

A Theory of Modal Control*

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Although considerable progress has been made in various aspects of control theory, there still appears to be no adequate theory for the control of large-scale linear time-invariant multivariable systems. If the engineering specifications required of the controlled system can be effectively summarized in a quadratic performance measure, then linear optimal control theory, in principle, provides a linear feedback controller which would perform the required task. Even under these circumstances the computational problems may be insurmountable. In an effort to circumvent these difficulties Rosenbrock suggested the use of modal control as a design aid. Modal control may be defined as control which changes the modes (i.e., the eigenvalues of the system matrix) to achieve the desired control objectives. This paper presents a complete and rigorous theory of modal control as well as recursive algorithms which permit modal control to be realized.

LIST OF COMMON SYMBOLS

A, \bar{A}, J, Λ	System Matrix in various representations
C, \bar{C}	Actuating Matrix in various representations
F, G	Feedback Matrix in various representations
c_i	i th Column of C
λ_i	i th Eigenvalue of A
$\det[\cdot]$	Determinant of $[\cdot]$
$[\cdot]^*$	Complex conjugate of $[\cdot]$
$[\cdot]^T$	Transpose of $[\cdot]$
$\langle a, b \rangle$	$a^T b$

1. INTRODUCTION

Although considerable progress has been made in various aspects of control theory, there still appears to be no adequate theory for the con-

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trol of large-scale linear time-invariant multi-variable systems. If the engineering specifications required of the controlled system can be effectively summarised in a quadratic performance measure, then linear optimal control theory, in principle, provides a linear feedback controller which would perform the required task. Even under these circumstances the computational problem associated with the solution of the Riccati equation to determine the feedback controller may be unsurmountable. For a realistic control problem, rarely can the engineering specifications be summarised in a single quadratic performance criterion, and even though the possibility exists of changing the weighting matrices in the quadratic form to improve the system responses, no systematic way of changing these matrices are known to date for multi-input systems. For single-input systems some qualitative results in this direction were obtained by Kalman (1964).

In an effort to circumvent these difficulties and in particular to obtain some insight into the dynamic behaviour of the system, Rosenbrock (1962) suggested the use of modal control as a design aid. Modal control may be defined as control which changes the modes (that is, the eigenvalues of the system matrix) to achieve the desired control objectives. It is our belief that Rosenbrock's suggestion has not found wide application primarily because no theory of modal control was available. In this paper a complete and rigorous theory of modal control as well as recursive algorithms which permit modal control to be realized are presented. Some theoretical work in this direction has been done independently by Wonham (1967), and Anderson and Luenberger (1967).

This paper may be divided into nine sections. In Section 2 we present the mathematical description of the system.

In Section 3 some new results on state controllability which are useful in the sequel are obtained. Section 4 discusses some canonical forms for linear systems. In Section 5 we introduce the new concept of mode controllability and discuss its relationship to state controllability.

In Section 6 we present a spectral technique for modal control and in Section 7 algorithms for the realization of modal controllers are presented. Sections 8 and 9 are devoted to the discussion of modal control when the complete state vector is not accessible for measurement.

Notation. Capital letters such as A , B denote matrices. Lower case letters x , $y \cdots z$ denote vectors, t denotes time. For functions of time $x(t)$, often the argument will be omitted. \dot{x} denotes $dx/dt \cdot \langle u, v \rangle = u^T v$.

2. SYSTEM DESCRIPTION

We shall study the time-invariant linear dynamical system represented by

$$\begin{aligned} (\mathcal{L}) \quad \frac{dx}{dt}(t) &= Ax(t) + Cm(t) \\ y(t) &= Hx(t) \end{aligned}$$

where t is the time, $x(t)$ is an n -vector, the state of the system; $m(t)$ is an r -vector, the input or control; $y(t)$ is an e -vector, the output of the system; A is a constant $n \times n$ matrix, the system matrix; $C = [C_1 \ C_2 \ \dots \ C_r]$ is a constant $n \times r$ matrix, the actuating matrix; and H is a constant $e \times n$ matrix, the measurement matrix. All quantities in the above equations are real. For given t , the set of all $x(t)$ is thus a real n -dimensional Euclidean space X , called the state space of the system and the set of all $y(t)$ is a real e -dimensional Euclidean space Y , called the output space of the system.

The function $t \rightarrow m(t)$ is assumed to be defined for all $t \in (-\infty, +\infty)$, is bounded in every bounded sub-interval of $(-\infty, +\infty)$ and considered to be a certain measurable function of t .

It is well known that given a control function $m(\cdot)$, to every initial state c and initial time t_0 there corresponds a flow

$$\phi_m(t; c, t_0) = \phi(t)$$

defined for all $t \in (-\infty, +\infty)$ which satisfies the identity

$$\phi_m(t_0; c, t_0) = c.$$

In the existing theory of linear control, the concept of controllability due to Kalman (1961) plays a very fundamental role. In the theory of modal control to be developed in this paper, the new concept of modal controllability plays an equally important role. The notion of controllability due to Kalman we shall refer to as state controllability (to distinguish from modal controllability). In the following section certain known results on state controllability which are to be used later are summarised. In order to relate the concepts of state and modal controllability, we need some new results on state controllability. These are also included in the following section.

3. STATE CONTROLLABILITY

DEFINITION 3.1 (Kalman). A state $x(0)$ is said to be controllable if there exists control $m(t)$ defined over a compact interval $[0, T]$ such

that $\phi_m(T; x(0)) = 0$. If every state $x(0)$ is controllable the system is said to be completely state controllable.

PROPOSITION 3.1. (\mathcal{L}) is completely state controllable if and only if there exists no representation of (\mathcal{L}) having $\dot{x}_t = \lambda x_t$ as one of its component state equations.

PROOF. (Necessity). The proof is obvious.

(Sufficiency). We shall prove the contrapositive, that is, if (\mathcal{L}) is not completely state controllable then a representation of (\mathcal{L}) exists which has as one of its component state equations

$$\dot{x}_t = \lambda x_t .$$

From Kalman's Canonical Structure Theorem (\mathcal{L}) admits a representation

$$(\mathcal{L}_K) \quad \begin{pmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_u \end{pmatrix} = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{pmatrix} \begin{pmatrix} \bar{x}_c \\ \bar{x}_u \end{pmatrix} + \begin{pmatrix} \bar{C}_{11} \\ 0 \end{pmatrix} m$$

and X admits a direct sum decomposition $X = X_c \oplus X_u$, where X_c is the controllable and X_u the uncontrollable subspace of X . Let n_u be the dimension of X_u . Since (\mathcal{L}) is not completely state controllable, \bar{A}_{22} is an $n_u \times n_u$ matrix with $n_u \geq 1$.

Consider the nonsingular linear transformation given by $x = T\bar{x}$, where

$$T = \begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix}.$$

Here T_{11} is a nonsingular matrix of appropriate order and T_{22} is selected such that $T_{22}^{-1}\bar{A}_{22}T_{22} = J[\bar{A}_{22}]$, the Jordan canonical form of \bar{A}_{22} .

The new system representation is now given by

$$(\mathcal{L}_1) \quad \dot{x} = A'x + C'm, \quad \text{where}$$

$$A' = \begin{pmatrix} T_{11}^{-1}\bar{A}_{11}T_{11} & T_{11}^{-1}\bar{A}_{12}T_{22} \\ 0 & J[\bar{A}_{22}] \end{pmatrix} \quad \text{and} \quad C' = \begin{pmatrix} T_{11}^{-1}\bar{C}_{11} \\ 0 \end{pmatrix}.$$

By inspecting the above representation it is clear that it has at least one component state equation of the form $\dot{x}_t = \lambda x_t$.

Remark. λ is actually an eigenvalue of A .

It is convenient to write the generic control term $m(t)$ as $m(t) = n(t) + m_f(x, t)$ where $m_f(x, t)$ is an explicitly defined feedback law.

DEFINITION 3.2. Let $m(t) = n(t) + m_f(x, t)$, where the feedback law $m_f(x, t)$ has the form $m_f(x, t) = Fx(t)$, where F is an $r \times n$ constant matrix. $m_f(x, t) = Fx(t)$ is then defined to be a constant linear feedback law.

PROPOSITION 3.2. *The dimension of the controllable subspace X_c remains invariant with respect to the application of constant linear feedback control to (\mathcal{L}) .*

Proof. The proof follows easily using Kalman's Canonical Structure Theorem.

COROLLARY 3.3. *If system (\mathcal{L}) characterized by the pair (A, C) is completely state controllable and F is any $r \times n$ matrix, then the system characterized by $(A + CF, C)$ is completely state controllable.*

For the next proposition it is convenient to consider the Jordan canonical form of \mathcal{L} given by

$$(\mathcal{L}_J) \quad \dot{z} = Jz + P^T m$$

where

$$J = \begin{pmatrix} J_1 & & & & \\ & J_2 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & J_\nu \end{pmatrix}, \quad J_i = \begin{pmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \ddots & \ddots \\ & & & & \lambda_i & 1 \end{pmatrix}$$

Each block J_i is of dimension n_i ($i = 1, 2, \dots, \nu$) and $\sum_{i=1}^{\nu} n_i = n$. Usually P^T is simply partitioned by rows of the form

$$P^T = \begin{pmatrix} p_1^T \\ p_2^T \\ \vdots \\ p_n^T \end{pmatrix}$$

However to prove the next proposition it is convenient to number the rows of P^T to correspond to the blocks of J in the following manner:

$$P^T = \begin{bmatrix} p_{1,1}^T \\ \dots \\ \cdot \\ \dots \\ p_{1,n}^T \\ \dots \\ \cdot \\ \cdot \\ \dots \\ p_{v,1}^T \\ \dots \\ \cdot \\ \cdot \\ \dots \\ p_{v,n_v}^T \end{bmatrix}$$

We then have,

PROPOSITION 3.4. (\mathcal{L}_J) is completely state controllable if and only if all rows of P^T corresponding to the last row of Jordan blocks containing the same-valued eigenvalue are linearly independent.

Proof. (Necessity). It will be shown that the existence of a set of dependent rows in P^T , corresponding to the last row of Jordan blocks containing the same-valued eigenvalue, implies that the system is not completely state controllable. For notational simplicity only, assume that $\lambda_1 = \lambda_i$ ($i = 1, 2, \dots, k \leq v$) and that the dependent rows in question correspond to Jordan blocks containing λ_1 . Then there exists a nontrivial set of scalars, δ_i ($i = 1, 2, \dots, k$), such that

$$\sum_{i=1}^k \delta_i p_{i,n_i} = 0.$$

Perform a nonsingular transformation of state $x = Nz$ such that one of the new state variables, say x_l , is defined by the relation

$$x_l = \sum_{i=1}^k \delta_i z_i.$$

This new state variable satisfies the equation $\dot{x}_l = \lambda x_l$. As a consequence of Proposition 3.1 it follows that the system is not completely state controllable.

(Sufficiency). It is sufficient to show that a system which is not completely state controllable must have a set of p_{i,n_i} corresponding to the same mode which are linearly dependent. Assume that the system is not completely state controllable. From Proposition 3.1 it follows that a representation of the system exists having $\dot{x}_t = \lambda x_t$ as one of its component state equations, where λ is an eigenvalue of the system. For simplicity of notation let $\lambda = \lambda_1$, where the first k Jordan blocks are the only blocks containing the eigenvalue λ_1 . Then x_t must be a non-trivial linear combination of the states corresponding to the first k Jordan blocks, i.e.,

$$x_t = \sum_{i=1}^{n_1} \delta_{1,i} z_{1,i}(t) + \cdots + \sum_{i=1}^{n_k} \delta_{k,i} z_{k,i}.$$

The state equation for x_t is

$$\begin{aligned} \dot{x}_t = \lambda x_t + \sum_{i=1}^{n_1-1} \delta_{1,i} z_{1,i+1} + \cdots + \sum_{i=1}^{n_k-1} \delta_{k,i} z_{k,i+1} \\ + \left[\sum_{i=1}^{n_1} \delta_{1,i} p_{1,i}^T + \cdots + \sum_{i=1}^{n_k} \delta_{k,i} p_{k,i}^T \right] m. \end{aligned}$$

This implies that

$$\begin{aligned} \delta_{1,i} = 0, \quad i = 1, 2, \dots, n_1 - 1, \\ \vdots \\ \delta_{k,i} = 0, \quad i = 1, 2, \dots, n_k - 1. \end{aligned}$$

Hence

$$\sum_{i=1}^k \delta_{i,n_i} p_{i,n_i}^T = 0.$$

Some obvious corollaries of Proposition 3.4 are listed below.

COROLLARY 3.5. *If $r = 1$, that is in the single-input case, a necessary condition for the system (\mathcal{L}_r) to be completely state controllable is that no two Jordan blocks contain the same eigenvalue.*

COROLLARY 3.6. *The minimum number of inputs for complete state controllability of (\mathcal{L}_r) is equal to the largest number of Jordan blocks containing the same eigenvalue.*

Remark. The geometric multiplicity of an eigenvalue λ_i is denoted

by g_{λ_i} , where g_{λ_i} = nullity of $[A - \lambda_i I]$ = number of Jordan blocks containing the eigenvalue λ_i . Thus the minimum number of inputs required for complete state controllability is equal to $\max_i g_{\lambda_i}$.

COROLLARY 3.7. *If the eigenvalues of A are distinct then (\mathcal{L}_J) is completely state controllable if and only if each row of P^T is nonzero.*

4. CANONICAL FORMS FOR \mathcal{L}

If (\mathcal{L}) is completely state controllable and $r = 1$, then a unique nonsingular transformation of the state $x = T\hat{x}$ exists, such that (\mathcal{L}) admits a companion matrix form representation

$$(\mathcal{L}_c) \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{C}m$$

where

$$\hat{A} = T^{-1}AT = \begin{pmatrix} 0 & 1 & 0 \cdots \cdots 0 \\ 0 & 0 & 1 \cdots \cdots 0 \\ \vdots & \vdots & \vdots \ddots \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & \cdots & \cdots & -a_1 \end{pmatrix} \quad \hat{C} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

For the subsequent development of the theory of modal control an appropriate generalisation of the companion matrix form for completely state controllable multi-input systems is necessary. We use the canonical form as generalised by Anderson and Luenberger (1967). Unfortunately the canonical form is not unique if $r > 1$; nevertheless a useful general structure is obtained for each derived nonsingular transformation of state $x = T\hat{x}$. The general form is

$$(\mathcal{L}_G) \quad \dot{\hat{x}} = \hat{A}\hat{x} + \hat{C}m$$

where

$$\hat{A} = T^{-1}AT = \begin{pmatrix} \hat{A}_{\xi\xi} & \cdots & \hat{A}_{\xi 1} \\ \vdots & & \vdots \\ \hat{A}_{1\xi} & \cdots & \hat{A}_{11} \end{pmatrix}; \quad \hat{C} = T^{-1}C = [\hat{C}_1 \cdots \hat{C}_r]$$

$$\xi \leq r$$

$$\hat{A}_{ik} = \begin{cases} \text{a zero matrix if } i > k \\ \text{a companion matrix if } i = k \\ \text{a matrix of zero except possibly for the first column if } i < k \end{cases}$$

and

$$\hat{C}_i = \begin{cases} \text{no special form if } i - \xi > 0 \\ \text{a zero vector except for unity in the row} \\ \text{corresponding to the last row of } A_{ii} \text{ if } i - \xi \leq 0. \end{cases}$$

If $r = 1$ the familiar companion matrix form is obtained.

5. MODE CONTROLLABILITY

In the subsequent sections we shall often use the term eigenvalues of the system when we mean eigenvalues of the corresponding system matrix.

DEFINITION 5.1. A set of complex numbers Λ is said to be

- (1) 'real' if $\lambda_i \in \Lambda$ implies λ_i is real
- (2) 'distinct' if $\lambda_i, \lambda_k \in \Lambda$ implies $\lambda_i \neq \lambda_k$ if $i \neq k$
- (3) 'self conjugate' if all complex quantities occur in complex conjugate pairs.

DEFINITION 5.2. Let $\Lambda_{\mathcal{E}} = \{\lambda: \lambda \text{ is an eigenvalue of } A\}$. Let $\Lambda_S \subset \Lambda_{\mathcal{E}}$ be a self-conjugate set of p , $p \leq n$ complex numbers. Let $\Gamma = \{\gamma_1 \cdots \gamma_p\}$ be an arbitrary set of self-conjugate complex numbers. (\mathcal{E}) is said to be mode controllable with respect to Λ_S if there exists an $n \times n$ constant matrix F such that the eigenvalues of $(A + CF)$ are $\Gamma \cup (\Lambda_{\mathcal{E}} \sim \Lambda_S)$. Such a matrix F is termed the modal controller.

DEFINITION 5.3. If $p = n$ and (\mathcal{E}) is mode controllable with respect to $\Lambda_{\mathcal{E}}$ then (\mathcal{E}) is said to be completely mode controllable.

PROPOSITION 5.1. *If there exists a representation of (\mathcal{E}) which is not completely mode controllable then (\mathcal{E}) is not completely mode controllable.*

Proof. The proof is by contradiction.

Let $(\mathcal{E}^{(1)})$: $\dot{x} = A_1x + C_1m$ be a representation of (\mathcal{E}) which is not completely mode controllable. Assume there exists another representation $(\mathcal{E}^{(2)})$: $\dot{y} = Qy + Rm$ which is completely mode controllable. Since these are representations of the same system the state vectors x and y are related by a nonsingular linear transformation $y = Tx$. Therefore,

$$\begin{aligned} A_1 &= T^{-1}QT \\ C_1 &= T^{-1}R, \end{aligned}$$

which shows that A and Q are similar. Since $(\mathcal{E}^{(2)})$ is completely mode

controllable, there exists a modal controller $m = Fy$ such that the eigenvalues of $(Q + RK)$ are T , where T is any arbitrary set of self-conjugate complex numbers. We then have

$$\begin{aligned} \dot{x} &= (A_1 + C_1KT)x \\ &= T^{-1}(Q + RK)Tx \end{aligned}$$

which shows that $(Q + RK)$ is similar to $A_1 + C_1KT$ and hence $(\mathcal{L}^{(1)})$ must be completely mode controllable. This is a contradiction and hence the proposition is proved.

Before proceeding to the general result relating state and mode controllability the case when $r = 1$ is worth considering. Assume that (\mathcal{L}) is completely state controllable. Then (\mathcal{L}) has the companion matrix representation (\mathcal{L}_c) . Let s be a complex variable. The eigenvalues of \hat{A} are the roots of

$$s^n + a_1s^{n-1} + \dots + a_n = 0 \tag{5.1}$$

To create a system-matrix whose eigenvalues are the roots of

$$s^n + a_1^0s^{n-1} + \dots + a_n^0 = 0 \tag{5.2}$$

it is known that the required unique feedback law is given by

$$m(t) = [a_n - a_n^0, \dots, a_1 - a_1^0]T^{-1}x(t).$$

We may then state

PROPOSITION 5.2. *If $r = 1$, a system which is completely state controllable is also completely mode controllable and the required modal control law is unique.*

The main theorem of this paper may now be stated.

THEOREM 5.3. *(\mathcal{L}) is completely state controllable if and only if it is completely mode controllable.*

Proof. (Sufficiency). Assume that the system is completely mode controllable but not completely state controllable. Then from Proposition 3.1 there exists a representation of (\mathcal{L}) which contains a component state equation of the form $\dot{x}_i = \lambda x_i$, where λ is an eigenvalue of A . Since this eigenvalue is uncoupled from the control it cannot be altered. Hence from Proposition 5.1 (\mathcal{L}) is not completely mode controllable which contradicts the initial assumption.

(Necessity). The proof of this part of the theorem is constructive in the sense that a modal controller is derived assuming that (\mathcal{L}) is com-

pletely state controllable. There are two distinct steps in the proof. Feedback is applied to transform (\mathcal{L}) into a system with distinct eigenvalues and then the required additional feedback to move the distinct eigenvalues to the desired locations is determined.

Step 1. Since (\mathcal{L}) is completely state controllable (\mathcal{L}) admits the generalised companion matrix representation (\mathcal{L}_G) . The eigenvalues of \hat{A} are the roots of the equation

$$\prod_{i=1}^{\xi} \det (SI - \hat{A}_{ii}) = 0. \quad (5.3)$$

Therefore the eigenvalues of the system can be changed by altering the elements of each \hat{A}_{ii} in a manner similar to the single input case. For example, suppose that it is required to change the eigenvalues associated with

$$\hat{A}_{kk} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ & 0 & \ddots & 1 \\ -a_{k,n_k} & -a_{k,n_k-1} & \cdots & -a_{k,1} \end{pmatrix}. \quad (5.4)$$

Let the new eigenvalues derived to be associated with \hat{A}_{kk} correspond to a matrix with elements $-a_{k,i}^0$, $i = 1, 2, \dots, n_k$ in the last row. Let the k th component of control be

$$m_k(t) = \sum_{i=1}^{n_k} (a_{k,i} - a_{k,i}^0) \hat{x}_{k,i}(t), \quad (5.5)$$

where the state vector \hat{x} is partitioned to correspond to the blocks on the diagonal of \hat{A} as follows:

$$\hat{x}^T = [\hat{x}_{\xi,n_\xi}, \dots, \hat{x}_{\xi,1} : \cdots : \hat{x}_{1,n} \cdots \hat{x}_{1,1}].$$

Then the only part of \hat{A} changed is \hat{A}_{kk} and the revised \hat{A}_{kk} is given by (5.2) with $a_{k,i}$ replaced by $a_{k,i}^0$ ($i = 1, 2, \dots, n_k$). In this way each \hat{A}_{ii} of \hat{A} could be changed. This construction is not completely general. The eigenvalues associated with each \hat{A}_{ii} cannot be moved to a completely arbitrary position. To preserve the realness of the closed loop system, it is required that each \hat{A}_{ii} of odd order of the closed loop system contain at least one real eigenvalue. Since Step 1 involves transforming (\mathcal{L}) to a system with distinct eigenvalues the above restriction is satisfied.

Step 2. Let the transformed system with distinct eigenvalues be

$$(\mathcal{L}_T) \quad \dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{C}\tilde{m}.$$

Since (\mathcal{L}) is completely state controllable (\mathcal{L}_T) is also completely state controllable. Fix the ratio of the components of $m(t)$, that is, let $m(t) = g\theta(t)$ where g is an $r \times 1$ vector and $\theta(t)$ is a scalar. Then if the pseudo single input system characterized by the pair $(\tilde{A}, \tilde{C}g)$ is completely state controllable, then from Proposition 5.1 the required feedback law for $\theta(t)$ can be obtained. It remains to show that a vector g exists such that (\mathcal{L}_T) can be transformed into a completely state controllable pseudo single-input system. This result is now stated in the form of a lemma.

LEMMA 5.3. *If (\mathcal{L}_T) is completely state controllable and has distinct eigenvalues, then there exists an $r \times 1$ vector g such that*

$$(\mathcal{L}_S) \quad \dot{\tilde{x}} = \tilde{A}\tilde{x} + d\theta, \quad \text{where } d = Cg$$

is also completely state controllable.

Proof. Consider the Jordan canonical form of (\mathcal{L}_T) .

$$\dot{z} = \Lambda z + P^T g \theta$$

It follows from Corollary 3.7 that each $p_i \neq 0$.

The existence of a vector g such that $P^T g$ does not have a zero element is shown by the following construction. As an initial guess for g , let g_0 be the r -dimensional vector defined by $g_0 = (1, 1, \dots, 1)^T$. Let $\alpha_{k0} = p_k^T g_0$, $k = 1, 2, \dots, n$. Let Π be the set of all p_k such that $\alpha_{k0} = \langle p_k, g_0 \rangle = 0$. If Π is empty, then the initial g_0 is acceptable, otherwise proceed as follows. Let p_β be an element of Π . Since $p_\beta \neq 0$ it has a nonzero element, call it $p_{i\beta}$. Increment the i th element of g_0 by $\epsilon > 0$ to form $g_0^{(1)}$. Then $\alpha_{\beta 0}^{(1)} = \langle p_\beta, g_0^{(1)} \rangle = \epsilon p_{i\beta} \neq 0$. Choose the value of ϵ so that the nonzero values of α_{k0} remain nonzero. This can always be done. First try an arbitrary $\epsilon > 0$, if this fails try 2ϵ , and so on. Since the number of vectors is finite an acceptable increment must be found before $(n - 1)$ steps. Once an acceptable ϵ is selected the number of elements in Π is decreased by at least one to form $\Pi_1 \subset \Pi$. Continue the process in the same manner by selecting an element of Π_1 . Since Π_1 has a finite number of elements a g_0 is eventually found.

Finally from Corollary 3.7 it follows that (\mathcal{L}_S) is completely state controllable.

6. A SPECTRAL TECHNIQUE FOR MODAL CONTROL

In this section we discuss constructive methods for achieving modal control, that is, the specification of a feedback controller such that some

or all of the eigenvalues of A may be moved to certain desired locations. The proof of Theorem 5.2, based on the generalised companion matrix canonical form (\mathcal{L}_σ) , is constructive in nature. However determination of a modal controller based on (\mathcal{L}_σ) has certain drawbacks. These are discussed below.

(1) A necessary condition for achieving the representation (\mathcal{L}_σ) is that (\mathcal{L}) be completely state controllable. Therefore before proceeding with the determination of the modal controller (\mathcal{L}) must be decomposed to isolate its controllable part.

(2) In general, the complete canonical form must be determined even if only a small set of controllable modes are to be moved. Since no distinction is made between the modes that have to be moved and those not, it is possible for a "nearly uncontrollable" part of the system, which may be of no interest, to create computational difficulties in deriving the modal controller.

(3) A component of the control vector is only employed to alter the eigenvalues of the associated companion matrix.

The contents of the companion matrix depend on the ordering of the columns in C . If the first column of C can influence every mode, i.e., the pair $[A, C_1]$ is completely state controllable then the canonical form in effect reduces to the companion matrix form (\mathcal{L}_c) with the first component of control as $m(t)$. The method delegates the control of an eigenvalue to the first column of C that can influence it. This rules out many designs in which a combination of control elements share the effort in shifting the eigenvalues. It also may create poor designs by having eigenvalues shifted by components of control with little influence over them.

The canonical coordinate system obtained is not well suited for studying the effect of shifting the eigenvalues on the feedback gains required on the original state measurements. Even if only one eigenvalue is shifted the use of T^{-1} makes it difficult to predict the resulting gains.

To alleviate some of the difficulties associated with that method a different eigenvalue shifting algorithm is now presented. The usefulness of these techniques can be better explained by first classifying linear stationary dynamical systems in the following way.

	Controllable	Uncontrollable
Distinct Eigenvalues	I	II
Repeated Eigenvalues	III	IV

Recall that the technique of Section 5 can immediately be applied only to systems I and III. The uncontrollable part existing in systems II and IV must be discarded before the canonical transformation can be obtained.

The spectral technique presented below is best suited to systems of class I and II. By using Theorem 5.2 an insignificant amount of control can always be found which will transform a system of class III into one of class I. However, to eliminate the effect of uncontrollable repeated eigenvalues in a system of class IV it is necessary to perform a decomposition as required in the other technique.

For ease of discussion it is assumed that the system has distinct eigenvalues¹ and is described in part by

$$\dot{x} = Ax + Cm \tag{6.1}$$

where

$$\begin{aligned} x &= Uz \\ U &= [u_1 \ u_2 \ \cdots \ u_n] \text{ eigenvectors of } A \\ V^T &= U^{-1} = \begin{pmatrix} v_1^T \\ \vdots \\ v_n^T \end{pmatrix} \text{ reciprocal basis vectors of } A \\ \Lambda &= V^T A U = \text{diagonal } [\lambda_1, \lambda_2, \dots, \lambda_n] \\ P^T &= V^T C = \begin{pmatrix} p_1^T \\ \vdots \\ p_n^T \end{pmatrix} \end{aligned}$$

Note. The eigenvectors are selected such that if $\lambda_i = \lambda_k^*$ then $u_i = u_k^*$. Hence $v_i = v_k^*$ and $p_i = p_k^*$.

Consider a feedback law

$$m = \sum_{i=1}^{l \leq n} g_i z_i = \sum_{i=1}^{l \leq n} g_i \langle v_i, x \rangle, \tag{6.2}$$

which is more generally written as $m = Gz = FX$, ($F = GV^T$). The closed-loop system may then be represented as

$$\dot{z} = \bar{\Lambda}z \quad (\bar{\Lambda} = \Lambda + P^T G) \tag{6.3}$$

¹ For the case of multiple eigenvalues see Simon (1967).

² To insure that m is real it is assumed that the set $\{\lambda_1, \lambda_2, \dots, \lambda_l\}$ is self-conjugate and that the r -dimensional vectors $g_i = g_i^*$ if $\lambda_i = \lambda_k^*$.

where

$$\bar{\Lambda} = \left[\begin{array}{c|c} \bar{\Lambda}_{11} & 0 \\ \hline \bar{\Lambda}_{21} & \bar{\Lambda}_{22} \end{array} \right]$$

$$= \left[\begin{array}{cccc|cc} \lambda_1 + \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n_m} & & \\ \alpha_{21} & \lambda_2 + \alpha_{22} & \cdots & \alpha_{2n_m} & & 0 \\ \vdots & \vdots & & \vdots & & \\ \alpha_{n_m 1} & \alpha_{n_m 2} & \cdots & \lambda_{n_m} + \alpha_{n_m n_m} & & \\ \hline \alpha_{n_m+1,1} & \alpha_{n_m+1,2} & \cdots & \alpha_{n_m+1,n_m} & \lambda_{n_m+1} & \\ \vdots & \vdots & & \vdots & 0 & \cdot \cdot \cdot 0 \\ \alpha_{n 1} & \alpha_{n 2} & \cdots & \alpha_{n,n_m} & 0 & \cdot \cdot \cdot \lambda_n \end{array} \right]$$

and $\alpha_{ik} = \langle p_i, g_k \rangle = p_i^T g_k$.

To determine the eigenvalues of the closed-loop system it is helpful to explore some properties of the characteristic equation

$$\det [sI - \bar{\Lambda}] = 0 \quad (6.4)$$

In this particular case the characteristic equation factors or "decomposes" into

$$\det [sI - \bar{\Lambda}_{11}] \prod_{i=\ell+1}^n (s - \lambda_i) = 0. \quad (6.5)$$

It is readily evident that only the first ℓ eigenvalues could change. Thus it is only necessary to examine the $\ell \times \ell$ matrix $\bar{\Lambda}_{11}$ to investigate the effects of the feedback on the eigenvalue locations.

The features of what may be termed a "modal decomposition property" can easily be extrapolated from the example above. For the case in which $\ell \leq n$ canonical state variables are employed in feedback it is found that the characteristic equation of the closed-loop system factors into $(n - \ell)$ linear terms and the determinant of an ℓ th order system where

- (1) The linear terms, $(s - \lambda_i)$, contain the eigenvalues unchanged by the feedback because their corresponding canonical state variables are not fed back.
- (2) The ℓ th order determinant is the determinant of the submatrix formed by crossing out the rows and columns in $[sI - \bar{\Lambda}]$ corresponding to the canonical state variables omitted in the feedback law.

Note that if any of the canonical state variables used in the feedback law correspond to uncontrollable modes, then a further simplification is possible. The resulting characteristic equation is formed in the manner described above with these state variables treated as if they were omitted in the feedback law.

Since the ordering of the eigenvalues in the canonical form is immaterial the feedback law given by (6.2) is a valid starting point for deriving the modal control law for (the first) ℓ eigenvalues. A set of r -dimensional vectors g_i , ($i = 1, 2, \dots, \ell$), causing the matrix \bar{A}_{11} of (6.3) to have the desired distribution of eigenvalues must now be determined. If the eigenvalues are controllable (and $r > 1$) many such sets exist. A general procedure for obtaining the g_i is to first find the characteristic equation of \bar{A}_{11}

$$\det [sI - \bar{A}_{11}] = s^\ell + f_1 s^{\ell-1} + \dots + f_{\ell-1} s + f_\ell = 0 \quad (6.6)$$

where the f_i , ($i = 1, 2, \dots, \ell$), are nonlinear (linear if $r = 1$) functions of the components of the g_i , and then compare it to the characteristic equation whose roots are the desired modes

$$\begin{aligned} (s - \gamma_1)(s - \gamma_2) \dots (s - \gamma_\ell) \\ = s^\ell + d_1 s^{\ell-1} + \dots + d_{\ell-1} s + d_\ell = 0. \end{aligned} \quad (6.7)$$

The condition that equations (6.6) and (6.7) have identical roots is that coefficients of like powers of s be equal, i.e.,

$$f_i = d_i, \quad i = 1, 2, \dots, \ell. \quad (6.8)$$

The simplicity of the expression (6.8) is deceiving because for $r > 1$ it represents a set of ℓ nonlinear equations in the $\ell \cdot r$ unknown components of the g_i . To algorithmize the procedure for deriving a modal control law a restriction is placed on the control which in effect reduces the system to a scalar-input system.

Let the vectors g_i in (6.2) be replaced by $\delta_i g_0$, i.e., let

$$m = g_0 \sum_{i=1}^{\ell} \delta_i z_i = g_0 \sum_{i=1}^{\ell} \langle \delta_i v_i, x \rangle \quad (6.9)$$

where the elements of the r -dimensional real vector g_0 , ($g_{10}, g_{20}, \dots, g_{r0}$), fix the ratio of control elements, and the scalar weights δ_i ³ are determined to achieve the desired changes in the first ℓ eigenvalues. Recall that controllability of the modes in the original system (6.1)

³ To insure that m is real let $\delta_i = \delta_i^*$ if $\lambda_i = \lambda_i^*$.

does not imply controllability in the reduced system. Therefore g_0 must be chosen so that $\alpha_{i0} = \langle p_i, g_0 \rangle \neq 0$ ($i = 1, 2, \dots, \ell$).

It may be shown (see Appendix 1) that for a feedback control law in the form of (6.9) the f_i in (6.8) are given by

$$f_i = (-1)^i [P_i(\lambda) + \sum_{k=1}^i \delta_i \alpha_{i0} P_{i-k}(\lambda | \lambda_i)], \tag{6.10}$$

$i = 1, 2, \dots, \ell$

where the P functions are defined as follows:

$P_i(\lambda_1, \lambda_2, \dots, \lambda_n)$ is the sum of the products, taken i at a time, of the elements from the set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

$P_i(\lambda)$ is a shortened notation which is used when the set of λ 's under discussion is clear.

$P_i(\lambda | \lambda_k)$ denotes $P_i(\lambda)$ with $\lambda_k = 0$.

$P_i(\lambda | \lambda_k, \lambda_\ell)$ denotes $P_i(\lambda)$ with $\lambda_k = \lambda_\ell = 0$.

$P_0(\lambda) \doteq 1$.

$P_i(\lambda) \doteq 0$, when i exceeds the number of elements in the set defined by λ .

The characteristic polynomial of the desired roots is

$$s^\ell - P_1(\gamma)s^{\ell-1} + \dots + (-1)^\ell P_\ell(\gamma).$$

Equating the corresponding f_i and d_i as in (6.8) yields

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ P_1(\lambda | \lambda) & P_1(\lambda | \lambda_2) & \dots & P_1(\lambda | \lambda_\ell) \\ \vdots & \vdots & & \vdots \\ P_{\ell-1}(\lambda | \lambda_1) & P_{\ell-1}(\lambda | \lambda_2) & \dots & P_{\ell-1}(\lambda | \lambda_\ell) \end{pmatrix} \begin{pmatrix} \delta_1 & \alpha_{10} \\ \delta_2 & \alpha_{20} \\ \vdots & \vdots \\ \delta_\ell & \alpha_{\ell 0} \end{pmatrix} \tag{6.11}$$

$$= \begin{pmatrix} P_1(\gamma) - P_1(\lambda) \\ P_2(\gamma) - P_2(\lambda) \\ \vdots \\ P_\ell(\gamma) - P_\ell(\lambda) \end{pmatrix}$$

which is symbolically written as $RD = Q$.

PROPOSITION 6.1. *A unique solution exists for the δ_i , ($i = 1, 2, \dots, \ell$), if and only if the modes $\lambda_1, \lambda_2, \dots, \lambda_\ell$ are distinct and $\alpha_{i0} \neq 0$, ($i = 1, 2, \dots, \ell$).*

Proof. See Appendix 2.

Since only real systems are considered all complex quantities must occur in complex conjugate pairs. It follows that $P_i(\lambda)$ and $P_i(\gamma)$ are always real, but $P_i(\lambda | \lambda_k)$ is real if and only if λ_k is real. Thus the matrix R in (6.11) is real if and only if all of the eigenvalues to be changed are real. It turns out, however, that the matrix equation (6.11) can always be transformed to an equation in real quantities. For example, if three eigenvalues are to be shifted and the first two are complex conjugates then the equivalent of equation (6.11) can be written as

$$\begin{pmatrix} 1 & 0 & 1 \\ \text{Re } P_1(\lambda | \lambda_1) & \text{Im } P_1(\lambda | \lambda_1) & P_1(\lambda | \lambda_3) \\ \text{Re } P_2(\lambda | \lambda_1) & \text{Im } P_2(\lambda | \lambda_1) & P_2(\lambda | \lambda_3) \end{pmatrix} \begin{pmatrix} 2 \text{Re}(\delta_1 \alpha_{10}) \\ -2 \text{Im}(\delta_1 \alpha_{10}) \\ \delta_3 \alpha_{30} \end{pmatrix} \tag{6.12}$$

$$= \begin{pmatrix} P_1(\gamma) - P_1(\lambda) \\ P_2(\gamma) - P_2(\lambda) \\ P_3(\gamma) - P_3(\lambda) \end{pmatrix}$$

The P_i 's may be defined recursively as follows

$$P_i(\lambda | \lambda_k) = P_i(\lambda) - \lambda_k P_{i-1}(\lambda | \lambda_k) \quad i = 1, 2, \dots, \ell - 1,$$

$$k = 1, 2, \dots, \ell$$

$$P_0(\lambda | \lambda_k) = 1$$

As a check on the numerical accuracy of the recursive procedure the following relation may be used

$$P_i(\lambda) - \lambda_k P_{i-1}(\lambda | \lambda_k) = 0 \quad k = 1, 2, \dots, \ell.$$

Analogous results may also be obtained for recursively defining $\text{Re } P_i(\lambda | \lambda_k)$ and $\text{Im } P_i(\lambda | \lambda_k)$ in terms of real quantities, Simon (1967).

OUTLINE OF ALGORITHM

- (1) Determine the primary information λ_i and v_i ($i = 1, 2, \dots, \ell \leq n$), i.e., the eigenvalues to be moved and their corresponding reciprocal basis vectors. Recall that $A^T v_i = \lambda_i v_i$.
- (2) Select the desired locations of the shifted eigenvalues γ_i , $i = 1, 2, \dots, \ell$, and the ratio of the control elements g_0 .

(3) Calculate the values of the

$$\alpha_{i0} = \langle p_i, g_0 \rangle = v_i^T C g_0 \quad (\text{check } \alpha_{i0} \neq 0) \quad \text{and}$$

the P_i required in (6.11).

(4) Solve (6.11) for the $\delta_i, i = 1, 2, \dots, \ell$, which completes the specification of the control law as defined in (6.9).

The only information required to define the control law is related to the eigenvalues which are to be shifted. It does not matter whether or not the other eigenvalues are controllable. A major portion of the solution effort is expended in inverting the matrix D in (6.11) which is only a function of the modes to be moved. Therefore the design parameters $\gamma_i, i = 1, 2, \dots, \ell$, and g_0 can be changed relatively easily to investigate other systems.

7. RECURSIVE DESIGN

The previous section presents an eigenvalue shifting algorithm which can be employed to shift ℓ ($\ell \leq n$) eigenvalues. A class of feedback controls is introduced which essentially reduces the problem to one of inverting an $\ell \times \ell$ matrix. In order to gain more insight into the effect of moving the eigenvalues in small groups and help alleviate the computational burden a recursive technique is developed. The technique is recursive in the sense that it allows a small number of eigenvalues to be moved to their desired locations at each iteration.

Consider again a system with distinct eigenvalues. For clarity it is necessary to use superscripts to distinguish between certain quantities at each stage of the design. Let the open-loop system be denoted by

$$(\mathcal{L}^{(0)}) \quad \dot{x} = A^{(0)}x + Cm \quad (7.1)$$

with the corresponding canonical representation

$$(\mathcal{L}_c^{(0)}) \quad \dot{z} = \Lambda^{(0)}z + P^{(0)T}m \quad (7.2)$$

where

$$\Lambda^{(0)} = V^{(0)T}A^{(0)}U^{(0)} = \text{diagonal } [\lambda_1, \lambda_2, \dots, \lambda_n],$$

and

$$P^{(0)T} = V^{(0)T}C.$$

The first step in the recursive design procedure consists of finding a

linear state variable feedback law

$$m^{(1)} = G^{(1)}z = F^{(1)}x, \quad (F^{(1)} \doteq G^{(1)}V^{(0)T}),$$

which moves a selected number of eigenvalues to specified locations while keeping the others fixed. For illustrative purposes assume that $m^{(1)}$ is chosen to change λ_1 and λ_2 to γ_1 and γ_2 respectively. Incorporating this feedback law into $(\mathcal{L}^{(0)})$ by formally replacing m by $m + m^{(1)}$ in (7.1) yields a new system, described by

$$(\mathcal{L}^{(1)}) \quad \dot{x} = A^{(1)}x + Cm \quad \text{where} \quad A^{(1)} \doteq A^{(0)} + CF^{(1)}, \quad (7.3a)$$

or

$$\dot{z} = \bar{\Lambda}^{(0)}z + P^{(0)T}m \quad \text{where} \quad \bar{\Lambda}^{(0)} \doteq \Lambda^{(0)} + P^{(0)T}G. \quad (7.3b)$$

If the eigenvalues of $(\mathcal{L}^{(1)})$ are distinct, then it may be represented in canonical form by

$$(\mathcal{L}_c^{(1)}) \quad \dot{z} = \Lambda^{(1)}z + P^{(1)T}m$$

where

$$\Lambda^{(1)} = V^{(1)T}A^{(1)}U^{(1)} = \text{diagonal } [\gamma_1, \gamma_2, \gamma_3 \quad \dots, \lambda_n],$$

and

$$P^{(1)T} = V^{(1)T}C.$$

Note that $\Lambda^{(1)}$ is the canonical form of $A^{(1)}$, and is thus a diagonal matrix having the eigenvalues of $A^{(1)}$ as its entries. Compare this with $\bar{\Lambda}^{(0)}$ defined in (7.3b).

The system denoted by $(\mathcal{L}^{(1)})$ is nothing more than the closed-loop system obtained by employing the feedback law $m^{(1)}$ in system $(\mathcal{L}^{(0)})$. For the purpose of recursive design it may also be viewed effectively as a new open-loop system with its system matrix given by $A^{(1)}$. If $(\mathcal{L}^{(1)})$ is found to be satisfactory the design is complete. Otherwise an additional feedback controller

$$m^{(2)} = G^{(2)}z = F^{(2)}x, \quad (F^{(2)} \doteq G^{(2)}V^{(1)T}),$$

can be derived to change the eigenvalues of $(\mathcal{L}^{(1)})$.

The recursive procedure can be continued indefinitely with any number of eigenvalues altered at each iteration. After σ steps the system

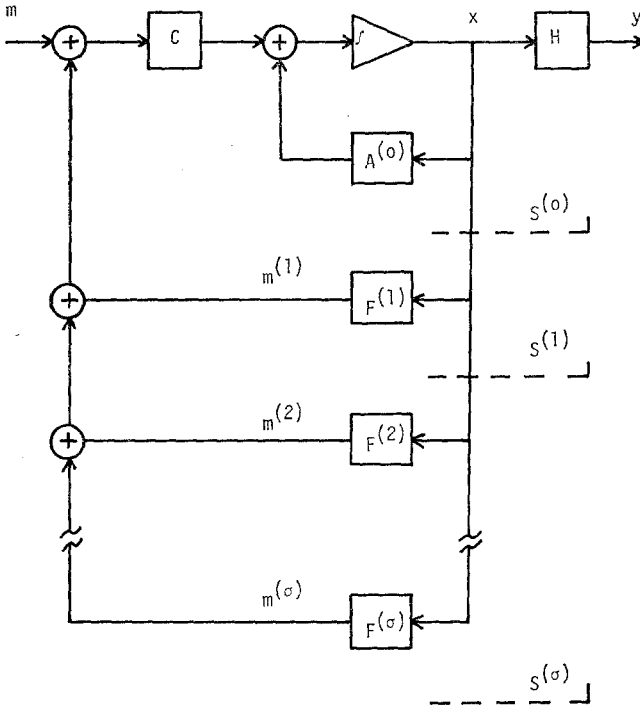


FIG. 1. Recursive design (σ stages)

is described by

$$\begin{aligned} \dot{x} &= A^{(0)}x + C(m + m^{(1)} + m^{(2)} + \dots + m^{(\sigma)}) \\ &= A^{(\sigma)}x + Cm, \quad \text{where } A^{(\sigma)} = A^{(0)} + C \sum_{i=1}^{\sigma} F^{(i)}. \end{aligned}$$

If the eigenvalues of the resulting matrix $A^{(\sigma)}$, are distinct, then its corresponding canonical representation is

$$(\mathcal{L}^{(\sigma)}) \quad \dot{z} = \Lambda^{(\sigma)}z + P^{(\sigma)T}m$$

where

$$\begin{aligned} \Lambda^{(\sigma)} &= V^{(\sigma)T}A^{(\sigma)}U^{(\sigma)} \\ P^{(\sigma)T} &= V^{(\sigma)T}C. \end{aligned}$$

A pictorial interpretation of the recursive design procedure may be found in Figures 1 through 3.

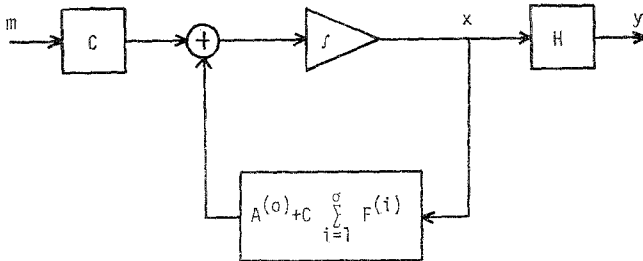


FIG. 2. Effective system (σ stages)

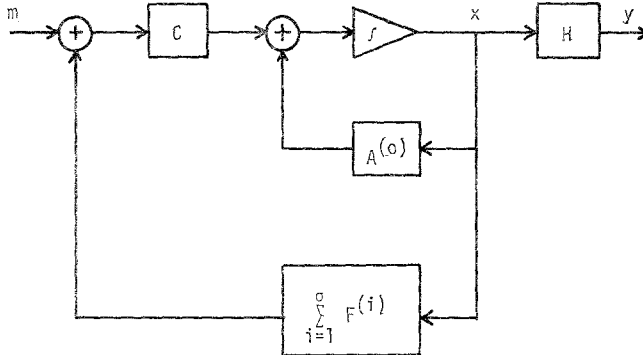


FIG. 3. Final design (σ stages)

Obviously if the recursive algorithm is to alleviate the overall computational effort each step in the design must be easier to perform than the single step required for the simultaneous movement of all of the eigenvalues. A tremendous savings in effort is afforded by the modal decomposition property when only a small number of eigenvalues is moved at one time because the amount of effort required to determine a control increases exponentially with the number of eigenvalues moved. It is necessary in the recursive algorithm to compute the $v_i^{(k)}$ (reciprocal basis vectors) corresponding to the eigenvalues moved in the k th step, but an efficient algorithm is developed to determine these quantities.

It is most convenient to work on one or two eigenvalues at a time. For a real eigenvalue the one-eigenvalue-shift may be used while the two-eigenvalue-shift allows real and complex conjugate pairs to be created, changed or destroyed. The algorithm of the one-eigenvalue-shift is derived below. Higher order eigenvalue shifts are derived in an analogous

way. A detailed derivation covering the four cases may be found in Simon (1967).

ALGORITHMS FOR SINGLE EIGENVALUE SHIFT

1. *Derivation of Control.* For notational convenience it is assumed that the first (real) eigenvalue, λ_1 , is to be changed to the (real) eigenvalue, γ_1 . Only a real to real change is treated to preserve the realness of the system at each stage in the design.

As a consequence of the modal decomposition property the form of $m^{(1)}$ is chosen as

$$m^{(1)} = g_1 z_1 = \begin{bmatrix} g_{11} \\ g_{21} \\ \vdots \\ g_{r1} \end{bmatrix} \langle v_1^{(0)}, x \rangle. \quad (7.4)$$

Substitution of (7.4) into (7.2) yields

$$\bar{A}^{(0)} = \left[\begin{array}{c|ccc} \lambda_1 + \alpha_{11}^{(0)} & & & 0 \\ \alpha_{21}^{(0)} & \lambda_2 & & 0 \\ \vdots & & \ddots & \\ \alpha_{n1}^{(0)} & 0 & & \lambda_n \end{array} \right]$$

where $\alpha_{k1}^{(0)} \doteq \langle p_k^{(0)}, g_1 \rangle$.

As expected $m^{(1)}$ only effects the first eigenvalue. The characteristic equation of the closed-loop system is

$$\det [sI - \bar{A}^{(0)}] = (s - \lambda_1 - \alpha_{11}^{(0)})(s - \lambda_2) \cdots (s - \lambda_n) = 0.$$

Thus any g_1 satisfying

$$\lambda_1 + \langle p_1^{(0)}, g_1 \rangle = \gamma_1 \quad (7.5)$$

will achieve the desired shift. Note that the result (7.5) is in perfect agreement with Corollary (3.7).

OBSERVATIONS

(i) For the case of scalar control g_1 and $p_1^{(0)} \doteq \langle v_1^{(0)}, c \rangle$ are also scalars. Therefore, $m^{(1)}$ is uniquely determined by

$$m^{(1)} = \frac{(\gamma_1 - \lambda_1)}{p_1^{(0)}} z_1 = \frac{(\gamma_1 - \lambda_1)}{\langle v_1^{(0)}, c \rangle} \langle v_1^{(0)}, x \rangle.$$

A measure of the controllability of the first eigenvalue is given by the absolute value of the normalized $p_1^{(0)}$. That is,

$$\overline{p_1^{(0)}} \doteq \left| \frac{\langle v_1^{(0)}, c \rangle}{\langle v_1^{(0)}, v_1^{(0)} \rangle^{1/2}} \right|$$

varies inversely with the absolute value of the gains required to yield a given displacement of the first mode. This agrees intuitively with the fact that when $\overline{p_1^{(0)}} = 0$ the eigenvalue λ_1 is uncontrollable.

(ii) In the multi-input case, the components of the control vector $m^{(1)}$ are proportional to the corresponding elements of the r -dimensional vector g_1 . A simple procedure for obtaining a unique solution to (7.5) is to specify a desired ratio for the components of $m^{(1)}$. Such a ratio may be based on the reliability, sensitivity, cost of the corresponding control variables, or be chosen to reduce the absolute value of the required feedback gains.

Assume that

$$g_1 \doteq \delta g_0$$

where δ is a scalar to be determined, and the vector g_0 specifies the ratio of control variables. Then

$$m^{(1)} = \delta g_0 \langle v_1^{(0)}, x \rangle$$

where $\delta = \gamma_1 - \lambda_1 / \langle p_1^{(0)}, g_0 \rangle$ is obtained from (7.5).

Recall that this procedure reduces the given system with r inputs to an effective scalar input system and that the first eigenvalue of this effective system is controllable if and only if g_0 is chosen such that $\langle p_1^{(0)}, g_0 \rangle \neq 0$. A measure of controllability for the first eigenvalue of the effective scalar input system may be defined as

$$\left| \frac{\langle p_1^{(0)}, g_0 \rangle}{\langle v_1^{(0)}, v_1^{(0)} \rangle^{1/2} \|g_0\|} \right|, \quad \text{where } \|g_0\| \doteq \max_{i=1, \dots, r} |g_{i0}|.$$

It may be shown (see Appendix 3) that the selection of the elements of g_0 by the rule

$$g_{i0} = (\text{sign } p_{i1}^{(0)}), \quad i = 1, 2, \dots, r \quad (7.6)$$

maximizes the measure of controllability. Hence the ratios defined by (7.6) require the least absolute value of feedback gains.

2. *Derivation of Updated Eigenvectors.* After determining an appropriate feedback law $m^{(1)}$, defined in (7.4), the open-loop system ($\mathcal{L}^{(0)}$)

is transformed into $(\mathcal{L}^{(1)})$ which is represented by

$$\dot{x} = A^{(1)}x + Cm.$$

The new system matrix satisfies the relations

$$\begin{aligned} A^{(1)} &= A^{(0)} + Cg_1v_1^{(0)T} \\ &= A^{(0)} + \sum_{k=1}^r c_k g_{k1} v_1^{(0)T} \end{aligned}$$

It is readily verified that

$$u_i^{(1)} = u_i^{(0)}, \quad i = 2, 3, \dots, n$$

because

$$A^{(1)}u_i^{(0)} = A^{(0)}u_i^{(0)} = \lambda_i u_i^{(0)}, \quad i = 2, 3, \dots, n.$$

The revised first eigenvector $u_1^{(1)}$ must satisfy

$$[A^{(0)} + \sum_{k=1}^r c_k g_{k1} v_1^{(0)T}]u_1^{(1)} = \gamma_1 u_1^{(1)}. \quad (7.7)$$

Since the $u_i^{(0)}$ form an n -dimensional basis $u_1^{(1)}$ can be represented by

$$u_1^{(1)} = \sum_{i=1}^n q_i u_i^{(0)} \quad (7.8)$$

where the q_i are scalars to be determined. Expanding (7.7) using (7.8) yields

$$\sum_{i=1}^n q_i \lambda_i u_i^{(0)} + q_1 \sum_{k=1}^r c_k g_{k1} = \gamma_1 \sum_{i=1}^n q_i u_i^{(0)}$$

but by definition $c_k = \sum_{i=1}^n p_{ki}^{(0)} u_i^{(0)}$, therefore

$$\sum_{i=1}^n [q_1 \alpha_{i1}^{(0)} - q_i (\gamma_1 - \lambda_i)] u_i^{(0)} = 0.$$

The fact that the $u_i^{(0)}$ are linearly independent implies that

$$q_1 \alpha_{i1}^{(0)} = q_i (\gamma_1 - \lambda_i) \quad i = 1, 2, \dots, n.$$

If a resulting set of eigenvalues, $\{\gamma_1, \lambda_2, \dots, \lambda_n\}$, remains distinct then $q_1 \neq 0$, and may be set equal to unity. Thus

$$q_i = \frac{\alpha_{i1}^{(0)}}{\gamma_1 - \lambda_i} \quad i = 2, 3, \dots, n \quad \text{with} \quad q_1 \doteq 1. \quad (7.9)$$

Note that if γ_1 is set equal to another eigenvalue it is possible for $u_1^{(1)}$ not to exist. This agrees with the well known fact that if an n th order matrix has repeated eigenvalues then it may have less than n linearly independent eigenvectors. ...

3. *Derivation of Updated Reciprocal Basis.* The updated set of eigenvectors can be calculated as described above if desired. By definition $V^{(1)\tau} = U^{(1)-1}$, however, it is more convenient to compute the reciprocal basis vectors using the relations $\langle u_i^{(1)}, v_k^{(1)} \rangle = \delta_{ik}$. Making use of the fact that $\langle u_i^{(0)}, v_k^{(0)} \rangle = \delta_{ik}$ it is found that

$$\begin{aligned} v_1^{(1)} &= v_1^{(0)} \\ v_k^{(1)} &= v_k^{(0)} - q_k v_1^{(0)}, \quad k = 2, 3, \dots, n. \end{aligned}$$

Note that it is not necessary to determine the eigenvectors. The q -coefficients are determined directly from Equation (7.9). As in the technique considered in the previous section the only primary information required is the location of the eigenvalues to be moved and their corresponding reciprocal basis vectors.

8. MINIMUM NUMBER OF STATE MEASUREMENTS REQUIRED FOR MODAL CONTROL

If the system (\mathcal{L}) is completely controllable⁴ we have shown that it is possible to synthesize the modal controller. Thus the minimum state measurement problem reduces to the algebraic problem of finding a feedback matrix F having the greatest number of zero columns such that the closed-loop system matrix $[A + CF]$ has the desired distribution of modes.

The characteristic equation of the closed-loop system matrix is

$$\det [sI - (A + CF)] = s^n + d_1 s^{n-1} + \dots + d_n = 0,$$

where the d_i coefficients are multinomials of the elements of F . To achieve the desired specification of the d_i and hence the eigenvalues it is necessary to have at least n independent parameters available in F . Therefore a lower bound on the number of state measurements required for modal control is equal to the smallest integer greater than or equal to n/r .⁵

⁴ It is no longer necessary to distinguish between completely state controllable and completely mode controllable systems.

⁵ A more realistic lower bound is obtained by replacing the number of inputs r with the rank of C when they are not equal.

The direct "brute force" approach for determining the elements of F proves to be very difficult. A set of nonlinear equations must be solved which increases in complexity with the number of nonzero columns of F . In the sequel another approach is taken for systems with distinct eigenvalues which involves only linear operations. The price paid for this simplicity is that only up to $r - 1$ state measurements can be eliminated.

Consider a system with distinct eigenvalues and let the control law be of the form

$$m = g_0 \sum_{i=1}^{\ell \leq r} \delta_i z_i = g_0 \langle v, x \rangle \quad (8.1)$$

where

$$v = \sum_{i=1}^{\ell \leq r} \delta_i v_i.$$

Obviously for each zero component of the n -dimensional vector v the corresponding component of the state does not have to be measured. Instead of specifying the r -dimensional vector g_0 initially as in (6.9) it now is made a function of the δ_i . The ℓ ($\ell \leq r$) scalar weights δ_i are chosen to make m real and independent of a set of states. The solution for the unknown parameters can be described by the following two step procedure.

Step 1. Determination of the δ_i , $i = 1, 2, \dots, \ell \leq r$.

Since the eigenvalues are assumed to be distinct the vectors v_i , $i = 1, 2, \dots, \ell$, are linearly independent. To insure that m is real let $\delta_i = \delta_k^*$ if $\lambda_i = \lambda_k^*$. A set of δ_i may thus always be found to create a real control law m independent of $\ell - 1$ components of state.

Step 2. Determination of the g_{k0} , $k = 1, 2, \dots, r$.

When control law (8.1) is substituted into (6.1) it is found that the elements of g_0 must satisfy (6.11). The vector D defined in (6.11) may now be expressed as

$$D = Mg_0 \quad (8.2)$$

where

$$M = \begin{pmatrix} \delta_1 p_{11} & \cdots & \delta_1 p_{1r} \\ \delta_2 p_{21} & \cdots & \delta_2 p_{2r} \\ \vdots & & \\ \delta_\ell p_{\ell 1} & \cdots & \delta_\ell p_{\ell r} \end{pmatrix}$$

and the p 's are as defined in (6.10).

PROPOSITION 8.1. *If the eigenvalues are distinct and the rank of M is ℓ , then a vector g_0 can be found such that the modal control law (8.1) is independent of $\ell - 1$ components of state.*

Proof. In the proof of Proposition 6.1, it was shown that the distinctness of the eigenvalues implies that the matrix R defined in (6.11) has an inverse. The fact that M is of full rank means that by an appropriate interchange of the elements of M and g_0 the right hand side of (8.2) can be written as

$$Mg_0 = (M_1 : M_2) \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$$

where the $\ell \times \ell$ submatrix M_1 is nonsingular. Then the components of g_0 which satisfy the required conditions are found from the relation

$$g_0 = M_1^{-1}[R^{-1}Q - M_2g_2]$$

where the components of g_0 in g_2 can be arbitrarily specified.

If more than ℓ ($\ell \leq r$) eigenvalues are to be moved a recursive design procedure must be employed. After the first stage of the design is completed a feedback control law is obtained which moves ℓ eigenvalues to desired locations while the others remain unchanged. The form of this feedback control is given by

$$m^{(1)} = F^{(1)}x$$

where $F^{(1)}$ has up to $(\ell - 1)$ zero columns corresponding to components of state that do not have to be measured. If the final design takes σ stages then

$$m_f = \sum_{i=1}^{\sigma} F^{(i)}x.$$

In this case it is only necessary that

$$F = \sum_{i=1}^{\sigma} F^{(i)}$$

contains a zero column corresponding to each component of state that is not to be measured.

9. MODAL CONTROL USING AN OBSERVER

In the previous development we have generally assumed that all the components of the state vector are available for measurement. One of

the few design procedures that explicitly accounts for the fact that some components of the state vector may not be accessible is presented in the previous section. In some applications however, it will be necessary to obtain estimates of the inaccessible components of the state vector to realise the modal controller. Luenberger (1964) showed how the available inputs and outputs may be used to obtain an asymptotic estimate of the state vector. The device which reconstructs the state vector is called an 'observer' or an 'asymptotic state estimator'. The observer itself is a time-invariant linear system, with arbitrary eigenvalues, driven by the inputs and outputs of the system it observes.

Following Luenberger, we shall show that the composite system, that is, the controlled system and the observer, are essentially two time-invariant linear differential systems in cascade. The shifting of the eigenvalues of the composite system can thus be carried out independently. We shall also demonstrate that the problem of constructing an observer with arbitrary dynamics is the dual of the modal control problem. The combination of these two facts allows us to employ the techniques for realising a modal controller to design the overall system.

In order to illustrate the design method consider the linear system (\mathcal{L}) and assume that the feedback control law is given by $m = Fx$. If the complete state vector is accessible, the closed-loop plant is described $\dot{x} = (A + CF)x$, so that the eigenvalues of the closed loop system are the eigenvalues of $A + CF$. Assume now that the complete state vector is not accessible. Let the observer be represented by

$$(0) \quad \dot{\hat{x}} = B\hat{x} + Dy + Cm$$

where B is an $n \times n$ matrix and D is an $n \times e$ matrix.

Since the state x is to be replaced by its estimate \hat{x} in the feedback law $m = Fx$, (\mathcal{L}) and (0) have the representations

$$(\mathcal{L}) \quad \dot{x} = Ax + CF\hat{x} \quad (9.1)$$

$$(0) \quad \dot{\hat{x}} = (B + CF)\hat{x} + DHx. \quad (9.2)$$

Subtracting (9.1) from (9.2), we obtain

$$\dot{\hat{x}} - \dot{x} = B\hat{x} - (A - DH)x. \quad (9.3)$$

Let us now choose $B = A - DH$. Therefore (9.3) becomes

$$\dot{\hat{x}} - \dot{x} = B(\hat{x} - x)$$

which has a solution

$$\hat{x}(t) = x(t) + e^{Bt}(\hat{x}(0) - x(0)). \quad (9.4)$$

If B is now designed to be a stability matrix, the estimate $\hat{x}(t)$ approaches $x(t)$ as $t \rightarrow \infty$.

To display the dynamics of the overall system explicitly, consider the nonsingular transformation of the state vector defined by

$$\begin{pmatrix} x \\ e \end{pmatrix} = \begin{pmatrix} I & 0 \\ -I & I \end{pmatrix} \begin{pmatrix} x \\ \hat{x} \end{pmatrix}, \quad \text{where } I \text{ is an } n \times n$$

identity matrix. The new representation of the composite system is

$$\begin{pmatrix} \dot{e} \\ \dot{\hat{x}} \end{pmatrix} = \begin{pmatrix} B & 0 \\ CF & A + CF \end{pmatrix} \begin{pmatrix} e \\ \hat{x} \end{pmatrix}.$$

The above representation clearly shows that the eigenvalues of the overall system are composed of the eigenvalues of the observer B and the eigenvalues of the closed-loop system matrix $A + CF$. Moreover the eigenvalues of the closed-loop system matrix are identical to those obtained when the complete state vector is accessible for use in the feedback law. Thus the only effect of the observer is the addition of more eigenvalues to the original system.

The eigenvalues of the observer matrix are essentially arbitrary. In addition to effecting the dynamics of the overall system these eigenvalues determine the accuracy of the estimator as shown in (9.4). Since A and H are given as part of the plant the specification of D uniquely determines the observer through the relation $B = A - DH$. Therefore it is important to select a matrix D which gives desirable properties to B . Presumably D would be chosen to insure the stability of the system, and make transients die out quickly.

The procedure for determining a matrix D to create arbitrary dynamics in B has already been treated. In some sense this is the dual of the control problem. Consider the system

$$\dot{e} = A^T e - H^T m. \quad (9.5)$$

If the pair (A, H) of the given plant is completely observable, then the system (9.5) is completely controllable. Assuming this to be true a feedback law $m = D^T e$ may be determined to create arbitrary dynamics in the closed-loop system

$$\dot{e} = (A^T - H^T D^T) e \doteq B^T e.$$

The system matrix of this system is the transpose of the system matrix of the required observer, and thus has the same matrix. Therefore the

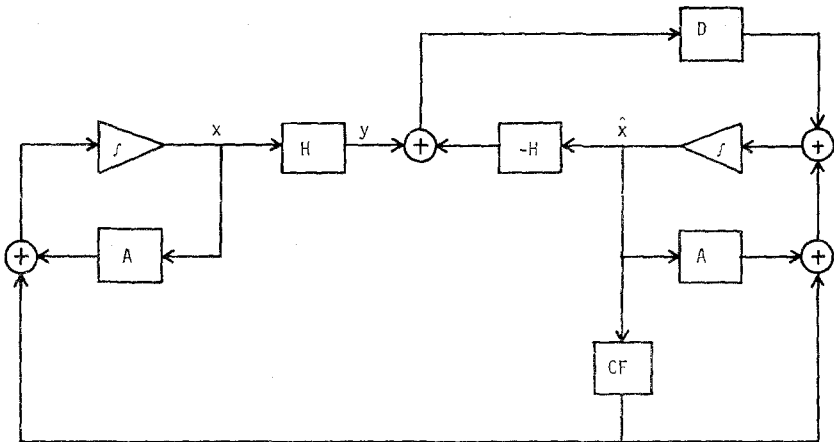


FIG. 4. Controlled plant and observer

required matrix D has been found. An illustration of the overall system is given in Figure 4.

Remark. As shown above the eigenvalues of the overall system consist of those of B (the open loop observer) and $A + CF$ (the controlled plant). The procedure of constructing the observer described in this section always enables these eigenvalues to be placed where desired. However, in certain cases, the design may not be physically satisfactory because the isolated closed loop observer (9.2) is not acceptable, e.g., the eigenvalues of $B + CF$ may be positive. In such a case the system design will have to be altered. It may be possible to remedy this by exploiting the freedom available in designing the observer and controller.

10. CONCLUSIONS

In this paper we have attempted to present a complete theory of modal control. Questions of existence and uniqueness of modal controllers as well as recursive algorithms for the realization of modal controllers have been investigated. For this purpose the new concept of modal controllability has been introduced. The existence question has also been discussed by Wonham (1967) using somewhat different methods.

In general, to achieve complete modal controllability it is necessary to employ state variable feedback and hence knowledge of the complete state vector is necessary.

In Section 8 we have presented some results pertaining to the minimum number of state vector component measurements necessary. Section 9 deals with the design of an observer or asymptotic estimator for the inaccessible components of the state vector. The design of the asymptotic estimator is in a certain sense the dual of the modal controller design problem.

Our result on the existence of a modal controller utilises a constructive method of proof. However the method used is not particularly suitable for use as a design technique. One of the main contributions of this paper has been the development of recursive algorithms for the realization of modal controllers. The algorithms are simple, require no complicated operations such as matrix inversion and may be easily programmed on a digital computer. Moreover to implement the algorithms complete knowledge of the eigenvalues and eigenvectors of the system matrix is not necessary if it is only desired to move a few eigenvalues to certain desired locations. No systematic development of algorithms for the design of modal controllers has previously been reported in the literature.

Assuming that a direct access computer is available to the designer, the techniques could perhaps best be utilised by developing a conversational mode program to perform the necessary calculations. In this way several designs could quickly be generated and a compromise design based on various engineering considerations arrived at.

APPENDIX 1

Denote the characteristic equation of the matrix

$$\begin{bmatrix} \lambda_1 + \delta_1 \alpha_1 & \delta_2 \alpha_1 & \cdots & \delta_n \alpha_1 \\ \delta_1 \alpha_2 & \lambda_2 + \delta_2 \alpha_2 & & \delta_n \alpha_2 \\ \vdots & \vdots & & \vdots \\ \delta_1 \alpha_n & \delta_2 \alpha_n & & \lambda_n + \delta_n \alpha_n \end{bmatrix} \quad (1.1)$$

by

$$s^n + f_1 s^{n-1} + f_2 s^{n-2} + \cdots + f_n = 0. \quad (1.2)$$

Note that the matrix (1.1) corresponds to the matrix used to define the f_i coefficients in (6.10) except for a slight simplification in notation.

In the derivation of (1.2) it is convenient to assume that $\alpha_n \neq 0$. The final result is independent of this assumption. Recall that a necessary condition for the mode λ_n not to be an eigenvalue of (1.1) is that $\delta_n \alpha_n \neq 0$.

Define

$$D_n = \det \begin{bmatrix} s - \lambda_1 - \delta_1 \alpha_1 & -\delta_2 \alpha_1 & \cdots & -\delta_n \alpha_1 \\ -\delta_1 \alpha_2 & s - \lambda_2 - \delta_2 \alpha_2 & \cdots & \delta_n \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_1 \alpha_n & -\delta_2 \alpha_n & \cdots & s - \lambda_n - \delta_n \alpha_n \end{bmatrix} \quad (1.3)$$

D_n may be written as

$$D_n = \det \begin{bmatrix} s - \lambda_1 - \delta_1 \alpha_1 & -\delta_2 \alpha_1 & \cdots & -\delta_n \alpha_1 \\ -\delta_1 \alpha_2 & s - \lambda_2 - \delta_2 \alpha_2 & \cdots & -\delta_n \alpha_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\delta_1 \alpha_n & -\delta_2 \alpha_n & \cdots & -\delta_n \alpha_n \end{bmatrix} + \det \begin{bmatrix} s - \lambda_1 - \delta_1 \alpha_1 & -\delta_2 \alpha_1 & \cdots & -\delta_{n-1} \alpha_1 & -\delta_n \alpha_1 \\ -\delta_1 \alpha_2 & s - \lambda_2 - \delta_2 \alpha_2 & \cdots & -\delta_{n-1} \alpha_2 & -\delta_n \alpha_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\delta_1 \alpha_{n-1} & -\delta_2 \alpha_{n-1} & \cdots & s - \lambda_{n-1} - \delta_{n-1} \alpha_{n-1} & -\delta_n \alpha_{n-1} \\ 0 & 0 & \cdots & 0 & s - \lambda_n \end{bmatrix} \quad (1.4)$$

The first determinant on the right hand side of (1.4) can be transformed into

$$\begin{bmatrix} s - \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & s - \lambda_2 & & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & & s - \lambda_{n-1} & \\ -\delta_1 \alpha_n & -\delta_2 \alpha_n & & -\delta_{n-1} \alpha_n & -\delta_n \alpha_n \end{bmatrix} \quad (1.5)$$

by multiplying the n th row by α_i/α_n , and subtracting it from the i th row for $i = 1, 2, \dots, n - 1$. Inspection of (1.4) and (1.5) then yields the recursive relation

$$D_n = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_{n-1})(-\delta_n \alpha_n) + (s - \lambda_n)D_{n-1}. \quad (1.6)$$

The expression for D_{n-1} in terms of D_{n-2} can be obtained by replacing n with $n - 1$ in (1.6). Substitution of this expression back into (1.6) yields

$$D_n = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_{n-1})(-\delta_n \alpha_n) + (s - \lambda_1) \cdots (s - \lambda_{n-2})(-\delta_{n-1} \alpha_{n-1})(s - \lambda_n) + (s - \lambda_n)(s - \lambda_{n-1})D_{n-2}. \quad (1.7)$$

Continued recursion on (1.6) with the terminal condition

$$D_1 = s - \lambda_1 - \delta_1 \alpha_1 \tag{1.8}$$

finally yields

$$\begin{aligned} D_n &= (s - \lambda_1) \cdots (s - \lambda_{n-1})(-\delta_n \alpha_n) \\ &\quad + (s - \lambda_1) \cdots (s - \lambda_{n-2})(-\delta_{n-1} \alpha_{n-1})(s - \lambda_n) \\ &\quad + \cdots + (-\delta_1 \alpha_1)(s - \lambda_2) \cdots (s - \lambda_n) \\ &\quad + (s - \lambda_1) \cdots (s - \lambda_n). \end{aligned} \tag{1.9}$$

Expanding (1.9) and collecting like terms gives the desired polynomial

$$s^n + f_1 s^{n-1} + f_2 s^{n-2} + \cdots + f_{n-1} s + f_n \tag{1.10}$$

where

$$f_i = (-1)^i [P_i(\lambda) + \sum_{k=1}^n \delta_k \alpha_k P_{k-1}(\lambda | \lambda_k)], \quad i = 1, 2, \dots, n.$$

The P functions are defined by

$P_i(\lambda_1, \lambda_2, \dots, \lambda_n)$ is the sum of the products, taken i at a time, of the elements from the set $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$. (1.11a)

$P_i(\lambda)$ is a shortened notation which is used when the set of λ 's under discussion is clear. (1.11b)

$P_i(\lambda | \lambda_k)$ denotes $P_i(\lambda)$ with $\lambda_k = 0$. (1.11c)

$P_i(\lambda | \lambda_k, \lambda_l)$ denotes $P_i(\lambda)$ with $\lambda_k = \lambda_l = 0$. (1.11d)

$P_0(\lambda) \doteq 1$. (1.11e)

$P_i(\lambda) \doteq 0$, when i exceeds the number of elements in the set defined by λ . (1.11f)

APPENDIX 2

It is sufficient to prove that the matrix

$$R = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ P_1(\lambda | \lambda_1) & P_1(\lambda | \lambda_2) & \cdots & P_1(\lambda | \lambda_n) \\ \vdots & \vdots & & \vdots \\ P_{n-1}(\lambda | \lambda_1) & P_{n-1}(\lambda | \lambda_2) & \cdots & P_{n-1}(\lambda | \lambda_n) \end{bmatrix} \tag{2.1}$$

has an inverse if and only if the modes $\lambda_1, \lambda_2, \dots, \lambda_n$ are distinct.

The proof is accomplished by performing a set of rank preserving operations on R which take it to a matrix (Vandermonde) that is known to be nonsingular if and only if the modes are distinct. The transformation takes place in $(n - 1)$ steps. At each step a row of R , starting with the second, is made to match its counterpart in the Vandermonde matrix.

Step 1. Multiply the first row of R by $P_k(\lambda)$ and subtract it from the $(k + 1)$ th row for $k = 1, 2, \dots, n - 1$. After utilizing the recursion relation

$$P_i(\lambda | \lambda_k) = P_i(\lambda) - \lambda_k P_{i-1}(\lambda | \lambda_k) \quad i = 1, 2, \dots, n - 1, k = 1, 2, \dots, n$$

$$P_0(\lambda | \lambda_k) = 1$$

and changing the sign of rows 2 through n the resulting matrix is

$$\begin{vmatrix} 1 & \dots & & & 1 \\ \lambda_1 & \dots & & & \lambda_n \\ \lambda_1 P_1(\lambda | \lambda_1) & & \dots & & \lambda_n P_1(\lambda | \lambda_n) \\ \vdots & & & & \vdots \\ \lambda_1 P_{n-2}(\lambda | \lambda_1) & & \dots & & \lambda_n P_{n-2}(\lambda | \lambda_n) \end{vmatrix} \quad (2.2)$$

Note that except for the first row each element of the matrix (2.2) is the product of the eigenvalue corresponding to that column and the element that was previously in the row above it.

Step 2. Multiply the second row of (2.2) by $P_k(\lambda)$ and subtract it from the $(k + 2)$ th row for $k = 1, 2, \dots, n - 2$. After utilizing the recursion relation and changing the sign of rows 3 through n the resulting matrix is

$$\begin{vmatrix} 1 & \dots & & & 1 \\ \lambda_1 & \dots & & & \lambda_n \\ \lambda_1^2 & \dots & & & \lambda_n^2 \\ \lambda_1^2 P_1(\lambda | \lambda_1) & & \dots & & \lambda_n^2 P_1(\lambda | \lambda_n) \\ \vdots & & & & \vdots \\ \lambda_1^2 P_{n-3}(\lambda | \lambda_1) & & \dots & & \lambda_n^2 P_n(\lambda | \lambda_n) \end{vmatrix} \quad (2.3)$$

Continuing in this manner after the i th step the resulting matrix is

given by

$$(i + 1) \text{ row} \rightarrow \begin{bmatrix} 1 & \cdots & & 1 \\ \lambda_1 & \cdots & & \lambda_n \\ \lambda_1^2 & \cdots & & \lambda_n^2 \\ \vdots & & & \vdots \\ \lambda_1^i & \cdots & & \lambda_n^i \\ \lambda_1^i P_1(\lambda | \lambda_1) & \cdots & \lambda_n^i P_1(\lambda | \lambda_n) \\ \vdots & & & \vdots \\ \lambda_1^i P_{n-(i+1)}(\lambda | \lambda_1) & \cdots & \lambda_n^i P_{n-(i+1)}(\lambda | \lambda_n) \end{bmatrix} \quad (2.4)$$

Step $(i + 1)$. Multiply the $(i + 1)$ th row of (2.4) by $P_k(\lambda)$ and subtract it from the $(k + i + 1)$ th row of (2.4) for $k = 1, 2, \dots, (n - i - 1)$. After utilizing the recursion relation and changing the sign of rows $(i + 2)$ through n the resulting matrix is

$$(i + 2) \text{ row} \rightarrow \begin{bmatrix} 1 & \cdots & & 1 \\ \lambda_1 & \cdots & & \lambda_n \\ \lambda_1^2 & \cdots & & \lambda_n^2 \\ \vdots & & & \vdots \\ \lambda_1^{i+1} & \cdots & & \lambda_n^{i+1} \\ \lambda_1^{i+1} P_1(\lambda | \lambda_1) & \cdots & \lambda_n^{i+1} P_1(\lambda | \lambda_n) \\ \vdots & & & \vdots \\ \lambda_1^{i+1} P_{n-(i+2)}(\lambda | \lambda_1) & \cdots & \lambda_n^{i+1} P_{n-(i+2)}(\lambda | \lambda_n) \end{bmatrix} \quad (2.5)$$

Finally after $(n - 1)$ steps the resulting matrix is given by the Vandermonde matrix

$$v_a = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2 \\ \vdots & \vdots & & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix} \quad (2.6)$$

The determinant of the Vandermonde matrix v_a is

$$\det[v_a] = \prod_{i>k} (\lambda_i - \lambda_k) \quad (2.7)$$

so that v_a is nonsingular if and only if the λ_i are all distinct.

As a result of properties of determinants it follows that $\det[R] \neq 0$. Actually it can be shown that

$$\det[R] = (-1)^{n(n-1)/2} \det[v_a] \quad (2.8)$$

APPENDIX 3

A typical element of the control vector $m^{(1)} = \delta g_0 \langle v_i^{(0)}, x \rangle$ where

$$\delta = \frac{\gamma_1 - \lambda_1}{\langle p_1^{(0)}, g_0 \rangle}$$

is

$$m_i^{(1)} = \frac{g_{i0}}{\langle p_1^{(0)}, g_0 \rangle} [(\gamma_1 - \lambda_1) \langle v_1^{(0)}, x \rangle], \quad i = 1, 2, \dots, r. \quad (3.1)$$

Note that $\langle p_1^{(0)}, g_0 \rangle = \sum_{i=1}^r g_{i0} \langle v_i^{(0)}, c \rangle$. Therefore $v_1^{(0)}$ can be multiplied by any arbitrary scalar without changing the feedback control law. Let $v_1^{(0)}$ be normalized so that $\langle v_1^{(0)}, v_1^{(0)} \rangle = 1$. Furthermore assume that $\|g_0\| = |g_{10}|$, i.e., the first element of g_0 has the largest absolute value. The measure of controllability is then given by

$$\left| \frac{\langle p_1^{(0)}, g_0 \rangle}{g_{10}} \right|. \quad (3.2)$$

It can be seen that this quantity is the absolute value of the inverse of the gain multiplying the fixed part of the first component of the control vector.

Obviously to maximize the measure of controllability, and hence minimize the absolute value of the required feedback gain it must be true that

$$g_{k0} = |\alpha_k| (\text{sign } p_{k1}^{(0)}), \quad k = 1, 2, \dots, r. \quad (3.3)$$

It is to be shown that $\alpha_k = 1$, $k = 1, 2, \dots, r$. With $g_{k0} = \text{sign } p_{k1}^{(0)}$, $k = 1, 2, \dots, r$, the expression in (3.2) reduces to

$$\sum_{k=1}^r |p_{k1}^{(0)}|. \quad (3.4)$$

Since $|\alpha_1| \geq |\alpha_k|$, by assumption, it follows readily that

$$\sum_{k=1}^r |p_{k1}^{(0)}| \geq \frac{1}{|\alpha_1|} \sum_{k=1}^r |\alpha_k p_{k1}^{(0)}|. \quad (3.5)$$

Hence the rule for obtaining the ratios given in (7.6) does indeed maximize the measure of controllability of the first mode.

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