FILTERING THEORY AND QUANTUM FIELDS[(1)*]

by

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I. - INTRODUCTION.

In this paper we describe certain remarkable connections that exist between mathematical developments in quantum field theory (and euclidean field theory) and filtering theory (and in general, system theory). Roughly speaking, the Kalman filter is the mathematical analog of a free quantum field (in a precise sense) and the study of non-linear filtering is the analog of the study of interacting quantum fields. Due to lack of space we only sketch this theory in this paper and the details of this work will be presented elsewhere. Some of these ideas were presented in an earlier paper of this author (MITTER [1]).

II. - THE FREE QUANTUM FIELD (after I.E. SEGAL).

We shall assume that the reader is familiar with the theory of gaussian measures on infinite dimensional spaces as developed by SEGAL and GROSS (cf. SEGAL [1] and GROSS [1]).

Let H' be a given real Hilbert space and let g denote the weak probability distribution on H' known as the centred isotropic normal distribution. This may be defined as follows :

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A cylinder set in H' can be described in

$$C = \{x \in H' \mid (\langle x, y_1 \rangle, \ldots, \langle x, y_n \rangle) \in A\}$$

where $y_1, \ldots, y_n$ are orthonormal, $A \in B(R^n)$ and $\langle \cdot, \cdot \rangle$ denotes the scalar product on H'. Let $P : H' \to \text{span}(y_1, \ldots, y_n)$ denote the orthogonal projection. Then

$$C = \{x \in H' \mid Px \in D\} \text{ where } D = C \cap \text{Range}(P) .$$

For each $\sigma > 0$, let

$$\mu_\sigma(C) = (2\pi\sigma)^{-n} \int_D e^{-\frac{\|x\|^2}{2\sigma}} dx \tag{2.1}$$

where $n = \dim(\text{Range}(P))$ and dx = Lebesque measure on Range (P). (The centred isotropic normal distribution corresponds to taking $\sigma = 1$).

The measure as defined above is countably additive on the σ-ring S_K (collection of cylinder sets based on $K = \text{span}(y_1, \ldots, y_n)$), but is only finitely additive on R, the ring of all cylinder sets. Also μ defined above does not have a countably additive extension on H'.

An integration theory over (H', g) can be developed and the space $L^2(H', g)$ can be constructed using appropriate completion arguments (cf. SEGAL [2] for a survey of these ideas).

Definition : The Free Boson field over a given complex Hilbert space H may be defined as $\Phi(H) = (K, W, \Gamma, v)$ consisting of :

(1) a complex Hilbert space K

(2) a Weyl system W on H, that is, a strongly continuous mapping $z \to W(z)$: $H \to \mathcal{U}(K)$ (the space of unitary operators on K) satisfying the Weyl relations

$$W(z)\, W(z') = \exp\left(\frac{i}{2} \operatorname{Im} \langle z, z' \rangle\right) W(z+z'), \ \forall z, z' \in H \tag{2.2}$$

(3) a continuous representation $\Gamma : \mathcal{U}(H) \to \mathcal{U}(K)$

$$\Gamma(U)\, W(z)\, \Gamma(U)^{-1} = W(Uz) \quad \forall U \in \mathcal{U}(H) \quad \forall z \in H \tag{2.3}$$

(4) a unit vector $v \in K$ s.t. $\Gamma(U)v = v$ and v is a cyclic vector for W

(5) Γ is positive in the sense that if $A \in L(H)$, $A \geq 0$ and self-adjoint, then $d\Gamma(A)$ = self-adjoint generator of the one-parameter unitary group. $[\Gamma(e^{itA}) | t \in R]$ is also non-negative. ∎

Using appropriate standardizations SEGAL [3] has proved

Theorem 1. - There exists a unique Weyl system W and a representation Γ of $U(H)$ such that $(L^2(H', g), W, \Gamma, v)$ represents the free Boson field. ∎

Remark : In terms of the above notation from Stone's theorem $W(z) = e^{i\psi(z)}$ where ψ is a mapping from H to the self-adjoint operators on K. Now if H_r is any real subspace of H such that $H = H_r \oplus iH_r$ (over the real field) then the restriction map $\psi|_{H_r}$, relative to the expectation values defined by the functional $E(A) = \langle Av, v\rangle$ for any operator A in the ring of operators generated by $e^{i\psi(x)}$, $x \in H_r$ is upto an isomorphism the centred isotropic normal process over H_r. This may be seen as follows. The $\psi(x)$ for $x \in H_r$ are mutually commutative and determine a maximal abelian ring of operators and hence may be identified with the multiplication operators associated with certain real measurable functions acting on $L^2(M, m)$ for a suitable measure space (M, m). Now $E(e^{i\psi(x)}) = e^{-\frac{1}{4}\|x\|^2} = E(e^{i\Phi(x)/\sqrt{2}})$ where Φ is the weak-distribution corresponding to white noise (centred isonormal process) and E denotes expectation. Hence Ω may be chosen as the probability space of white noise such that expectation values for operators in K and for random variables on Ω are equal. This isomorphism is due to Segal.

The generator $d\Gamma(A)$ as defined previously is the "number operator" of quantum mechanics. It is best studied not on $L^2(H', g)$ but on the isomorphic Fock space $e^H = H^o \oplus H \oplus H \otimes_S H \oplus \ldots$ considered as a direct sum of tensor products of Hilbert spaces. Here $H^o = \mathbb{C}$ and H is the complexification of H'. The action of $d\Gamma(A)$ on e^H is essentially the action of $A \oplus \ldots \otimes_S A$ on the Hermite polynomials, which span e^H. If we take $A = I$ then $d\Gamma(I)$ essentially represents the dynamics of an infinite number of harmonic oscillators.

Before we close this section we shall state a theorem of Shale (SHALE [1]).

We have now constructed the free field starting with a Guaussian measure on H' with covariance $\|x\|^2$. Let $A : H' \to H'$ be a bounded positive operator with bounded inverse and let g' denote Gauss measure on H' with covariance $\frac{1}{2}\|Ax\|^2$. We can now construct the corresponding quantum field where the representation space K is $L^2(H', g')$. We can ask the question whether the two fields are unitary equivalent. This is really a question on relative absolute continuity of measures and the answer is provided completely :

Theorem 2.- A necessary and sufficient condition for the two fields to be unitary equivalent are that $A - I$ be Hilbert Schmidt. ∎

III.- KALMAN FILTERING AND THE FREE QUANTUM FIELD.

Let $(\Omega, \mathcal{A}, P)$ be the probability space on which all random variables are defined. Consider the linear filtering problem

$$(3.1) \qquad \begin{cases} y_t = \int_0^t z_s \, ds + \eta_t \\ z_s = (h, x_s) \\ x_t = \int_0^t F x_s \, ds + \int_0^t g \, dW_s \end{cases}$$

In the above, y is the scalar observation process, z is the scalar signal process, x is an R^n-valued state process, F is an $n \times n$ matrix and g is an n-vector. η and W are independent standard Wiener processes.

We first consider the linear filtering problem of recursively obtaining $\hat{x}_t \triangleq E(x_t \mid \mathcal{F}_t^y)$, where $\mathcal{F}_t^y$ is the σ-field generated by $\{y_s, 0 \leq s \leq t\}$. As is well known the innovation process

$$(3.2) \qquad \nu_t = y_t - \int_0^t (h, \hat{x}_s) \, ds$$

is standard Brownian motion. Now, from standard theory

$$\hat{x}_s = \int_0^s k_s(\tau) \, dy_\tau \quad , \text{ where } k_s(.) \in L^2(0, s ; R^n) .$$

Hence equation (3.2) can be written as

(3.3) $\nu_t = y_t - \int_o^t (h, \int_o^s k_s(\tau)\, dy_\tau) ds$, which we write in obvious operator form as

(3.4) $\nu = (I-K)y$, where we regard y as an element of $H^1(0,1;\mu_y)$ and ν as an element $H^1(0,1;\mu_\nu)$ (H^1 is the space of absolutely continuous functions with derivatives in L^2 and μ_y is the Gauss measure corresponding to the y-process and μ_ν is the Gauss measure corresponding to the ν process). The operator K is obviously Hilbert-Schmidt. The Kalman filter is essentially the action of the operator I-K on the observation process y to produce the innovation process ν . By the Segal-Feldman theorem μ_y and μ_ν are equivalent and by Theorem 2 (Shale) the quantum field's corresponding to the Gaussian measures μ_y and μ_ν are unitarily equivalent. Hence the Kalman filter is isomorphic to the free quantum field obtained from the innovations process ν .

A different view of this fact can be obtained by considering the partial differential equation describing the unnormalized conditional density. For this purpose take

$$z_s = x_s \ , \ F \equiv 0 \ \text{and} \ g \equiv 1 \ \text{in (3.1)}.$$

Then the unnormalized conditional density $\rho_t(x,\omega)$ (given $y_s, 0 \leq s \leq t$) is given by the stochastic partial differential equation

$$\text{(3.5)} \qquad d\rho_t(x,\omega) = \frac{1}{2}\frac{\partial^2}{\partial x^2}\rho_t(x,\omega)dt + x\rho_t(x,\omega)\, dy_t \ , \ t \geq 0 \ .$$

This equation may be written in Stratanovich form as the ordinary partial differential equation

$$\text{(3.6)} \qquad \frac{\partial \rho_t}{\partial t} = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} - x^2\right)\rho_t + x\dot{y}_t \rho_t \ .$$

It has been emphasized by BROCKETT [1] and the author (cf. MITTER [1]) that the oscillator algebra has an important role to play in explicitly integrating this equation. If we denote by $H = -\frac{1}{2}\frac{\partial^2}{\partial x^2} + \frac{1}{2}x^2$, $Q = x$ and $P = \frac{d}{dx}$, then we can compute (on some common dense invariant domain say $C_c^\infty(\mathbb{R})$) :

$$(3.7)\qquad \begin{cases} [H, Q] = P \\ [H, P] = -Q \\ [P, Q] = I \end{cases}$$

The abstract Lie algebra represented by (3.7) is known as the oscillator algebra. All its unitary irreducible representations can be obtained (1), for example, using the theory of AUSLANDER-KOSTANT [1]. On some common invariant domain $S \subset L^2(\mathbb{R})$ it may be represented by

$P \to -\frac{d}{dx}$

$Q \to -ix$

$I \to i$

$H \to -\frac{i}{2}(P^2 + Q^2)$, which is the Harmonic oscillator representation of quantum physics. This representation can be considered to be a unitary dilation of the Kalman filter representation. Equation (3.6) can thus be justifiably called an assembly of (euclidean) forced harmonic oscillators, one for each ω. We can show rigorously that the Lie algebra (3.7) can be integrated and this specifies the filter.

IV. OTHER APPLICATIONS.

The isomorphism exhibited between Kalman filtering and the free quantum field has other applications.

Fock space ideas have applications to certain non-linear filtering problems (cf. MARCUS-MITTER-OCONE).

Suppose $A : H' \to H'$ is a contraction. Then $\Gamma(A)$ is a contraction from $L^q(H')$ to $L^p(H')$ for $1 \leq q \leq p \leq \infty$ provided $\|A\| \leq \sqrt{\frac{q-1}{p-1}}$.

This is a theorem of Nelson (NELSON [1]). This theorem has applications to obtain estimates on the product of Ito multiple integrals and to expansion of non-linear functionals of white noise. These ideas have been exploited in obtaining expansions for filter representations (cf. MITTER-OCONE [2]).

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(1) after complexification

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