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Sensitivity reduction over a frequency band

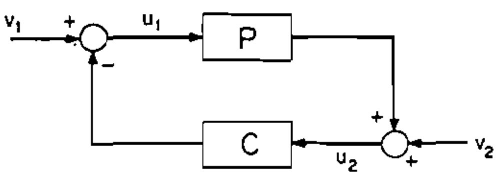
YOSHITO OHTA†‡, GILEAD TADMOR†§ and SANJOY K. MITTER†

The problem of reducing the sensitivity of a possibly infinite-dimensional linear single-input single-output system over a finite frequency interval by feedback is considered. Specifically, the following are proven: (i) if one wants to bound the overall sensitivity, the existence of a non-trivial inner part inhibits the reduction of the sensitivity over the interval; (ii) in a system that is continuous and has at most countably many zeros on the imaginary axis, one can reduce the sensitivity over an interval to be arbitrarily small, while the overall sensitivity is kept bounded if and only if the system is outer and has no zeros on the interval. These extend results for rational transfer functions.

1. Introduction

This paper considers the problem of reducing the sensitivity of a linear single-input single-output system over a finite frequency interval by feedback.

The feedback system is described by the Figure. P is a given system and we assume $P \in H^\infty$ (i.e. stable) and C is a feedback. We say that the feedback stabilizes the system if the transfer functions from (v_1, v_2) to (u_1, u_2) all belong to H^∞.



The closed-loop sensitivity S is the transfer function from v_2 to u_2 and is given by

$$S(s) = [1 + P(s)C(s)]^{-1} \tag{1.1}$$

The problem of sensitivity reduction over a frequency band X is stated as follows. Let χ be the characteristic function of a given bounded set $X \subset (-\infty, \infty)$, on the imaginary axis, i.e.

$$\chi(j\omega) = \begin{cases} 1 & \text{if } \omega \in X \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

For given $\varepsilon > 0$ and $M > 1$, find a stabilizing feedback for which the sensitivity satisfies

$$\|\chi S\|_\infty < \varepsilon, \quad \|S\|_\infty < M \tag{1.3}$$

The main results follow.

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Theorem 1

Suppose $P \in H^\infty$ has a non-trivial inner part and χ is the characteristic function of a subset of the imaginary axis that has positive Lebesgue measure. Then

$$\inf_{\|S\|_\infty < M} \|\chi S\|_\infty > 0$$

where $M > 1$ and the infimum is taken over all stabilizing compensators.

Theorem 2

Suppose $P \in H^\infty$ is continuous on the imaginary axis and has at most countably many zeros on the imaginary axis. Let χ be the characteristic function of a compact set $X \subset (-\infty, \infty)$ on the imaginary axis. Then for any $1 > \varepsilon > 0$ and any $M > 1$ there exist a stabilizing compensator such that

$$\|\chi S\|_\infty < \varepsilon, \quad \|S\|_\infty < M$$

if and only if P is outer and has no zeros on jX.

Previous discussions of this problem appear in Zames and Bensoussan (1983), Bensoussan (1984), Francis and Zames (1984), Pandolfi and Olbrot (1986) and Francis (1987). Bensoussan (1984) showed that if the plant P is analytic, is bounded, has no zero in $\operatorname{Re} s \geqslant 0$, and satisfies an attenuation condition at $s = \infty$, then for any $\varepsilon > 0$ and $M > 1$, the problem has a solution. Especially the problem is solvable when P is of minimum phase. Theorem 2 generalizes this result, and seems to illuminate more on the structural aspects of the sensitivity reduction problem. In Francis and Zames (1984), in the framework of rational plants, it was shown that if the plant P has a right half-plane zero then there exists a positive number k such that

$$\|\chi S\|_\infty \geqslant \|S\|_\infty^k$$

Hence given $M > 1$, there is $\varepsilon > 0$ such that the problem has no solution. Theorem 1 is a natural extension of this statement. Pandolfi and Olbrot (1986) showed that if the plant is analytic and has no zero in some region containing $\operatorname{Re} s \geqslant 0$, and satisfies some intricate condition near $s = \infty$, then for any $\varepsilon > 0$ and $M > 2 + L$ (L is determined by the condition), the problem has a solution. However, the condition seems rather difficult to check. The difficulty was demonstrated by the authors' wrong conclusion that for $P(s) = \exp(-s)/(s+1)$ (which has a non-trivial inner part), and some $M > 2$, the problem has a solution for any $\varepsilon > 0$.

2. Preliminaries and notations

2.1. *Parametrization of stabilizing feedbacks*

We parametrize feedbacks achieving stability. The parametrization was introduced by Youla *et al.* (1976) and modified by Desoer *et al.* (1980). The following is a corollary of Fagnani *et al.* (1987) for a stable system.

Proposition 1

Assume P is stable ($P \in H^\infty$). Then a feedback C stabilizes the system if and only if there exists $h \in H^\infty$, $Ph \neq 1$ such that

$$C = \frac{h}{1 - Ph} \tag{2.1}$$

Substituting (2.1) into (1.1), we have

$$S = 1 - Ph \tag{2.2}$$

Therefore our problem is reduced to that of finding $h \in H^\infty$ satisfying

$$\|\chi(1 - Ph)\|_\infty < \varepsilon, \quad \|1 - Ph\|_\infty < M \tag{2.3}$$

for given $\varepsilon > 0$, $M > 1$.

2.2. *H^p functions*

H^2 and H^∞ are the Hardy spaces of analytic functions on the right half-plane with L^2 and L^∞ boundary values, respectively. Hoffman (1962) and Douglas (1972) are good sources on H^p spaces, inner–outer factorizations, etc. The following is from Douglas (1972) and is worthy of note.

Proposition 2 (Douglas 1972)

Assume $P \in H^\infty$ and let $K = H^2 \ominus PH^2$ (or $K = (PH^2)^\perp$). Then $K = \{0\}$ if and only if P is outer.

2.3. *σ-Inner product and σ-norm*

The Laplace transformation L defines an isometric isomorphism from $L^2[0, \infty)$ to H^2. We shall use both the time domain and the frequency domain in our analysis.

We denote the usual inner product and norm of H^2 (respectively, $L^2[0, \infty)$) by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. For future use, we introduce also a whole family of additional inner products and norms as follows. Given $\sigma > 0$, and $f, g \in H^2$, define the σ-inner product and the σ-norm by

$$\langle f, g \rangle_\sigma = (2\pi)^{-1} \int_{-\infty}^{\infty} f(\sigma + j\omega)\overline{g(\sigma + j\omega)}\, d\omega \tag{2.4}$$

$$\|f\|_\sigma = \langle f, f \rangle_\sigma^{1/2} \tag{2.5}$$

Since L is an isometry, there hold

$$\langle f, g \rangle_\sigma = \int_0^\infty L^{-1}(f)(t)\overline{L^{-1}(g)(t)} \exp(-2\sigma t)\, dt \tag{2.6}$$

$$\|f\|_\sigma = \left(\int_0^\infty |L^{-1}(f)(t)|^2 \exp(-2\sigma t)\, dt \right)^{1/2} \tag{2.7}$$

If $L^{-1}(f)$ and $L^{-1}(g)$ are supported within the compact interval $[0, T]$, given some $T > 0$, then

$$\exp(-\sigma T)\|f\| \leqslant \|f\|_\sigma \leqslant \|f\| \tag{2.8}$$

and

$$|\langle f, g \rangle - \langle f, g \rangle_\sigma| \leqslant (1 - \exp(-2\sigma T))(\|f\| + \|g\|)^2 \tag{2.9}$$

For $x \in L^2[0, \infty)$, define $x_T \in L^2[0, \infty)$ to be the truncation of x at time T, $T > 0$, i.e.

$$x_T(t) = \begin{cases} x(t) & t \leqslant T \\ 0 & \text{otherwise} \end{cases} \tag{2.10}$$

For $f = L(x)$, we denote $f_T = L(x_T)$. Notice that $x_T \to x$ and $f_T \to f$ as $T \to \infty$, in the usual topology of $L^2[0, \infty)$ and H^2.

3. Proofs of theorems 1 and 2

In proving Theorem 1 we use the following observations.

Lemma 1

Let $\{g_n\} \subset H^\infty$ be a sequence such that $\|g_n\|_\infty < M$. Let χ be the characteristic function of a set $X \subset [-\infty, \infty]$ of positive measure on the imaginary axis. Suppose $\|\chi g_n\|_\infty \to 0$ as $n \to \infty$. Then for any compact set Y in the open right half-plane, $|g_n(s)| \to 0$, uniformly for $s \in Y$.

Proof

It seems convenient to establish the lemma in the disc. For $g \in H^\infty$, define

$$g_D(z) = g\left(\frac{1+z}{1-z}\right) \tag{3.1}$$

Then $g_D \in H^\infty(D)$ and $\|g\|_\infty = \|g_D\|_\infty$, where D is the unit disc on the complex plane. Let $Y_D = \{z \mid (1+z)/(1-z) \in Y\}$, and $X_D = \{z \mid (1+z)/(1-z) \in jX\}$ be the inverse image of Y and X by the Mobius transformation, respectively, and χ_D be the characteristic function of X_D. Since Y is a compact set in the open right half-plane, $Y_D \subset B(0, r)$ ($\equiv$ the closed disc of radius r) for some $0 < r < 1$. By Jensen's inequality, we have

$$\log |g_D(z)| \leqslant \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |g_D(\exp(j\theta))| \operatorname{Re}\left[\frac{\exp(j\theta) + z}{\exp(j\theta) - z}\right] d\theta \tag{3.2}$$

Note that for $z \in Y_D$ and $\theta \in [-\pi, \pi]$

$$\frac{1+r}{1-r} \geqslant \frac{1+|z|}{1-|z|} \geqslant \operatorname{Re}\left[\frac{\exp(j\theta) + z}{\exp(j\theta) - z}\right] \geqslant \frac{1-|z|}{1+|z|} \geqslant \frac{1-r}{1+r} \tag{3.3}$$

Using the inequalities (3.3), for $z \in Y_D$ we can find a uniform upper bound for the right-hand side of (3.2).

$$\begin{aligned}
&\log |g_D(z)| \\
&\leqslant \frac{1}{2\pi}\left[\frac{1+r}{1-r} \int_{-\pi}^{\pi} \log^+ |g_D(\exp(j\theta))| \, d\theta - \frac{1-r}{1+r} \int_{-\pi}^{\pi} \log^+ |g_D(\exp(j\theta))|^{-1} \, d\theta\right] \\
&\leqslant \frac{1}{2\pi}\left[\frac{1+r}{1-r} \int_{-\pi}^{\pi} |g_D(\exp(j\theta))| \, d\theta - \frac{1-r}{1+r} \int_{X_D} \log^+ |g_D(\exp(j\theta))|^{-1} \, d\theta\right] \\
&\leqslant \frac{1+r}{1-r} \|g_D\|_\infty - \frac{1}{2\pi} \frac{1-r}{1+r} \log^+(\|\chi_D g_D\|_\infty^{-1}) \mu(X_D)
\end{aligned} \tag{3.4}$$

where

$$\log^+ x = \log\{\max(1, x)\} \geqslant 0 \tag{3.5}$$

and μ is the Lebesgue measure on the unit circle.

Let $\{g_{nD}\}$ be the sequence in $H^\infty(D)$ obtained from $\{g_n\}$ by the transformation (3.1). Note that

$$\|g_{nD}\|_\infty < M \quad \text{and} \quad \|\chi_D g_{nD}\|_\infty \to 0, \quad \text{as } n \to \infty \tag{3.6}$$

Applying (3.4) to the sequence $\{g_{nD}\}$, we see that the right-hand side of (3.4) tends to $-\infty$ uniformly for $z \in Y_D$. Hence $|g_n(s)| \to 0$ uniformly for $s \in Y$. □

Corollary 1

Let $\{g_n\}$ be as in Lemma 1, $f \in H^2$ and $\sigma > 0$. Then $\|g_n f\|_\sigma \to 0$ as $n \to \infty$.

Proof

Fix $\varepsilon > 0$. Since $f \in H^2$, there exists $\Omega > 0$ such that

$$(2\pi)^{-1} \int_{|\omega| > \Omega} |f(\sigma + j\omega)|^2 \, d\omega < \frac{\varepsilon}{2M^2} \tag{3.7}$$

This implies that

$$(2\pi)^{-1} \int_{|\omega| > \Omega} |g_n f(\sigma + j\omega)|^2 \, d\omega < \frac{\varepsilon}{2} \tag{3.8}$$

for all n, since $\|g_n\|_\infty < M$.

Applying Lemma 1 to the sequence $\{g_n\} \subset H^\infty$ and $Y = \{s \mid s = \sigma + j\omega, |\omega| \leqslant \Omega\}$, and Lebesgue's dominant convergence theorem, we see that

$$(2\pi)^{-1} \int_{|\omega| \leqslant \Omega} |g_n f(\sigma + j\omega)|^2 \, d\omega \to 0 \quad \text{as } n \to \infty \tag{3.9}$$

Thus, for n large enough

$$\|g_n f\|_\sigma^2 = (2\pi)^{-1} \int_{-\infty}^{\infty} |g_n f(\sigma + j\omega)|^2 < \varepsilon \tag{3.10}$$

This implies that $\|g_n f\|_\sigma \to 0$ as $n \to \infty$. □

Claim 1

Let $\{g_n\} \subset H^\infty$, $\|g_n\|_\infty < M$ and $f \in H^2$. Then for each $\lambda > 0$, there exist some $T > 0$ and $\sigma > 0$ such that $|\langle f, g_n f\rangle - \langle f_T, g_n f\rangle_\sigma| < \lambda$ for all n.

Proof

$$\begin{aligned} |\langle f, g_n f\rangle - \langle f_T, g_n f\rangle| &= |\langle f - f_T, g_n f\rangle| \\ &\leqslant \|f - f_T\| \, \|g_n f\| \\ &\leqslant \|g_n\|_\infty \, \|f - f_T\| \, \|f\| \\ &\leqslant M \|f - f_T\| \, \|f\| \end{aligned} \tag{3.11}$$

Utilizing (2.9), we also have

$$\begin{aligned}|\langle f_T, g_n f\rangle - \langle f_T, g_n f\rangle_\sigma| &= |\langle f_T, (g_n f)_T\rangle - \langle f_T, (g_n f)_T\rangle_\sigma| \\ &\leqslant (1 - \exp(-2\sigma T))[\|f_T\| + \|(g_n f)_T\|]^2 \\ &\leqslant (1 - \exp(-2\sigma T))[\|f\| + \|g_n f\|]^2 \\ &\leqslant (1 + M)^2(1 - \exp(-2\sigma T))\|f\|^2 \qquad (3.12)\end{aligned}$$

Recall that $f_T \to f$ as $T \to \infty$. Choose T sufficiently large so that $\|f - f_T\| < \lambda/2M\|f\|$. Then choose σ sufficiently small so that $(1 - \exp(-2\sigma T)) < \lambda/2(1 + M)^2\|f\|^2$. Combining (3.11) and (3.12), we have the desired inequality. □

For ease of reference, we repeat our results.

Theorem 3

Suppose $P \in H^\infty$ has a non-trivial inner part and χ is the characteristic function of a subset of the imaginary axis that has positive Lebesgue measure. Then

$$\inf_{\|S\|_\infty < M} \|\chi S\|_\infty > 0 \qquad (3.13)$$

where $M > 1$ and the infimum is taken over all stabilizing compensators.

Proof

On the contrary, assume that there exists a sequence $\{S_n\}$ of sensitivity functions, $S_n = 1 - Ph_n$, $h_n \in H^\infty$ with $\|S_n\|_\infty < M$, $\|\chi S_n\|_\infty \to 0$ as $n \to \infty$.

Set $K = H^2 \ominus PH^2$. The subspace K is non-trivial ($K \neq \{0\}$), by Proposition 2, since P has a non-trivial inner part. For any $f \in K$ and any $g \in H^2$, we have

$$\langle f, S_n g\rangle = \langle f, (1 - Ph_n)g\rangle = \langle f, g\rangle - \langle f, Ph_n g\rangle = \langle f, g\rangle \qquad (3.14)$$

in particular, for $f \in K, f \neq 0$

$$\langle f, S_n f\rangle = \|f\|^2 > 0 \qquad (3.15)$$

Hence, in view of Claim 1, there exist $T > 0$, $\sigma > 0$ and $\delta > 0$ such that

$$\langle f_T, S_n f\rangle_\sigma > \delta \qquad (3.16)$$

for all n.

On the other hand, by Corollary 1

$$|\langle f_T, S_n f\rangle_\sigma| \leqslant \|f_T\|_\sigma \|S_n f\|_\sigma \to 0 \quad \text{as } n \to \infty \qquad (3.17)$$

which is a contradiction. □

Theorem 4

Suppose $P \in H^\infty$ is continuous on the imaginary axis and has at most countably many zeros on the imaginary axis. Let χ be the characterization function of a compact set jX; $X \subset (-\infty, \infty)$ on the imaginary axis. Then for any $1 > \varepsilon > 0$ and any $M > 1$ there exists a stabilizing compensator such that

$$\|\chi S\|_\infty < \varepsilon, \quad \|S\|_\infty < M \qquad (3.18)$$

if and only if P is outer and has no zeros on jX.

Proof

Necessity. If P is not outer, then the conclusion follows from Theorem 1. Let P have a zero on X. Suppose then that there exists $h \in H^\infty$ such that

$$\|\chi(1-Ph)\|_\infty < \varepsilon, \quad \|1-Ph\|_\infty < M \tag{3.19}$$

Fix $\delta > 0$. As P is continuous on the imaginary axis

$$\mu[X \cap \{\omega \mid |P(i\omega)| < \delta \|h\|_\infty^{-1}\}] > 0$$

that is

$$\mu[X \cap \{\omega \mid |P(j\omega)h(j\omega)| < \delta\}] > 0$$

where μ is the Lebesgue measure on the imaginary axis. Since δ was arbitrary $\|\chi(1-Ph)\|_\infty \geqslant 1$, a contradiction.

Sufficiency. The proof is by construction of $h \in H^\infty$ such that

$$\|\chi(1-Ph)\|_\infty < \varepsilon, \quad \|1-Ph\|_\infty < M \tag{3.20}$$

for given $0 < \varepsilon < 1$, $M > 1$.

Let

$$U = \{u \mid u = \infty \text{ or } P(ju) = 0\} \tag{3.21}$$

From the assumption, U is at most countable and $U \cap X = \varnothing$. Let $U = \{u_n, n = 1, 2, \ldots\}$ be an enumeration of U.

Define $r_{n,a}$ by

$$r_{n,a}(s) = \begin{cases} \left[\dfrac{a(s-ju_n)}{a(s-ju_n)+1}\right]^{(1/2^{n+1})} & u_n \neq \infty \\ \left[\dfrac{a}{s+a}\right]^{(1/2^{n+1})} & u_n = \infty \end{cases} \tag{3.22}$$

where $a > 0$ is a parameter to be fixed later, and the branch of the $2^{-(n+1)}$th complex root is decided in such a way that the positive real line is mapped into itself. Equation (3.22) defines an analytic function on the open right half-plane, since the function $a(s-ju_n)/[a(s-ju_n)+1]$ (or $a/(s+a)$) maps the open right half-plane into itself. Furthermore, the following properties of $r_{n,a}$ are easily proved: (i) $r_{n,a} \in H^\infty$, $\|r_{n,a}\|_\infty = 1$; (ii) $r_{n,a}$ is outer; (iii) $r_{n,a}$ is continuous on the imaginary axis including ∞; (iv) $r_{n,a}(ju_n) = 0$; (v) $\|\chi(1-r_{n,a})\|_\infty \to 0$ as $a \to \infty$ (note that $X \cap U = \varnothing$); and (vi) $|\arg r_{n,a}(s)| \leqslant \pi 2^{-(n+2)}$, for any $s \in \{\operatorname{Re} s \geqslant 0\} \cup \{\infty\}$.

Given $\varepsilon > 0$ (from (3.20)), we choose $\eta > 0$ such that $|\log z| < \eta$, $z \in C$, implies

$$|z-1| < \varepsilon \tag{3.23}$$

For each n, we choose the parameter a, according to the property (v), in such a way that

$$\|\chi \log r_{n,a}\|_\infty < \frac{\eta}{2^n} \tag{3.24}$$

is satisfied. For brevity we denote r_n instead of $r_{n,a}$, henceforth.

The properties (iii) and (iv) imply that there exists a neighbourhood W_n of u_n in the one point compactification of R such that $W_n \cap X = \varnothing$ and $|r_n(j\omega)| < M-1$, $\omega \in W_n$.

(Note that a neighbourhood of ∞ is $\{\omega \mid |\omega| > \Omega\}$ for some $\Omega > 0$.) U is compact since it is a closed subset of a compact set, and hence the cover $U \subset \bigcup_{n=1}^{\infty} W_n$ has a finite subcover, say $U \subset W = \bigcup_{n \in N} W_n$, where N is a finite index set.

Since P and r_n are outer

$$P(s) = \lambda \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \log|P(j\omega)| \frac{\omega s + j}{\omega + js}\frac{d\omega}{1+\omega^2}\right] \tag{3.25}$$

and

$$r_n(s) = \lambda_n \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \log|r_n(j\omega)| \frac{\omega s + j}{\omega + js}\frac{d\omega}{1+\omega^2}\right] \tag{3.26}$$

for $\lambda, \lambda_n \in C$, $|\lambda| = |\lambda_n| = 1$.

Given $\delta > 0$, let

$$D_\delta = \{\omega \mid |P(j\omega)| \leqslant \delta\} \tag{3.27}$$

and, define $h_\delta(s)$ by

$$h_\delta(s) = \bar{\lambda} \prod_{n \in N} \bar{\lambda}_n \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} [-C_\delta(\omega)] \frac{\omega s + j}{\omega + js}\frac{d\omega}{1+\omega^2}\right] \tag{3.28}$$

where

$$C_\delta(\omega) = \begin{cases} 0 & \omega \in D_\delta \\ C(\omega) & \omega \notin D_\delta \end{cases} \tag{3.29}$$

$$C(\omega) = \log|P(j\omega)| - \sum_{n \in N} \log|r_n(j\omega)| \tag{3.30}$$

The proof will be completed if we show that $h_\delta \in H^\infty$ and that for sufficiently small $\delta > 0$, this function satisfies (3.19).

As is known (Hoffman 1962, p. 53), the right-hand side of (3.28) defines an H^∞ function if and only if $\exp(-C_\delta) \in L^\infty$ and C_δ is integrable with respect to $d\omega/(1+\omega^2)$. A sufficient condition holds in particular if C_δ is bounded. C_δ is bounded because $|P|$ and $|r_n|$ are bounded and bounded away from zero off D_δ.

In verifying (3.19), we use the following equalities.

$$\begin{aligned} Ph_\delta(s) &= \prod_{n \in N} \lambda_n \exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty} \sum_{n \in N} \log|r_n(j\omega)| \frac{\omega s + j}{\omega + js}\frac{d\omega}{1+\omega^2} + \frac{1}{\pi}\int_{D_\delta} C(\omega)\frac{\omega s + j}{\omega + js}\frac{d\omega}{1+\omega^2}\right] \\ &= \prod_{n \in N} r_n(s) \exp\left[\frac{1}{\pi}\int_{D_\delta} C(\omega)\frac{\omega s + j}{\omega + js}\frac{d\omega}{1+\omega^2}\right] \end{aligned} \tag{3.31}$$

and

$$|Ph_\delta(j\omega)| = \begin{cases} \prod_{n \in N} |r_n(j\omega)| & \omega \notin D_\delta \\ |P(j\omega)| & \omega \in D_\delta \end{cases} \tag{3.32}$$

Indeed, it is also known (Hoffman 1962, p. 53) that the boundary value of $h_\delta(s)$ as $s \to j\omega$ satisfies $|h_\delta(j\omega)| = \exp[-C_\delta(\omega)]$, almost everywhere, and from this (3.32) follows.

Note now that if $\omega \in W$ then $|r_n(j\omega)| < M - 1$ for some index $n \in N$, and $|r_n(j\omega)| \leqslant 1$, for all $n \in N$. Thus, for $\delta < M - 1$ there holds

$$|Ph_\delta(j\omega)| = \begin{cases} < M - 1 & \omega \in W \\ \leqslant 1 & \omega \notin W \end{cases} \tag{3.33}$$

Consequently, for $\omega \in W$

$$|1 - Ph_\delta(j\omega)| \leqslant 1 + M - 1 = M \tag{3.34}$$

For handling ω in the complement of W, we first observe that

$$\left|\arg \prod_{n \in N} r_n(j\omega)\right| \leqslant \sum_{n \in N} |\arg r_n(j\omega)| < \sum_{n=1}^{\infty} \pi 2^{-(n+2)} = \frac{\pi}{4} \tag{3.35}$$

from property (vi) of r_n. Thus, from Claim 2 below, for sufficiently small δ

$$|\arg Ph_\delta(j\omega)| < \frac{\pi}{4} \tag{3.36}$$

Hence (from (3.33), (3.36)) for $\omega \in {}^cW$

$$|1 - Ph_\delta(j\omega)| \leqslant 1 < M \tag{3.37}$$

Equations (3.34) and (3.37) imply $\|1 - Ph_\delta\|_\infty < M$.

Finally we consider $\omega \in X$; then

$$\left|\log \prod_{n \in N} r_n(j\omega)\right| \leqslant \sum_{n \in N} |\log r_n(j\omega)| \leqslant \sum_{n \in N} \frac{\eta}{2^n} < \sum_{n=1}^{\infty} \frac{\eta}{2^n} = \eta \tag{3.38}$$

by (3.24). From Claim 2 below it follows that for sufficiently small δ, we have

$$|\log Ph_\delta(j\omega)| < \eta \tag{3.39}$$

for $\omega \in X$. Thus, (3.23) and (3.39) imply $\|\chi(1 - Ph_\delta)\|_\infty < \varepsilon$, as required. □

Claim 2

$$Ph_\delta \to \prod_{n \in N} r_n \quad \text{as } \delta \to 0 \text{ in } L^\infty({}^cW)$$

Proof

Notice that the continuity of P implies $\bar{D}_\delta \cap {}^cW = \varnothing$ for small δ. Given that δ is indeed small, we have (from (3.32))

$$|Ph_\delta(j\omega)| = \prod_{n \in N} |r_n(j\omega)| \quad \text{for } \omega \in {}^cW \tag{3.40}$$

Hence, it remains to check that

$$\arg Ph_\delta(j\omega) \to \arg \prod_{n \in N} r_n(j\omega) \quad \text{as } \delta \to 0 \tag{3.41}$$

in $L^\infty({}^cW)$.

From (3.30), it suffices to show that

$$\int_{D_\delta} C(\omega) \frac{\omega\theta + 1}{\omega - \theta} \frac{d\omega}{1 + \omega^2} \to 0 \quad \text{as } \delta \to 0 \tag{3.42}$$

uniformly for θ in ${}^cW$.

Since D_δ lies strictly within the interior of W, the kernel $(\omega\theta+1)/(\omega-\theta)$ is uniformly bounded over the domain $\omega \in D_\delta$ and $\theta \in {}^cW$. Setting $d\mu(\omega) = d\omega/(\omega^2+1)$, we know that $\log|P(\cdot)|$ and $C(\cdot)$ belong to $L^1(d\mu)$. By the first fact, it is necessary that $\mu(D_\delta)\to 0$ as $\delta \to 0$. Consequently, the second implies (3.42). This proves the claim. □

Remark 1

Notice that we did not require continuity of $P(j\omega)$ at $\omega = \pm\infty$. In fact, the assumptions on the continuity of $P(j\omega)$ and the compactness of the subset X can be relaxed in various ways without requiring considerable changes in the analysis. The current setup was chosen for simplicity.

Remark 2

A major part of the proof of Theorem 2 is dedicated to the construction of the 'roll-off' functions r_n, that are needed when $2 \geqslant M > 1$. For $M > 2$, the assumptions can be further relaxed, e.g. if $P(j\omega)$ is continuous, to the requirement that P be outer and have no zeros in X.

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