were formed, it would be lower triangular because of the zeros on the diagonal of D.

Remark: For arbitrary $l(\leq n-1)$ and $r(\leq n)$ (2) reduces to

$$\dot{x} = Ax + A_1 x(t-h) + B_0 u + B_1 u(t-h) + \cdots + B_\nu u(t-\nu h) \quad (3)$$

$$y = cx + du$$

where $\nu = l + 1 + r - n$ under the convention that negative ν is taken as zero. Note that if $l + 1 + r \leq n$ $\hat{B}$ is the zero matrix eliminating delays in the control.

It can be shown that for some cases it is possible by a change in state variables to eliminate delays in u (i.e., make $\hat{B} = 0$). The interesting fact is that the new state variables are not related to the normal form state variables by a nonsingular matrix of time constants, as in the ordinary case, but by a matrix composed of constants and operators such as D_h. The question of states being isomorphic under certain types of transformations then becomes quite interesting. (Some investigation in this area is currently in progress.)

Again, the important point of this correspondence is to show that delays in the control are inherent in many state representations and should be considered in stating meaningful control problems.

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Synthesis of Transfer Function Matrices with Invariant Zeros

***Abstract*—A class of multi-input multi-output systems is considered in which the zeros of the transfer function matrix remain invariant with respect to a form of state variable feedback.**

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Consider the linear control process

$$\frac{dx}{dt} = Ax(t) + Bu(t)$$
$$y(t) = Hx(t) \qquad (1)$$

where the state of the system $x(t) \in X = R^n$, and the output of the system $y(t) \in Y = R^p$. The control functions $t \to u(t)$ are assumed to be Lebesgue measurable functions which are bounded in every compact interval in R such that for each $t, u(t) \in U = R^m$. A is an $n \times n$, B is an $n \times m$, and H is a $p \times n$ constant matrix.

For the system (1) the transfer function $T(s)$ where s is a complex variable is given by

$$T(s) = H[sI - A]^{-1}B. \qquad (2)$$

Let $\Gamma = \{\gamma_1 \dots \gamma_n\}$ be an arbitrary set of n complex numbers with the proviso that each $\gamma_i \in \Gamma$ which is complex occurs in conjugate pairs. It is now well known [1], [2] if (1) is completely controllable, the pole locations of $T(s)$ may be moved to those determined by Γ by the application of affine state variable feedback, that is by the application of a control law

$$u(t) = k + Kx(t) \qquad (3)$$

where k is a constant vector and K a constant matrix.

It is also well known that if $m = 1$, the zeros of the transfer function matrix remain invariant with respect to affine state variable feedback. For multi-input systems this result is not true in general.

Assume that A has distinct eigenvalues. In applications, the matrix H will usually be fixed by the availability of transducers to measure the state variables. More freedom may be available in the selection of the activating matrix B. Each column of B may be considered as an activating vector for a different component of control.

Let b_k denote the kth column of B. Let $\{\lambda^{(k)}\}$ denote the set of eigenvalues of A which may be influenced by b_k. The class of multi-input systems considered in this paper has the following properties:

$$\bigcup_{k=1}^{m} \{\lambda^{(k)}\} = \{\lambda_1, \lambda_2, \cdots, \lambda_n\} \qquad (4)$$

$$\{\lambda^{(i)}\} \cap \{\lambda^{(k)}\} = 0, \; i \neq k. \qquad (5)$$

Condition (4) guarantees that the system is completely controllable while condition (5) indicates that each actuating vector influences a different set of eigenvalues. The class of systems satisfying conditions (4) and (5) will be referred to as systems with disjoint control.

Since we are interested in the properties of the transfer function matrix, we may consider any convenient representation of (1). In terms of the canonical form for multi-input systems [1], [3], [4], the state equation of (1) admits the representation[1] of the following

$$\frac{d\hat{x}}{dt} = \hat{A}\hat{x}(t) + \hat{B}u(t) \qquad (6)$$

[1] In what follows we could also use the Jordan canonical form representation for (1).

where

$$\hat{A} = \begin{bmatrix} \hat{A}_m & & & & \\ & \hat{A}_{m-1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & \hat{A}_1 \end{bmatrix}$$

$$\hat{B} = \left[\begin{array}{c|c|c|c} 0 & \cdots & 0 & \begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix} \\ \hline 0 & \cdots & \begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix} & 0 \\ \hline & \vdots & & \\ \hline \begin{matrix}0\\ \vdots\\ 0\\ 1\end{matrix} & \cdots & 0 & 0 \end{array}\right].$$

Each $\hat{A}_k (k = 1,2, \cdots, m)$ is a companion matrix corresponding to the eigenvalues influenced by b_k. This class of systems has a unique representation except for the orde - ing of the $\hat{A}_k$. A typical submatrix A_k of A is given by

$$\hat{A}_k = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 \\ -a_{k,n_k} & -a_{k,n_k-1} & & -a_{k,1} \end{bmatrix} \qquad (7)$$

with the corresponding characteristic equation

$$s^{n_k} + a_{k,1}s^{n_k-1} + \cdots + a_{k,n_k} = 0. \qquad (8)$$

Since the system is completely controllable,

$$\sum_{k=1}^{m} n_k = n.$$

The characteristic equation of $\hat{A}$ is given by

$$\det [sI - \hat{A}] = \prod_{k=1}^{m} \det [sI - \hat{A}_k] = 0.$$

Every eigenvalue of $\hat{A}$ may be altered by changing the characteristic equation of the companion matrix it is associated with. For example, suppose that the eigenvalues of $\hat{A}_k$ are to be moved to new locations determined by the characteristic equation

$$s^{n_k} + \bar{a}_{k,1}s^{n_k-1} + \cdots + \bar{a}_{k,n_k} = 0. \qquad (9)$$

Let the state vector be partitioned according to the companion matrices $\hat{A}_k$ in $\hat{A}$ and let the kth component of the control vector be

$$u_k(t) = [a_{k,n_k} - \bar{a}_{k,n_k}, \cdots, a_{k,1} - \bar{a}_{k,1}]\hat{x}_k. \quad (10)$$

If all the eigenvalues are changed in this manner, the resulting system $\hat{A}$ is still of the form (6), excepting that the last row of each A_k is changed due to the shifting of the eigenvalues.

Due to the block diagonal structure of (6)

$$[sI - \hat{A}]^{-1} = \begin{bmatrix} [sI - \hat{A}_m]^{-1} & & & & \\ & [sI - \hat{A}_{m-1}]^{-1} & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & [sI - \hat{A}_1]^{-1} \end{bmatrix} \quad (11)$$

where a typical block $[sI - \hat{A}_k]^{-1}$ is of the form

$$[sI - \hat{A}_k]^{-1} = \frac{1}{\det\,[sI - \hat{A}]} \begin{bmatrix} \cdots 1 \\ \cdots s \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdot \quad \cdot \\ \cdots s^{n_k - 1} \end{bmatrix}. \quad (12)$$

The matrix (12) is not completely described since only the last column proves to be of importance. The transfer function matrix corresponding to the representation (6) is

$$T(s) = \hat{H}[sI - \hat{A}]^{-1}\hat{B}$$

$$= \hat{H} \left[\begin{array}{c:c:c:c} 0 & 0 & \cdots & \begin{matrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^{n_m - 1} \end{matrix} \\ \hdashline \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} & \begin{matrix} \cdot \\ \cdot \\ \cdot \end{matrix} \\ \hdashline 0 & \begin{matrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^{n_2 - 1} \end{matrix} & \cdots & 0 \\ \hdashline \begin{matrix} 1 \\ s \\ \cdot \\ \cdot \\ \cdot \\ s^{n_1 - 1} \end{matrix} & 0 & \cdots & 0 \end{array}\right]. \quad (13)$$

Let the ith row of $\hat{H}$ be given as

$$\hat{h}^i = [\hat{h}^i_{m,0}, \cdots, \hat{h}^i_{m,n_m-1} \mid \cdots \mid \hat{h}^i_{2,0}, \cdots, \hat{h}^i_{2,n_2-1} \mid \hat{h}^i_{1,0} \cdots \mid \hat{h}^i_{1,n_1-1}] \quad (14)$$

where the partitions are introduced to correspond to the horizontal partition of $[sI - \hat{A}]^{-1}\hat{B}$. Then a typical element of the transfer function matrix relating the kth input to the ith output is

$$\frac{y_i(s)}{u_k(s)} = \frac{\hat{h}^i_{k,0} + s\hat{h}^i_{k,1} + \cdots + s^{n_k - 1}\hat{h}^i_{k,n_k-1}}{\det[sI - \hat{A}_k]}. \quad (15)$$

The zeros of this typical element of the transfer function matrix depend only on $\hat{H}$ which is fixed by the system. Det $[sI - \hat{A}_k]$ is only a function of the last row of elements of $\hat{A}_k$. Hence feedback of the type described in (10) may be used to create arbitrary pole locations of the transfer function matrix while leaving the zeros invariant.

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Identification of Process Delay Time

***Abstract*—A method of identification of a process delay time is suggested in conditions when the delay is varying slowly and cannot be estimated by direct measure, and in which also the process input is not available for measurement. A cross correlation function is suggested as a means of evaluating the process delay time so that the system can be adapted to the new conditions necessary for acceptable performance.**

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The type of system to be considered is that generally represented in Fig. 1. Such a system can be controlled by a classical three-term controller but not in an optimum manner, although the results may be reasonably satisfactory.

In the literature on the control of delay time processes it has often been assumed that the function $G(p)$ is of the form $1/(1 + \gamma p)$. Reference [6] discusses the simplest condition when $\gamma = 0$, [7] that of $\gamma/T_1 < 1$ and [8] that of $\gamma/T_1 > 1$. The system shown in Fig. 1 is that proposed by Smith [1], [2]; this system enables the output integral error to be minimized without any cost being allocated to the manipulated variable $M(p)$. For stability reasons it is necessary in this system to maintain $T_2 = T_1$. This note is concerned with a possible method whereby T_2, the delay time of the simulator, can be matched to T_1, the actual process delay time that may be varying. $F(p)$ represents the function of the process output measurement transducer.

The output–disturbance transfer function for the system in Fig. 1 is

$$\frac{C(p)}{D(p)} = \frac{G(p)\,e^{-T_1 p}}{1 - F(p)e^{-T_1 p}/(1 - F(p)e^{-T_2 p})}.$$

If $F(p) = 1$, or if $F(p)$ contains a time constant that is very small compared with T_1, the necessary and sufficient condition for stability is that $T_2 = T_1$ [3]–[5]. If $G(p) = 1/(1 + \gamma p)$ and $F(p) = 1$, then the output/disturbance transfer function can be expressed in the form

$$\frac{e^{-p}(1 - e^{-\alpha p})}{(1 + ap)(1 - e^{-\alpha p} + e^{-p})}$$

in which $a = \gamma/T_1$ and $\alpha = T_2/T_1$ and T_1 is considered as one unit of time. Then by consideration of the characteristic equation it can be shown that, if $a > 0$, for stability $\alpha = 1$. If $G(p)^{-1}$ were replaced, more realistically, by a function $(1 + ap)/(1 + bp)$ in which $a \gg b$, then the same condition for stability holds as it does if the element $G(p)^{-1}$ is not present in the control system.

A possible means of obtaining an indirect estimate of T_1 is by use of a cross-correlation function $\Phi_{ze}(\tau)$ between $z(t)$ and $e(t)$, the output of the simulator and the actuating signal of the system, so that

$$\Phi_{ze}(\tau) = \frac{1}{T}\int_0^T z(t)e(t + \tau)\,dt.$$

The evaluation of this function over a range of τ provides an oscillating profile with clearly defined maxima and minima directly indicating values of T_1 and T_2 and also the difference $T_1 - T_2 = \xi$. The presence of noise in the system should ensure the existence of these peaks even if the command $r(t)$ remains constant.

For the class of system in which $F(p) = 1$ and $G(p) = 1/(1 + aT_1 p)$, results for values of $a = 0.15$ and $a = 0.3$, $R(p) = 0$, and a unit step or sine input (period $12T_1$) applied at $t = 0$ as a disturbance enable certain conclusions to be clearly drawn:

the first noticeable minimum of $\Phi_{ze}(\tau)$ occurs for $\tau = \xi$