi}(r, s))\, dr.\end{aligned}$$

We differentiate the above expression with respect to t and set s equal to t in the resulting expression to obtain (5.18). ▧

5.5. Integral and differential equations for operator $P(t)$. In this section we derive equations for operator $P(t)$. We shall use the operators $D(t, s)$, $P(t, \alpha)$ and $P(t)$ of Definition 4.7. Given $G \in \mathscr{L}(M^2)$ we denote by $G^0 \in \mathscr{L}(M^2, H)$ the operator defined by $G^0h = (Gh)(0)$ and by

$$\begin{bmatrix} G^{00} & G^{01} \\ G^{10} & G^{11} \end{bmatrix}$$

the corresponding matrix of operators defined on $H \times L^2(-b, 0; H)$.

PROPOSITION 5.6. *The operator P of Definition 4.7 is the unique solution of the following system of equations*:

$$\begin{aligned}(5.19)\qquad &P^0(t) = \Psi(T, t)\tilde{F}^0\tilde{\Phi}(T, t) + \int_t^T \Psi(r, t)[\tilde{Q}(r)^0 + P^{00}(r)R(r)P^0(r)]\tilde{\Phi}(r, t)\, dr,\\ &[P(t)h](\alpha) = \begin{cases} P^0(t-\alpha)\tilde{\Phi}(t-\alpha, t)h, & t - \alpha \leqq T,\\ 0, & otherwise.\end{cases}\end{aligned}$$

$\tilde{\Phi}(t, s)$ is the semigroup generated by the solutions of

$$\begin{aligned}(5.20)\qquad &\frac{dz}{dt}(t) = [\tilde{A}(t) - \tilde{R}(t)P(t)]z(t) \quad a.e.\ in\ [s, T],\\ &z(s) = h \in \mathscr{D}, \quad s \in [0, T[,\\ &\tilde{\Phi}(t, s)h = z(s),\end{aligned}$$

and $\Psi(t, s) \in \mathscr{L}(H)$ is generated by the solutions of

$$\begin{aligned}(5.21)\qquad &\frac{dy}{ds}(s) + [\tilde{A}_t^0(s) - P^0(s)\tilde{R}(s)]y_s = 0 \quad a.e.\ in\ [0, t], \quad y(t) = k^0,\\ &\Psi(t, s)k^0 = y(s),\end{aligned}$$

where $y_s(\theta)$ is equal to $y(s - \theta)$ when $s - \theta \leqq t$ and 0 when $s - \theta > t$.

Proof. (i) Notice that (5.20) and (5.21) are only functions of $P^0(t)$. Assume that $P^0(t)$ of Definition 4.7 is the unique solution of the first equation of (5.19). We have shown that

$$\psi(r, s) = D(r, s)h = D(r, r)\tilde{\phi}(r, s), \quad r \geqq s$$

(Corollary 4.4 and Lemma 4.5). The above equation can be rewritten

$$D(r, s)h = D(r, r)\Phi(r, s)h.$$

We let $r = t - \alpha$ and $s = t$ in the above equation and use Definition 4.7 to obtain the second equation of (5.19).

(ii) *Uniqueness.* Let $\bar{P}^0(t)$ be a solution of systems (5.19)–(5.21). Let $\tilde{\phi}(\cdot, s)$ be the solution in $[s, T]$ of the following equation:

$$(5.22)\qquad \begin{aligned} &\frac{\partial}{\partial t}\tilde{\phi}(t, s) = [\tilde{A}(t) - \tilde{R}(t)\bar{P}(t)]\tilde{\phi}(t, s) \quad \text{in } [s, T], \\ &\tilde{\phi}(s, s) = h. \end{aligned}$$

By definition,

$$(5.23)\qquad \tilde{\phi}(t, s) = \tilde{\Phi}(t, s)h.$$

We define

$$(5.24)\qquad \psi(t, s) = \bar{P}^0(t)\tilde{\phi}(t, s).$$

We use (5.21) to obtain

$$(5.25)\qquad \psi(t, s) = \Psi(T, t)F\phi(T, s) + \int_t^T \Psi(r, t)[\tilde{Q}(r)^0 + \bar{P}^{00}(r)R(r)\bar{P}^0(r)]\tilde{\phi}(r, s)\,dr.$$

We differentiate with respect to t both sides of (5.25):

$$(5.26)\qquad \begin{aligned} &\frac{\partial}{\partial t}\psi(t, s) + \tilde{A}_T^0(t)\tilde{\psi}(t, s) + Q(t)\phi(t, s) = 0 \quad \text{in } [s, T], \\ &\psi(T, s) = F\phi(T, s), \end{aligned}$$

where $\tilde{\psi}(t, s)(\theta)$ is equal to $\psi(t - \theta, s)$ when $t - \theta \leqq T$ and 0 when $t - \theta > T$.

We can also rewrite (5.22) using (5.24):

$$(5.27)\qquad \begin{aligned} &\frac{\partial}{\partial t}\tilde{\phi}(t, s) = \tilde{A}(t)\tilde{\phi}(t, s) - \tilde{R}(t)\tilde{\psi}(t, s) \quad \text{in } [s, T], \\ &\tilde{\phi}(s, s) = h. \end{aligned}$$

System (5.26)–(5.27) is the optimality system (4.17)–(4.18) and we know (Lemma 4.5 and Definition 4.7) that

$$(5.28)\qquad \psi(s, s) = P^0(s)h.$$

Let $t = s$ in (5.24) and (5.28):

$$(5.29)\qquad \bar{P}^0(s)h = \psi(s, s) = P^0(s)h.$$

Since this is true for all $s \in [0, T[$, we have established that a solution (if it exists) of system (5.19)–(5.21) is necessarily unique and equal to P^0.

(iii) *Existence.* Let P be as in Definition 4.7. The optimality system (4.17)–(4.18) can be put in the form (5.26)–(5.27) and (5.28) is true by Lemma 4.5 and Definition 4.7. As a result, $\tilde{\Phi}(t, s)$ and $\Psi(t, s)$ are well-defined and equation (5.26) can be rewritten as follows:

$$\frac{\partial}{\partial t}\psi(t,s) + [\tilde{A}_T^0(t) - P^0(t)\tilde{R}(t)]\tilde{\psi}(t,s) + [\tilde{Q}(t)^0 + P^0(t)\tilde{R}(t)P(t)]\tilde{\phi}(t,s) = 0 \quad \text{a.e. in } [s,T], \tag{5.30}$$

$$\psi(T,s) = F\phi(T,s). \tag{5.31}$$

Using (2.22) we can rewrite system (5.30)–(5.31) in integral form:

$$\psi(t,s) = \Psi(T,t)F\phi(T,s) + \int_t^T \Psi(r,t)(\tilde{Q}(r)^0 + P^0(r)\tilde{R}(r)P(r))\tilde{\phi}(r,s)\,dr. \tag{5.32}$$

By using the relations

$$\tilde{\phi}(t,s) = \tilde{\Phi}(t,s)h \quad \text{and} \quad P^0(t)\tilde{\phi}(t,s) = \psi(t,s), \tag{5.33}$$

(5.32) becomes

$$P^0(t)\tilde{\phi}(t,s) = \left[\Psi(T,t)F\Phi(t,s) + \int_t^T \Psi(r,t)(\tilde{Q}(r)^0 + P^0(r)\tilde{R}(r)P(r))\tilde{\Phi}(r,s)\,dr\right]h. \tag{5.34}$$

Let $t = s$ in (5.34) and use the fact that $\tilde{\phi}(s,s) = h$ to obtain (5.21). This shows that P^0 is a solution of system (5.19)–(5.21). ▧

COROLLARY 5.7. *The operators P^0 and D of Definition 4.7 are solutions of the following coupled system:*

$$D(r,s) = P^0(r)\tilde{\Phi}(r,s), \quad r \geqq s, \tag{5.35}$$

and for all h,

$$\begin{aligned} &\frac{d}{dt}[P^0(t)h] + [P^0(t)\tilde{A}(t) + A_{00}(t)^*P^0(t) \\ &\quad + \sum_{i=1}^{N} \begin{cases} A_i(t-\theta_i)^*D(t-\theta_i,t), & t-\theta_i \leqq T \\ 0, & \textit{otherwise} \end{cases} \\ &\quad + \int_{-b}^{0} \begin{cases} A_{01}(t-\theta,\theta)^*D(t-\theta,t), & t-\theta \leqq T \\ 0, & \textit{otherwise} \end{cases} d\theta - P^{00}(t)R(t)P^0(t) \\ &\quad + \tilde{Q}(t)^0]h = 0 \quad \textit{a.e. in } [0,T], \\ &P^0(T) = \tilde{F}^0. \end{aligned} \tag{5.36}$$

Proof. Consider equation (5.30). Using (5.28) and (5.20) we can compute

$$\frac{\partial \psi}{\partial t}(t,s) = \frac{\partial}{\partial u}[P^0(u)\tilde{\phi}(t,s)]_{u=t} + P^0(t)[\tilde{A}(t) - \tilde{R}(t)P(t)]\tilde{\phi}(t,s). \tag{5.37}$$

Let $s = t$ in (5.30) and (5.37). Using the definition of $\tilde{A}_T^0$ and $\tilde{\psi}(t,t)$ we obtain (5.36). ▧

Remark. $P(t)$ can be obtained from $D(r,s)$ (Definition 4.7):

$$[P(t)h](\alpha) = P(t,\alpha)h = D(t-\alpha,t)h, \quad t-\alpha \leqq T.$$

To complete the picture we need an equation for D. This equation can be formally obtained provided that the semigroup $\tilde{\Phi}(r, s)$ of system (5.20) is a solution of the following equation:

$$\frac{\partial}{\partial s}\tilde{\Phi}(r, s)^* + [\tilde{A}(s) - \tilde{R}(s)P(s)]^*\tilde{\Phi}(r, s)^* = 0 \quad \text{a.e. in } [0, r],$$

$$\tilde{\Phi}(r, r)^* = I.$$

This is equivalent to postulating the existence of a topological adjoint system for system (5.20). Under this hypothesis we formally differentiate (5.35) with respect to s to obtain the desired differential equation for D:

$$\frac{\partial}{\partial s}D(r, s) + D(r, s)[\tilde{A}(s) - \tilde{R}(s)P(s)] = 0 \quad \text{a.e. in } [0, r],$$

$$D(r, r) = P^0(r).$$

Let $(r, s) = (t - \alpha, t)$ in the above equation. Since $P(t, \alpha) = D(t - \alpha, t)$ we can obtain the following differential equation for $P(t, \alpha)$:

$$\left[\frac{\partial}{\partial t} + \frac{\partial}{\partial \alpha}\right]P(t, \alpha) + P(t, \alpha)[\tilde{A}(t) - \tilde{R}(t)P(t)] = 0$$

in the region $\{(t, \alpha) \in [0, T] \times I(-b, 0) | t - \alpha \leqq T\}$ with boundary conditions

$$P(s, 0) = P^0(s), \quad s \in [0, T].$$

For completeness we also include the following result which is obtained by decomposition of (5.17) and the first equation of (5.19).

COROLLARY 5.8. *The operator* Π *defined in Corollary* 4.9 *is the unique solution of the following system of equations*:

$$\begin{aligned}\Pi^{00}(t) = {} & \Psi(T, t)F\tilde{\Phi}^{00}(T, t) + \int_t^T \Psi(r, t)Q(r)\tilde{\Phi}^{00}(r, t)\,dr \\ & + \int_t^T \Psi(r, t)\Pi^{00}(r)R(r)[\Pi^{00}(r)\tilde{\Phi}^{00}(r, t) + \Pi^{01}(r)\tilde{\Phi}^{10}(r, t)]\,dr,\end{aligned} \tag{5.38}$$

$$\begin{aligned}\Pi^{01}(t) = {} & \Psi(T, t)F\tilde{\Phi}^{01}(T, t) + \int_t^T \Psi(r, t)Q(r)\tilde{\Phi}^{01}(r, t)\,dr \\ & + \int_t^T \Psi(r, t)\Pi^{00}(r)R(r)[\Pi^{00}(r)\tilde{\Phi}^{01}(r, t) + \Pi^{01}(r)\tilde{\Phi}^{11}(r, t)]\,dr,\end{aligned} \tag{5.39}$$

$$\Pi^{10}(t) = \Pi^{01}(t)^*, \tag{5.40}$$

$$\begin{aligned}\Pi^{11}(t) = {} & \tilde{\Phi}^{01}(T, t)^*F\tilde{\Phi}^{01}(T, t) + \int_t^T \tilde{\Phi}^{01}(r, t)^*Q(r)\tilde{\Phi}^{01}(r, t)\,dr \\ & + \int_t^T [\Pi^{00}(r)\tilde{\Phi}^{01}(r, t) + \Pi^{01}(r)\tilde{\Phi}^{11}(r, t)]^*R(r)[\Pi^{00}(r)\tilde{\Phi}^{01}(r, t) \\ & + \Pi^{01}(r)\tilde{\Phi}^{11}(r, t)]\,dr,\end{aligned} \tag{5.41}$$

where $\tilde{\Phi}(t, s)$, *defined by* (5.16), *and* $\Psi(t, s)$, *defined by* (5.21), *only depend on* Π^{00} *and* Π^{01}.

Proof. To derive (5.38) and (5.39) we decompose (5.19) and use the fact that $\Pi^0(t) = P^0(t)$, and to derive (5.41) we decompose (5.17). ▧

Remark. Equation (5.38) relates Π^{00} to Π^{00} and Π^{01}, equation (5.39) relates Π^{01} to Π^{00} and Π^{01} and equation (5.41) explicitly relates Π^{11} to Π^{00} and Π^{01}.

Acknowledgment. It is a pleasure to acknowledge interesting and enlightening discussions we have had with Professor J. L. Lions of the University of Paris.

Notes added in proof.

1. It can be shown that in Proposition 5.4 and Corollary 5.5 the map $s \mapsto \Pi(s):[0, T] \to \mathscr{L}(M^2)$ is continuous and for all $h, \bar{h}$ in $\mathscr{D}$ the map $s \mapsto (h|\Pi(s)\bar{h})$ is in $AC^1(0, T; R)$. Somewhat similar remarks can be made for the map $t \mapsto P^0(t)$ in Proposition 5.6 and Corollary 5.7.

2. The relationship between controllability, stabilizability and the infinite time quadratic cost problem has been clarified. See:

(a) M. C. Delfour and S. K. Mitter, *L^2-stability, stabilizability and the infinite time quadratic cost problem for linear autonomous hereditary differential systems*, Rep. C.R.M.-132, Centre de Recherches Mathématiques, Université de Montréal, 1971; submitted to this Journal.

(b) H. F. Vandevenne, *Qualitative properties of a class of infinite dimensional systems*, Doctoral thesis, Electrical Engineering Dept., M.I.T., Cambridge, Mass., 1972.

3. The following reference which appears to be relevant to the present work was pointed out to us by the referee:

A. Manitius, *Optimum control of linear time-lag processes with quadratic performance indices*, Proc. 4th IFAC Congress, Warsaw, Poland, 1969.

REFERENCES

[1] Y. Alekal, P. Brunovsky, D. H. Chyung and E. B. Lee, *The quadratic problem for systems with time delay*, IEEE Trans. Automatic Control, AC-16 (1971), pp. 673–687.

[2] A. F. Buckalo, *Explicit conditions for controllability of linear systems with time lag*, Ibid., AC-13 (1968), pp. 193–195.

[3] D. H. Chyung and E. B. Lee, *Linear optimal control problems with time delays*, this Journal, 4 (1966), pp. 548–575.

[4] G. Da Prato, *Equations d'évolution dans des algèbres d'opérateurs et application à des équations quasi-linéaires*, J. Math. Pures Appl., 48 (1969), pp. 59–107.

[5] M. C. Delfour, *Function spaces with a projective limit structure*, J. Math. Anal. Appl., to appear.

[6] M. C. Delfour and S. K. Mitter, *Hereditary differential systems with constant delays. I—General case*, Centre de Recherches Mathématiques, Université de Montréal, Canada, 1970; J. Differential Equations, to appear.

[7] ———, *Hereditary differential systems with constant delays. II—A class of affine systems and the adjoint problem*, Centre de Recherches Mathématiques, Université de Montréal, Canada, 1970; submitted to J. Differential Equations.

[8] ———, *Systèmes d'équations différentielles héréditaires à retards fixes. Théorèmes d'existence et d'unicité*, C. R. Acad. Sci. Paris, 272 (1971), pp. 382–385.

[9] ———, *Systèmes d'équations différentielles héréditaires à retards fixes. Une classe de systèmes affines et le problème du système adjoint*, Ibid., 272 (1971), pp. 1109–1112.

[10] ———, *Le contrôle optimal des systèmes gouvernés par des équations différentielles héréditaires affines*, Ibid., 272 (1971), pp. 1715–1718.
[11] ———, *Controllability and observability for infinite-dimensional systems*, this Journal, 10 (1972).
[12] N. DUNFORD AND J. T. SCHWARTZ, *Linear Operators. I*, Interscience, New York, 1967.
[13] D. H. ELLER, J. K. AGGARWAL AND H. T. BANKS, *Optimal control of linear time-delay systems*, IEEE Trans. Automatic Control, AC-14 (1969), pp. 678–687.
[14] H. O. FATTORINI, *Some remarks on complete controllability*, this Journal, 4 (1966), pp. 686–694.
[15] ———, *On complete controllability of linear systems*, J. Differential Equations, 3 (1967), pp. 391–402.
[16] J. K. HALE, *Functional Differential Equations*, Springer-Verlag, New York, 1971.
[17] A. HALANAY, *On a boundary-value problem for linear systems with time lag*, J. Differential Equations, 2 (1966), pp. 47–56.
[18] ———, *On the controllability of linear difference-differential systems*, Mathematical Systems Theory and Economics, II, H. W. Kuhn and G. P. Szegö, eds., Springer-Verlag, Berlin, 1970, pp. 329–335.
[19] D. R. HALEY, *R^n-Controllability for differential-delay equations*, Doctoral thesis, Mathematics Dept., University of Maryland, College Park, 1970.
[20] V. JURDJEVIC, *Abstract control systems: Controllability and observability*, this Journal, 8 (1970), pp. 424–439.
[21] R. E. KALMAN, *Mathematical description of linear dynamical systems*, this Journal, 1 (1963), pp. 152–192.
[22] F. M. KIRILLOVA AND S. V. CHURAKOVA, *The problem of the controllability of linear systems with an after-effect*, Differential Equations, 3 (1967), pp. 221–225.
[23] N. N. KRASOVSKII, *On the analytic construction of an optimal control in a system with time lags*, Prikl. Mat. Meh., 26 (1962), pp. 39–51 = J. Appl. Math. Mech., 26 (1962), pp. 50–67.
[24] ———, *Optimal processes in systems with time lags*, Proc. 2nd IFAC Conf., Basel, Switzerland, 1963.
[25] H. J. KUSHNER AND D. I. BARNEA, *On the control of a linear functional-differential equation with quadratic cost*, this Journal, 8 (1970), pp. 257–272.
[26] J. L. LIONS, *Contrôle optimal de systèmes gouvernés par des équations aux dérivées partielles*, Dunod, Paris, 1968; English transl., Springer-Verlag, Berlin, New York, 1971.
[27] D. L. LUKES AND D. L. RUSSELL, *The quadratic criterion for distributed systems*, this Journal, 7 (1969), pp. 101–121.
[28] S. K. MITTER, *Optimal control of distributed parameter systems*, Joint Automatic Control Conference (Boulder, Colo., 1969), ASME, New York, 1969, pp. 13–48.
[29] V. M. POPOV, Private communication.
[30] D. W. ROSS AND I. FLÜGGE-LOTZ, *An optimal control problem for systems with differential-difference equation dynamics*, this Journal, 7 (1969), pp. 609–623.
[31] L. M. SILVERMAN AND H. E. MEADOWS, *Controllability and observability in time-variable linear systems*, this Journal, 5 (1967), pp. 64–73.
[32] G. S. TAHIM, *Controllability of linear systems with time lag*, Proc. 3rd Allerton Conf. on Circuit and System Theory, Urbana, Ill., 1965, pp. 346–353.
[33] L. WEISS, *On the controllability of delay-differential systems*, this Journal, 5 (1967), pp. 575–587.
[34] ———, *An algebraic criterion for controllability of linear systems with time delay*, IEEE Trans. Automatic Control, AC-15 (1970), pp. 443–444.
[35] ———, *New aspects and results in the theory of controllability for various system models*, Joint Automatic Control Conference (Boulder, Colo., 1969), ASME, New York, 1969, pp. 453–456.
[36] ———, *Lectures on controllability and observability*, C.I.M.E. Seminar on Controllability and Observability, Edizioni Cremonese, Rome, 1969, pp. 205–289.
[37] W. M. WONHAM, *On a matrix Riccati equation of stochastic control*, this Journal, 6 (1968), pp. 681–697.