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A note on essential spectra and norms of mixed Hankel–Toeplitz operators

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Abstract: In this brief, Hankel–Toeplitz operators which occur in feedback theory, e.g., in the minimization of mixed H^∞ sensitivity and complementary sensitivity, will be considered. A method of computing their spectra, eigenvectors, and norms will be presented for infinite-dimensional systems subject to continuous weightings.

Keywords: H-infinity optimization, Infinite-dimensional mixed sensitivity, Essential spectra.

1. Introduction

Mixed sensitivity optimization was considered by Kwakernaak [9]. Francis et al. [5,4,6] gave various characterizations of the problem, e.g., in terms of the distance from $[{W \atop 0}]$ to $[{M \atop V}]H^\infty$, where W, M, V, are in H^∞ and M is inner. Jonckheere and Verma [7,12] described the problem in terms of the norm of the Hankel–Toeplitz operator displayed in (1) (below). Implicit methods of minimization, e.g. the ε-iteration [6], were introduced by these authors. Apart from the highly implicit nature of the minimization, the theory remains incomplete for irrational plants, for which a method of determining essential spectra has yet to be provided.

Here an explicit formula for the essential spectra of such operators will be derived, as well as a method of computing discrete eigenvalues in which the only implicit step involves the evaluation of the zeros of a 'characteristic determinant' function of the real variable λ, which is analytic in λ. The results extend those of Foias et al. [13,2,3] and Flamm [1] for (unmixed) sensitivity minimization. In particular, essential spectra are computed by viewing the operators in question as compact perturbations of multiplication operators, as in [13].

Recently, some results related to the present paper were obtained independently by Juang and Jonckheere [8] but are limited to rational plants.

2. Essential spectrum

Let Π_+ and Π_- denote projections from L^2 (half-plane) to H^2 and $H^2_- := L^2 \ominus H^2$ respectively. For any $W \in H^\infty$, W denotes the multiplication operator in L^2, $W(x) = Wx$. For any symbol $W \in L^\infty$, the *Hankel* operator $\Gamma_W : H^2 \to H^2_-$ is the operator

$$\Gamma_W := \Pi_- W \mid H^2$$

where $\mid H^2$ denotes restriction to H^2, and the *Toeplitz* operator Θ_W is the operator

$$\Theta_W := \Pi_+ W \mid H^2.$$

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We wish to compute the norm of the operator $G: H^2 \to H^2$,

$$G := \Gamma^*_{WM^*}\Gamma_{WM^*} + \Theta^*_V\Theta_V \tag{1}$$

where $W \in H^\infty$ and $V \in H^\infty$ are continuous, and $M \in H^\infty$ is a (possibly discontinous) inner function in H^∞. The superscript * denotes the adjoint of an operator or, when applied to a function $y \in L^2$, denotes the involution $y^*(s) = \bar{y}(-\bar{s})$. Expression (1) can be stated in the form

$$G = (\Pi_+ MW^*\Pi_- WM^* + \Pi_+ V^*V) \mid H^2. \tag{2}$$

Decompose H^2, $H^2 = K \oplus MH^2$, and let Π_K, Π_M be the projection operators from L^2 to K and MH^2 respectively. Let $\sigma(X)$, $\sigma_e(X)$ denote the spectrum and essential spectrum of any operator X. For any inner function $M \in H^\infty$, $\sigma_e(M)$ is the set of imaginary points which are essential singularities of M.

Theorem 1. Essential spectrum of G.

$$\sigma_e(G) = \{ \mid W(j\omega) \mid^2 + \mid V(j\omega) \mid^2 : j\omega \in \sigma_e(M)\} \cup \left[\inf_\omega \mid V(j\omega)\mid^2,\ \sup_\omega \mid V(j\omega) \mid^2\right].$$

Proof. Let $Z \in H^\infty$ be the outer function satisfying $\overline{W}(j\omega)W(j\omega) + \overline{V}(j\omega)V(j\omega) = \overline{Z}(j)Z(j\omega)$. We will establish the following three identities:

$$\sigma_e(G) = \sigma_e[\Pi_K(W^*W + V^*V) \mid K] \cup \sigma_e[V^*V], \tag{2a}$$

$$\begin{aligned}\sigma_e[\Pi_K(W^*W + V^*V) \mid K] &= \{ \mid \overline{W}(j\omega)W(j\omega) + \overline{V}(j\omega)V(j\omega) \mid : j\omega \in \sigma_e(M)\} \\ &= \sigma_e(\Gamma^*_{ZM^*}\Gamma_{ZM^*} \mid K),\end{aligned} \tag{2b}$$

$$\sigma_e(V^*V) = \left[\inf_\omega \mid V(j\omega) \mid^2,\ \sup_\omega \mid V(j\omega) \mid^2\right]. \tag{2c}$$

Theorem 1 follows from (2a, b, c).

If X, Y are any pair of operators in a Hilbert space, then $X \sim Y$ means that $X - Y$ is compact. The symbol $\sim$ denotes equivalence modulo the compact operators (i.e., in a Calkin Algebra). It follows from the definition of essential spectrum that if $X \sim Y$, then X and Y have identical essential spectra.

To prove (2a), observe that

$$\begin{aligned}\Pi_+ MW^*\Pi_- WM^*\Pi_+ &= \Pi_K W^*M\Pi_- M^*W\Pi_K \quad \text{as } K \perp \ker(\Pi_- WM^*\Pi_+) \\ &= \Pi_K W^*W\Pi_K - \Pi_K W^*M\Pi_+ WM^*\Pi_K \\ &\sim \Pi_K W^*W\Pi_K.\end{aligned} \tag{3}$$

The last equivalence is true because

$$\Pi_K W^*M\Pi_+ WM^*\Pi_K = \Pi_K W^*M(\Pi_+ W\Pi_-)M^*\Pi_K = \Pi_K W^*M\Gamma^*_{W^*}M^*\Pi_K$$

which is compact because it contains the factor [1] $\Gamma^*_{W^*}$, which is compact as W is continuous. Next

$$\begin{aligned}\Pi_+ V^*V\Pi_+ &= \Pi_K V^*V\Pi_K + \Pi_M V^*V\Pi_M + \Pi_K V^*V\Pi_M + \Pi_M V^*V\Pi_K \\ &\sim \Pi_K V^*V\Pi_K + \Pi_M V^*V\Pi_M\end{aligned} \tag{4}$$

because

$$\Pi_M V^*V\Pi_K = M(\Pi_+ M^*V^*VM\Pi_-)M^*\Pi_+ = M\Gamma^*_{(V^*V)^*}M^*\Pi_+ \tag{5}$$

which is compact since $\Gamma_{(V^*V)^*}$ is. Similarily the adjoint of (5), $\Pi_K V^*V\Pi_M$, is compact. From (2), (3), (4),

$$G \sim [\Pi_K(W^*W + V^*V)\Pi_K + \Pi_M V^*V\Pi_M] \mid H^2 =: G_1. \tag{6}$$

[1] The adjoint $X^*: H^2_- \to H^2$ of any operator $X: H^2 \to H^2_-$ satisfies $\langle y, Xx\rangle_{H^2_-} = \langle X^*y, x\rangle_{H^2}$. In particular, $\Gamma^*_{W^*} = \Pi_+ W\Pi_-$.

Clearly the subspaces K and $H^2 \ominus MH^2$ reduce G_1, and $\sigma_e(G)$ is therefore the union of the essential spectra of the operators $\Pi_K(W^*W+V^*V)\,|\,K$ and $\Pi_M V^*V\,|\,MH^2$. It remains only to show that

$$\sigma_e\left(\Pi_M V^*V\,|\,MH^2\right)=\sigma_e(V^*V). \tag{7}$$

Recall that any selfadjoint operator X, $\lambda\in\sigma_e(X)$ iff there is a normalized ($\|x_i\|=1$) sequence, $x_i\to 0$ weakly, $(X-\lambda)x_i\to 0$ strongly. Therefore $\lambda\in\sigma_e(\Pi_M V^*V\,|\,MH^2)$ iff there is a normalized sequence $Mx_i\in MH^2$, $Mx_i\to 0$ weakly and

$$\Pi_M V^*VMx_i-\lambda Mx_i\to 0 \tag{8}$$

strongly. Now (8) is equivalent to

$$V^*VMx_i-\lambda Mx_i\to 0 \tag{9}$$

strongly, because the difference

$$V^*V\Pi_M-\Pi_M V^*V\Pi_M=\Pi_- V^*V\Pi_M+\Pi_K V^*V\Pi_M \tag{10}$$

is compact by (5) and the compactness of $\Gamma_{V^*V}\Pi_M$, and therefore $Mx_i\to 0$ weakly and is bounded implies that $V^*VMx_i\to\Pi_M V^*VMx_i$ strongly. Finally, the weak (strong) convergence to 0 of x_i is equivalent to the weak (strong) convergence to 0 of Mx_i, so (9) is equivalent to $V^*Vx_i-\lambda x_i\to 0$ strongly, $x_i\to 0$ weakly, $\|x_i\|=1$, which means that $\lambda\in\sigma_c(V^*V)$. This proves (2a).

To prove (2b), observe that

$$\Pi_K(W^*W+V^*V)\,|\,K=(\Pi_K Z^*M\Pi_- M^*Z+\Pi_K Z^*M\Pi_+ M^*Z)\,|\,K\sim\Gamma_{ZM^*}^*\Gamma_{zM^*}\,|\,K$$

because

$$\Pi_K Z^*M\Pi_+ M^*Z\,|\,K=\Pi_K Z^*M\Gamma_{Z^*}^*M^*\,|\,K \tag{11}$$

and Γ_{Z^*} is compact so (11) is compact.

We now employ the essential spectral mapping theorem for continuous functions of the shift, see [10, p. 125]: If F is any continuous complex-valued function on $(-\infty,\infty)$, then $\sigma_e(\Pi_K F\,|\,K)=F(\sigma_e(M))$. By letting $F=\overline{W}W+\overline{V}V$ we obtain the first identity of (2b); the proof of the second identitiy is similar to that of (3), but with Z replacing W,

$$\sigma_e(\Gamma_{ZM^*}^*\Gamma_{ZM^*}\,|\,K)=\left\{\,|Z(\mathrm{j}\omega)|^2:\mathrm{j}\omega\in\sigma_e(M)\right\}.$$

(2c) is a standard result for multiplication operators in L^2 which are real valued on the imaginary axis [11, p. 55]. □

3. Eigenvectors and norm of G

The essential spectral radius of any operator X is $\rho_e(X):=\sup|\sigma_e(X)|$. By Theorem 1,

$$\rho_e(G)=\max\left[\,\|V\|_\infty^2,\ \sup\{(|W(\mathrm{j}\omega)|^2+|V(\mathrm{j}\omega)|^2):\ \omega\in\sigma_e(M)\}\right].$$

Since G is a self-adjoint bounded operator, it follows from the definition of essential spectrum [10, pp. 304, 313] that $\|G\|\ge\rho_e(G)$, and the inequality is strict iff G has an eigenvalue λ^2, $\lambda^2>\rho_e(G)$, in which case $\|G\|=\max\{\lambda^2>\rho_e(G):\ Gx=\lambda^2x,\ x\in H^2\}$. We seek a test for such eigenvalues, which are necessarily of finite muliplicity and isolated in $\sigma(G)$ (or they would belong to $\sigma_e(G)$).

λ^2 is an eigenvalue of G if the equation

$$(\Gamma_{WM^*}^*\Gamma_{WM^*}+\Pi_+V^*V)x=\lambda^2x,\quad x\in H^2, \tag{12}$$

has a solution for x.

Henceforth suppose that W and V are rational though the inner function M may be irrational and that $\lambda > \|V\|_\infty$. Denote the order of any rational F by N_F, and let $N := N_W + N_V$. Let B_λ be the Blaschke product whose zeros are those zeros of $\lambda^2 - V^*(s)V(s)$ lying in $\mathrm{Re}(s) > 0$.

Lemma 1. *If λ^2 is an eigenvalue of G, then the associated eigenvector lies in $H^2 \ominus B_\lambda MH^2 =: K_\lambda$.*

Proof. If λ^2 is an eigenvalue, then (12) gives

$$\Gamma_{WM^*}^*\Gamma_{WM^*}x - \Pi_- V^*Vx = (\lambda^2 - V^*V)x.$$

Als $\lambda > \|V\|_\infty$, this is equivalent to

$$B_\lambda^* M^* x = (\lambda^2 - V^*V)^{-1} B_\lambda^* (M^*\Gamma_{WM^*}^*\Gamma_{WM^*}x - M^*\Pi_- V^*Vx). \tag{13}$$

Note that the factors $(M^*\Gamma_{WM^*}^*\Gamma_{WM^*}x - M^*\Pi_- V^*Vx)$ and $(\lambda^2 - V^*V)^{-1}B_\lambda^*$ on the right-hand side of (13) are in H_-^2. Therefore, $x \in B_\lambda MH_-^2 \cap H^2 = H^2 \ominus B_\lambda MH^2$. □

Lemma 2. *$G \mid K_\lambda$ is a finite-rank perturbation of the multiplication operator $(W^*W + V^*V) \mid K_\lambda$. Indeed,*

$$G\Pi_\lambda = (W^*W + V^*V)\Pi_\lambda - \Delta_\lambda \tag{14}$$

where $\mathrm{rank}(\Delta_\lambda) \le 2N$,

$$\Delta_\lambda = \left[\Gamma_{W^*W+V^*V} + (W^* - \Gamma_{W^*})M\Gamma_{W^*B_\lambda^*}^* B_\lambda^* M^*\right]\Pi_\lambda, \tag{15}$$

and Π_λ denotes the projection operator form L^2 to K_λ.

Proof. We have

$$\begin{aligned} G\Pi_\lambda &= \Pi_+ W^*M\Pi_- M^*W\Pi_\lambda + \Pi_+ V^*V\Pi_\lambda \\ &= \Pi_+(W^*W + V^*V)\Pi_\lambda - \Pi_+ W^*M\Pi_+ M^*W\Pi_\lambda \\ &= (W^*W + V^*V)\Pi_\lambda - \Pi_-(W^*W + V^*V)\Pi_\lambda - \Pi_+ W^*M\Pi_+ M^*W\Pi_\lambda. \end{aligned}$$

Hence (14) is true with Δ_λ given by

$$\Delta_\lambda = \Pi_-(W^*W + V^*V)\Pi_\lambda + (W^*M - \Pi_- W^*M)\Pi_+ WB_\lambda \Pi_- B_\lambda^* M^* \Pi_\lambda \tag{16}$$

which coincides with (15), and where we have used the identity $B_\lambda^* M^* \Pi_\lambda = \Pi_- B_\lambda^* M^* \Pi_\lambda$. The rank bound follows from the bounds $\mathrm{rank}(\Gamma_{W^*W-V^*V}) \le N_W + N_V$, $\mathrm{rank}(\Gamma_{W^*}B_\lambda^*) \le N_W + N_{B_\lambda}$, $N_{B_\lambda} \le N_V$ applied to the expression (15) for Δ_λ. □

For simplicity, we will asume the generic case in which the poles and zeros of W, W^*, V, V^*, and M are simple, distinct from each other, and the (possibly multiple) zeros of $(W^*W + V^*V - \lambda^2)(s)$, $0 \le \lambda < \infty$, are isolated from the poles of M. The more general case can be treated as in [13].

Lemma 3. *The range Δ_λ admits a basis of functions $\psi_i \in L^\infty$, $i = 1, \ldots, 2N$, which are explicitly given in the Appendix, and which are analytic at all complex points at which W, W^*, V^*, B_λ, and M are nonsingular.*

Proof. It is shown in the Appendix that the range of Δ_λ is spanned by functions $\psi_i(s)$ which are finite forms in $W(s)$, $M(s)$, and $(s + \eta_i)^{-1}$, $i = 1, \ldots, 2N$, where η_i are singularities of W, W^*, V^*, B_λ or $M(s)$. Each $\psi_i(s)$ is therefore meromorphic in $\mathrm{Re}(s) \ne 0$ and analytic except at these singularities. The set $\{\psi_i(s)\}$ is independent in L^∞ and therefore forms a basis, as each ψ_i has a pole not present in the others under the genericity assumption. □

By Lemmas 1–3, if λ^2 is an eigenvalue of G with eigenvector x, then the equation $Gx = \lambda^2 x$ is equivalent to

$$[(W^*W + V^*V - \lambda^2)x](s) = (\Delta_\lambda x)(s) = \sum_{i=1}^{2N} \zeta_i^\lambda \psi_i^\lambda(s). \tag{17}$$

Let s_i^λ, $i = 1, \ldots, 2N$, denote the $2N$ zeros of $(W^*W + V^*V - \lambda^2)(s)$. If λ^2 is discrete, then these zeros are isolated from $\sigma_e(M)$, because $\lambda^2 \notin \sigma_e(W^*W + V^*V \mid K_\lambda)$, by Theorem 1. Therefore each s_i^λ is a point of analyticity of $M(s)$ and, under the genericity assumption, a point of analyticity of each $\psi_i(s)$ and hence of (17). If s_i^λ lies in the half-plane $\mathrm{Re}(s_i^\lambda) \geq 0$, where $x \in H^2$ is bounded, s_i^λ must be a zero of (17). A similar conclusion is reached for the other half-plane by multiplying (17) by $(M^*B_\lambda^*)(s)$, noting that $M^*B_\lambda^* x \in H_-^2$ for $x \in K_\lambda$, and that s_i^λ is disjoint from the singularities of $(M^*B_\lambda^*)(s)$. Hence we get $2N$ equations in as many coefficients ζ_i^λ,

$$\sum_{j=1}^{2N} \zeta_j^\lambda \psi_j^\lambda(s_i^\lambda) = 0, \quad i = 1, \ldots, 2N. \tag{18}$$

Introduce the $2N \times 2N$ matrix $A(\lambda) := [\psi_j^\lambda(s_i^\lambda)]$ and the $2N \times 1$ matrix $\zeta^\lambda := [\zeta_j^\lambda]$ to get the matrix equation

$$A(\lambda)\zeta^\lambda = 0. \tag{19}$$

The zeros s_i^λ lie on the root-locus of $(W^*W + V^*V - \lambda^2)(s)$, and are distinct except at a finite number of values of λ^2. At any λ^2 at which s_i^λ is a zero of multiplicity r, $(r-1)$ derivatives of (18) must vanish at s_i^λ. Write the resulting matrix equation as

$$A'(\lambda)\zeta^\lambda = 0. \tag{20}$$

It follows that if λ^2 is a discrete eigenvalue then $\det A(\lambda) = 0$ and, if any root s_i^λ of $(W^*W + V^*V - \lambda^2)(s) = 0$ is multiple, then $\det A'(\lambda) = 0$. Conversely, if these determinants are null then there exists ζ_i^λ satisfying (18), (19). In that case the roots s_i^λ are zeros of both sides of (17), and the ratio

$$x := \left(\sum_{j=1}^{2N} \zeta_j^\lambda \psi_j^\lambda(s_i^\lambda) \right) (W^*W + V^*V - \lambda^2)^{-1}$$

defines a function $x \in H^2$ which satisfies $Gx = \lambda^2 x$. Therefore we get the following result.

Theorem 2. *The discrete eigenvalues of G are the values of λ^2 in the complement of $\sigma_e(G)$ at which* $\det A(\lambda) = 0$, *and at which* $\det A'(\lambda) = 0$ *whenever s_i^λ is a multiple zero of $W^*(s)W(s) + V^*(s)V(s) - \lambda^2$. Moreover,*

$$\|G\| = \max(\rho_e(G), \lambda^2_{\max})$$

where $\lambda^2_{\max}$ is the largest eigenvalue.

Note that $\sigma_e(G)$ is determined by Theorem 1. The characteristic determinant is analytic in λ except where s_i^λ is a multiple zero. (Alternatively, the function $\prod_{i \neq j}(s_i^\lambda - s_j^\lambda)^{-1} \det A(\lambda)$ is analytic in λ for all $\lambda \notin \sigma_e(G)$, and the zeros of this function are the discrete eigenvalues of G.)

Appendix. Evaluation of the basis $\psi_i^\lambda(s)$

For any rational $F \in L^\infty$, the notation η_i^F, $i = 1, \ldots, N_F$, will denote an ordered enumeration of the poles of F, and $R^F(\eta_i)$ the residue of F at the pole η_i. For any $x \in K_\lambda$, we evaluate the components of $\Delta_\lambda x$

appearing on the right-hand side of (16) by contour integration. We get

$$(\Pi_- W^*Wx)(s) = \sum_{i=1}^{N_{W^*}} R^{W^*}(\eta_i^{W^*})W(\eta_i^{W^*})(s-\eta_i^{W^*})^{-1}x(\eta_i^{W^*}), \tag{A1}$$

$$(\Pi_- V^*Vx)(s) = \sum_{i=1}^{N_{V^*}} R^{V^*}(\eta_i^{V^*})V(\eta_i^{V^*})(s-\eta_i^{V^*})x(\eta_i^{V^*}), \tag{A2}$$

$$\begin{aligned} &-(\Pi_- W^*M\Pi_+ WB_\lambda\Pi_- B_\lambda^* M^* x)(s) \\ &= -\sum_{i=1}^{N_{W^*}} (s-\eta_i^{W^*})^{-1}R^{W^*}(\eta_i^{W^*})M(\eta_i^{W^*}) \\ &\quad\times\left\{\sum_{j=1}^{N_W}(\eta_i^{W^*}+\eta_j^W)^{-1}M^*(\eta_j^W)x(\mu_j^W) + \sum_{k=1}^{N_{B_\lambda}}(\eta_i^{W^*}-\eta_k^{B_\lambda})^{-1}W(\eta_k^{B_\lambda})M^*(\eta_k^{B_\lambda})R^x(\eta_k^{B_\lambda})\right\}, \end{aligned} \tag{A3}$$

$$\begin{aligned} (W^*M\Pi_+ WB_\lambda\Pi_- B_\lambda^* M^* x)(s) = W^*(s)M(s)&\left\{\sum_{i=1}^{N_W}(s-\eta_i^W)^{-1}R^W(\eta_i^W)M^*(\eta_i^W)x(\eta_i^W)\right. \\ &\left.+\sum_{j=1}^{N_{B_\lambda}}(s-\eta_j^{B_\lambda})^{-1}W(\eta_j^{B_\lambda})M^*(\eta_j^{B_\lambda})R^x(\eta_j^{B_\lambda})\right\}. \end{aligned} \tag{A4}$$

Now let η_i^λ, $i=1,\ldots,2N$, be the ordered set of poles formed from the sets $\{\eta_i^{W^*}\}$, $\{\eta_j^W\}$, $\{\eta_k^{V^*}\}$, and $\{\eta_l^{B_\lambda}\}$ in sequence, and let ζ_i^λ be the coefficients

$$\zeta_i^\lambda = \begin{cases} x(\eta_i^\lambda), & i=1,\ldots,N_V+2N_W, \\ R^x(\eta_i^{B_\lambda}), & i=N_V+2N_W+1,\ldots,2N. \end{cases}$$

(A1)–(A4) are summed to get the result that $(\Delta_\lambda x)(s) = \sum_{i=1}^{2N}\zeta_i^\lambda\psi_i^\lambda(s)$ where ψ_i^λ is the sum of all terms in (A1)–(A4) multiplying ζ_i^λ, namely:

For $\zeta_i^\lambda = x(\eta_i^{W^*})$,

$$\psi_i^\lambda(s) = R^{W^*}(\eta_i^{W^*})W(\eta_i^{W^*})(s-\eta_i^{W^*})^{-1}.$$

For $\zeta_i^\lambda = x(\eta_j^W)$,

$$\begin{aligned}\psi_i^\lambda(s) = &-\sum_{k=1}^{N_{W^*}}(s-\eta_k^{W^*})^{-1}R^{W^*}(\eta_k^{W^*})M(\eta_k^{W^*})(\eta_k^{W^*}-\eta_j^W)^{-1}M^*(\eta_j^W) \\ &+W^*(s)M(s)(s-\eta_j^W)^{-1}R^W(\eta_j^W)M^*(\eta_j^W).\end{aligned}$$

For $\zeta_i^\lambda = x(\eta_k^{V^*})$,

$$\psi_i^\lambda(s) = R^{V^*}(\eta_k^{V^*})V(\eta_k^{V^*})(s-\eta_k^{V^*}).$$

For $\zeta_i^\lambda = R^x(\eta_l^{B_\lambda})$,

$$\begin{aligned}\psi_i^\lambda(s) = &-\sum_{j=1}^{N_{W^*}}(s-\eta_j^{W^*})^{-1}R^{W^*}(\eta_j^{W^*})M(\eta_j^{W^*})(\eta_j^{W^*}-\eta_l^{B_\lambda})^{-1}W(\eta_l^{B_\lambda})M^*(\eta_l^{B_\lambda}) \\ &+W^*(s)M(s)(s-\eta_l^{B_\lambda})^{-1}W(\eta_l^{B_\lambda})M^*(\eta_l^{B_\lambda}).\end{aligned}$$

References

[1] D.S. Flamm. Control of delay systems for minimax sensitivity, M.I.T. Ph.D. Thesis and L.I.D.S. Report (June 1986).

[2] C. Foias, A. Tannenbaum and G. Zames, Some explicit formulae for the singular values of certain Hankel operators with factorizable symbol, *SIAM J. Math. Anal.*, submitted (1987).

[3] C. Foias, A. Tannenbaum and G. Zames, On the H^∞-optimal sensitivity problem for systems with delays *SIAM J. Control* **25** (3) (1987) 686–705.

[4] A. Feintuch and B.A. Francis, Uniformly optimal control of linear systems, *Automatica* **21** (5) (1985) 563–574.

[5] B.A. Francis, Notes on H^∞-optimal linear feedback systems, Lecture Notes Linköping University (1983).

[6] B.A. Francis and J. Doyle. Linear control theory with an H^∞ optimality criterion, Systems Control Group Report #8501, University of Toronto (Oct. 1985). To appear in *SIAM J. Control.*

[7] E.A. Jonckheere and M. Verma, A spectral characterization of H^∞ optimal feedback performance and its efficient computation, *Systems Control Lett.* **8** (1986) 13–22.

[8] J.C. Juang and E.A. Jonckheere, On computing the spectral radius of the Hankel plus Toeplitz operator, U.S.C. Dept. of E.E. Report (March 1987).

[9] H. Kwakernaak, Minimax frequency domain performance and robustness optimization of linear feedback systems, *IEEE Trans. Automat. Control* **30** (1985) 994–1004.

[10] N.K. Nikolskii, *Treatise on the Shift Operator* (Springer, Berlin–New York, 1980).

[11] S.C. Power, *Hankel Operators on Hilbert space* (Pitman, London, 1982).

[12] M.S. Verma and E. Jonckheere, L^∞ compensation with mixed sensitivity as a broad band matching problem, *Systems Control Lett.* **4** (1984) 125–129.

[13] G. Zames, A. Tannenbaum and C. Foias, Optimal H^∞ interpolation: A new approach, *Proc. 25th Conf. Dec. Control* (1986) pp. 350–355.