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Linear systems over Noetherian rings in the behavioural approach

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Abstract

In this paper we will study dynamical systems over Noetherian rings. We will follow the behavioural approach. We first study two particular cases: autonomous systems and controllable systems. In the first case we will be able to connect autonomous systems with finitely generated systems. Moreover we will propose several different concepts of controllability and analyze how they are connected each other. Finally we will define the concept of state space module. In this way we will say that the system is realizable if its state module is finitely generated. The last part of the paper is devoted to the analysis of the properties that ensure the realizability of the system.

1 Introduction

The strict relationship between system theory and coding theory is something known from the early 70's. Actually it is easy to see that an encoder for a convolutional code can be seen as a linear system over a finite field and so much of the knowledge on the properties of linear systems could be used in the development of convolutional codes [6].

Since codes over finite fields can be sometimes too restrictive, some researchers tried to develop a system theory over groups and rings [2, 13, 12] which produced a number of very interesting results. They showed that many properties of linear systems over a field hold true also for systems over groups or over rings.

In the last years the behavioural approach to dynamical systems has been the object of much investigation. Actually, this approach constitutes an alternative

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framework for modelling phenomena that seems to be more effective when there exists an unclear distinction between causes and effects (see [15]). Also optimal control techniques and modelling procedures have been proposed based on this approach.

Very recently it has been also realized that the behavioural approach to system theory is a very useful framework where one can develop a more general theory of convolutional codes (see [7]). Therefore it seems quite interesting to extend this approach, originally introduced for systems over general fields and in particular over the real field, to more general structures, such as groups or rings. In [11, 7] it is shown that many properties of linear systems over fields hold true even for over noncommutative groups. The most important result in this sense is the fact that even in this generality it is possible to define a canonical state space group. It is shown moreover how the state group can be connected with the trellis diagrams of the convolutional code.

It is clear that the same can be done for systems over a generic ring and that in this case the state space is a module. The condition that the state space module is finitely generated is a reasonable requirement. This is similar to the concept of realizability of a input/output map in the classical input/output approach as defined in [2, 13, 12].

In this paper we study linear systems over Noetherian rings in the behavioural approach focusing our attention on the state realizability problem. We first analyze the properties of two particular kinds of systems: the autonomous systems and the controllable systems. The main result about autonomous systems is provided by the theorem showing that there exists a strict relationship between an autonomous system over a Noetherian domain and a corresponding linear system over the field of fractions. This seems to give a first extension to the behavioural approach of the classical Rouchaleau-Kalman-Wyman theorem [2, 13, 12]. Moreover we propose different concepts of controllability and give the relations existing between them. We show then that, as for systems over fields, there exists a nice image representation for controllable systems. We define moreover the concept of controllable subsystem. Finally, in the last part of the paper, we study the realizability of a linear system over a Noetherian ring, i.e. when the canonical state space is finitely generated module.

2 Basic definitions

In this section we will first recall some basic concepts of behavioural theory of dynamical systems as it has been introduced by Willems in [15] and then we will introduce some more specific ideas for systems over rings according this approach.

Before we give some notation that will be used in the sequel. In this paper we will consider only commutative Noetherian rings. For the elementary results of commutative algebra we will need in this paper awe will refer to [1]. Given a ring

R, we say that $x \in R$ is a zero-divisor in R, if there exists $y \neq 0$ in R such that $xy = 0$. If R is a domain, then no nonzero element of R is zero-divisor. With the symbol $R[z, z^{-1}]$ we will denote the ring of all Laurent polynomials with coefficients in R, i.e. the ring of polynomials for which we allow positive and negative powers of the indeterminate z. More formally the ring $R[z, z^{-1}]$ can be considered the ring of fraction of $R[z]$ with respect to the multiplicatively closed set $S = \{z^i : i \in \mathbb{N}\}$ (see chapter 3 of [1]). It follows from Hilbert basis theorem and from prop. 7.3 in [1] that $R[z, z^{-1}]$ is Noetherian, if R is. Every nonzero $p \in R[z, z^{-1}]$ can be represented as

$$p = \sum_{i=l}^{L} p_i z^i,$$

where $p_i \in R$ and p_l, p_L are nonzero. In this case, we denote by $\deg p$ the nonnegative integer $L - l$. If both the nonzero coefficients p_l, p_L are not zero-divisors in R, then p is called normal.

A dynamical system is defined as a triple $\Sigma = (T, W, \mathcal{B})$, where T is the time set, W is the signal alphabet, i.e. the space where the signals take their values and finally $\mathcal{B}$ is a subset of the set W^T of all the signals and it describes the dynamics of the system simply specifying what are the signals that are allowed. In this paper we will consider a particular kind of dynamical systems, i.e. linear shift-invariant systems over rings.

More precisely we will require that

- The time set T is the set of integers $\mathbb{Z}$. Dynamical systems whose time set is $\mathbb{Z}$ are called discrete.

- The signal alphabet W is a finite generated module over a Noetherian ring R.

- On the set of all signals $W^{\mathbb{Z}}$ can be introduced a module structure over the ring $R[z, z^{-1}]$ of the Laurent polynomials over R as follows:

  Both the sum of two signals in $W^{\mathbb{Z}}$ and the product of an element of R and a signal in $W^{\mathbb{Z}}$ are done pointwise while for every $w \in W^{\mathbb{Z}}$ we define $z^h w \in W^{\mathbb{Z}}$ as

  $$(z^h w)(t) := w(t + h)$$

  and then we define the product of a Laurent polynomial in $R[z, z^{-1}]$ and a signal in $W^{\mathbb{Z}}$ by extending by linearity the previous definitions. More precisely if $p = \sum_{i=l}^{L} p_i z^i$ is a polynomial in $R[z, z^{-1}]$ and $w \in W^{\mathbb{Z}}$, then

  $$(pw)(t) = \sum_{i=l}^{L} p_i w(t + i).$$

  We will require that the behaviour is a $R[z, z^{-1}]$ submodule of $W^{\mathbb{Z}}$.

In case that R is a field, we obtain the linear shift-invariant systems as they have been introduced by Willems.

An important property of linear shift-invariant systems that is useful to consider is the completness.

Definition 1 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be any dynamical system. Then*

1. *Σ is complete if*

$$w \in \mathcal{B} \quad \Leftrightarrow \quad w_{|I} \in \mathcal{B}_{|I} \ for\ all\ finite\ I \subseteq \mathbb{Z}.$$

2. *Σ is L-complete with $L \in \mathbb{N}$ if*

$$w \in \mathcal{B} \quad \Leftrightarrow \quad w_{|I} \in \mathcal{B}_{|[t,t+L)} \ for\ all\ t \in \mathbb{Z}.$$

3. *Σ is strongly complete if Σ is L-complete for some $L \in \mathbb{N}$.*

It is clear that a complete linear shift-invariant system is fixed by the countable family of finitely generated R-submodules $\mathcal{B}_{|[-n,n]} \subseteq W^{2n+1}$ for $n = 0, 1, 2, ...$ while if Σ is a L-complete linear shift-invariant system, then it is fixed by the finitely generated R-submodule $\mathcal{B}_{|[0,L]} \subseteq W^{L+1}$.

As shown in [15], for linear shift-invariant systems over fields completeness and strongly completeness are equivalent. This assertion can be generalized for systems over rings whose signal alphabet W is a module satisfying the descending chain condition. Examples of such modules are given by the finite modules. The proof of this equivalence is essentially similar to the one given in [14] for systems over fields.

Proposition 1 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant complete system over a ring R. If the module W satisfies the descending chain condition, then Σ is strongly complete.*

Proof Consider the modules

$$\mathcal{M}_n := \{w(n) : w \in \mathcal{B}, w_{|[0,n-1]} = 0\}.$$

It is clear that $\mathcal{M}_1 \supseteq \mathcal{M}_2 \supseteq \mathcal{M}_3 \supseteq \cdots$ and so there exists $N \in \mathbb{N}$ such that $\mathcal{M}_N = \mathcal{M}_{N+1} = \mathcal{M}_{N+2} = \cdots$. We want to show that Σ is N-complete. To this pourpose it is enough to show that if $w_{|[t,t+N]} \in \mathcal{B}_{|[t,t+N]}$ for all $t = 0, 1, \ldots, n$, then we have that $w_{|[0,n+N]} \in \mathcal{B}_{|[0,n+N]}$. We show this by induction on n. For $n = 0$ this is true. Suppose that the assertion is true for $n-1$ and suppose that $w_{|[t,t+N]} \in \mathcal{B}_{|[t,t+N]}$ for all $t = 0, 1, \ldots, n$. Then, by induction, we have that $w_{|[0,n+N-1]} \in \mathcal{B}_{|[0,n+N-1]}$ and so there exists $w_1 \in \mathcal{B}$ such that $w_{1|[0,n+N-1]} = w_{|[0,n+N-1]}$. On the other hand, since $w_{|[n,n+N]} \in \mathcal{B}_{|[n,n+N]}$, then there exists $w_2 \in \mathcal{B}$ such that $w_{2|[n,n+N]} = w_{|[n,n+N]}$. Let $w' := w_2 - w_1 \in \mathcal{B}$. Then $w'_{|[n,n+N-1]} = 0$. Since $\mathcal{M}_N = \mathcal{M}_{N+n}$, then there exists

$\bar{w} \in \mathcal{B}$ such that $\bar{w}_{|[0,n+N-1]} = 0$ and $\bar{w}(n+N) = w'(n+N)$. Let $w'' := w_1 + \bar{w} \in \mathcal{B}$. Then $w''_{|[0,n+N-1]} = w_{1|[0,n+N-1]} = w_{|[0,n+N-1]}$ and $w''(n+N) = w_1(n+N) + \bar{w}(n+N) = w_1(n+N) + w'(n+N) = w_1(n+N) + w_2(n+N) - w_1(n+N) = w_2(n+N)$. We can argue that $w''_{|[0,n+N]} = w_{|[0,n+N]}$ and so $w_{|[0,n+N]} \in \mathcal{B}_{|[0,n+N]}$.

■

Completeness and strongly completeness are not equivalent in general even for systems over the principal ideal domain $\mathbb{Z}$, as shown in [4].

3 Finitely generated and autonomous systems

In this section we want to study the properties of a particular class of linear shift-invariant systems, i.e. the linear shift-invariant system whose behaviour is a finitely generated R-submodule of $W^{\mathbb{Z}}$. First we show that this kind of systems admit a very nice representation that is a state space representation.

Suppose that $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a finitely generated linear shift-invariant system and let $\{w_1, \dots, w_n\}$ be a family of generators of the finitely generated R-module $\mathcal{B}$. Then for every generator w_i we have that $zw_i \in \mathcal{B}$ and so there exist $a_{i1}, \dots, a_{in} \in R$ such that

$$zw_i = a_{i1}w_1 + \dots + a_{in}w_n.$$

Consequently, if we define the matrix $A \in R^{n \times n}$ as $A := \{a_{ij}\}_{i,j=1}^{n}$, then we have that

$$z\,[\,w_1 \quad \cdots \quad w_n\,] = [\,w_1 \quad \cdots \quad w_n\,]\,A.$$

Note that A must be invertible and so $\det A$ is a unit in R.

Let w be any signal in $\mathcal{B}$. Then

$$w = \alpha_1 w_1 + \dots + \alpha_n w_n = [\,w_1 \quad \cdots \quad w_n\,]\,x_0,$$

where $x_0 := [\alpha_1, \dots, \alpha_n]^T \in R^n$. For all $t \in \mathbb{Z}$ we have that

$$\begin{aligned} w(t) &= (z^t w)(0) = ([\,w_1 \quad \cdots \quad w_n\,]\,A^t x_0)(0) = [\,w_1(0) \quad \cdots \quad w_n(0)\,]\,A^t x_0 \\ &= CA^t x_0, \end{aligned}$$

where we denote $C := [\,w_1(0) \quad \cdots \quad w_n(0)\,] \in W^{1 \times n}$. Therefore we have shown that $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a finitely generated linear shift-invariant system if and only if for some $n \in \mathbb{N}$ there exist an invertible $A \in R^{n \times n}$ and $C \in W^{1 \times n}$ such that

$$\mathcal{B} = \{w \in W^{\mathbb{Z}} \;:\; \exists x_0 \in R^n,\; w(t) = CA^t x_0,\; \forall t \in \mathbb{Z}\}. \tag{1}$$

In other words, all finitely generated linear shift-invariant systems admit a representation of the following kind:

$$\begin{cases} x(t+1) &= Ax(t) \\ w(t) &= Cx(t) \end{cases},$$

where $x \in (R^n)^{\mathbb{Z}}$.

The concept of autonomous system plays a central role in the description of finite dimensional systems over fields. This concept have been introduced in [15] in the general behavioural framework. We will give also a symmetric version of it.

Definition 2 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over R. Then*

1. *Σ is autonomous if*

$$w \in \mathcal{B},\ w_{|(-\infty,0]} = 0 \quad \Rightarrow \quad w = 0.$$

2. *Σ is symmetrically autonomous if*

$$w \in \mathcal{B},\ w_{|(-\infty,0]} = 0 \quad \Rightarrow \quad w = 0$$

and

$$w \in \mathcal{B},\ w_{|[0,+\infty)} = 0 \quad \Rightarrow \quad w = 0.$$

We have distinguished two different concepts of autonomous systems since they are not equivalent in general. The following proposition shows that if R is Noetherian and the system strongly complete, then the two concepts coincides. Moreover it clarifies the relation between finitely generated systems and autonomous systems.

Proposition 2 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then the following facts are equivalent:*

1. *Σ is finitely generated.*
2. *Σ is symmetrically autonomous and strongly complete.*
3. *Σ is autonomous and strongly complete.*

Proof (1$\Rightarrow$3) Since Σ is finitely generated, it admits the representation (1). We want to show now that Σ is $n-1$-complete. Let $p \in R[z, z^{-1}]$ be the characteristic polynomial of the matrix A. Then $p = p_0 + p_1 z + \cdots + p_n z^n$, with p_0, p_n invertible elements in R and by Caley-Hamilton theorem it is easy to see that

$$\bar{\mathcal{B}} := \{w \in W^{\mathbb{Z}} : pw = 0\} \supseteq \mathcal{B}.$$

It is clear that $\bar{\Sigma} = (\mathbb{Z}, W, \bar{\mathcal{B}})$ is n-complete. Suppose that $w_{|[t,t+n)} \in \mathcal{B}_{|[t,t+n)}$ for all $t \in \mathbb{Z}$. Then $w_{|[t,t+n)} \in \bar{\mathcal{B}}_{|[t,t+n)}$ for all $t \in \mathbb{Z}$ and so $w \in \bar{\mathcal{B}}$. Since $w_{|[0,n)} \in \mathcal{B}_{|[0,n)}$, there exists $w' \in \mathcal{B}$ such that $w'_{|[0,n)} = w_{|[0,n)}$. Let $\delta := w' - w$. Then $\delta \in \bar{\mathcal{B}}$ and $\delta_{|[0,n)} = 0$. It is easy to verify that, since p_0, p_n are invertible, this implies that $\delta = 0$ and so $w = w' \in \mathcal{B}$ Suppose that $w \in \mathcal{B}$ and that $w_{|(-\infty,0]} = 0$. Then $w \in \bar{\mathcal{B}}$ and

so $pw = 0$. Then it is easy to see that this implies that w must be zero and so $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is autonomous.
(3$\Rightarrow$2) Let

$$\mathcal{M}_k := \{w_{|[-L,0]} \in W^{L+1} : w_{|[k,+\infty)} = 0,\ w \in \mathcal{B}\}.$$

Since for all $k \in \mathbb{N}$ we have $\mathcal{M}_k \subseteq \mathcal{M}_{k+1}$, then there exists $T \in \mathbb{N}$ such that $\mathcal{M}_T = \mathcal{M}_{T+1} = \mathcal{M}_{T+2} = \cdots$. Suppose now that there exists $w \in \mathcal{B}$ such that $w_{|[T,+\infty)} = 0$. If we show that this implies that $w(T-1) = 0$, then we are done. In these hypotheses we have that $w_{|[-L,0]} \in \mathcal{M}_T$. Let $w' := z^{-1}w$. Then $w'_{|[-L,0]} \in \mathcal{M}_{T+1} = \mathcal{M}_T$ and so there exists $w'' \in \mathcal{B}$ such that $w''_{|[-L,0]} = w'_{|[-L,0]}$ and $w''_{|[T,+\infty)} = 0$. Let $\delta := w'' - w' \in \mathcal{B}$. Since $\delta_{|[-L,0]} = 0$, then by L-completeness we have that $\bar{\delta} \in W^{\mathbb{Z}}$ such that $\bar{\delta}_{|(-\infty,0]} = 0$ and $\bar{\delta}_{[-L,+\infty)} = \delta_{[-L,+\infty)}$ is in $\mathcal{B}$ and since Σ is autonomous, then $\bar{\delta} = 0$. We can argue that $\delta_{[-L,+\infty)} = 0$ and that $w(T-1) = w'(T) = w''(T) = 0$.
(2$\Rightarrow$1) Suppose that Σ is symmetrically autonomous and L-complete. Consider the R-homomorphism

$$\Phi\ :\ \mathcal{B}\ \rightarrow\ W^L\ :\ w\ \mapsto\ w_{|[0,L-1]}.$$

It is easy to see that this homomorphism is injective. Actually, suppose that $w \in \mathcal{B}$ and that $w_{|[0,L-1]} = 0$. Then, by L-completness, we have that there exists $\bar{w} \in \mathcal{B}$ such that $\bar{w}_{|(-\infty,L-1]} = 0$ and $\bar{w}_{|[L,+\infty)} = w_{|[L,+\infty)}$. We argue that $\bar{w} = 0$ and so $w_{|[L,+\infty)} = 0$. Consequently we have that $w = 0$. Now, since Φ is injective, we have that $\mathcal{B}$ is isomorphic to a submodule of W^L and so it is Noetherian (by prop. 6.7 in [1]).

■

We will show now another useful characterization ensuring that a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is finitely generated. This is connected with the ideal

$$Ann(\mathcal{B}) := \{p \in R[z, z^{-1}]\ :\ pw = 0,\ \forall\ w \in \mathcal{B}\},$$

of $R[z, z^{-1}]$.

Proposition 3 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then Σ is finitely generated if and only if $Ann(\mathcal{B})$ contains a normal polynomial.*

Proof Suppose first that Σ is finitely generated. Then it admits the representation (1). Let $p = \sum_{i=0}^{n} p_i z^i$ be the characteristic polynomial of A. Then by the Caley-Hamilton theorem (see [10]) we have that $p(A) = 0$ and so it is easy to see that $p \in Ann(\mathcal{B})$. Since p is a characteristic polynomial of a matrix, then $p_n = 1$. Moreover, since A is invertible, then it is easy to see that also $p_n = \det A$ is invertible. Hence p is a normal polynomial.
Suppose conversely that $p \in Ann(\mathcal{B})$ and that p is normal, i.e.

$$p = p_l z^l + p_{l+1} z^{l+1} + \cdots + p_{L-1} z^{L-1} + p_L z^L,$$

where both p_l, p_l are not zero-divisors in R. Then it can be seen that $\bar{\mathcal{B}} := \{w \in W^{\mathbb{Z}} \ : \ pw = 0\}$ is a finitely generated R-module containing $\mathcal{B}$. Actually, consider the map

$$\Phi \ : \ \mathcal{B} \ \rightarrow \ W^{L-l} \ : \ w \ \mapsto \ w_{|[l,L)}.$$

It is easy to see that it is an R homomorphism and that this homomorphism is injective. Suppose that $\Phi(w) = 0$. Then, since $w \in \bar{\mathcal{B}}$, then it satisfies the difference equation

$$p_l w(t+l) + p_{l+1} w(t+l+1) + \cdots + p_{L-1} w(t+L-1) + p_L w(t+L) = 0, \quad \forall t \in \mathbb{Z}.$$

If we apply this equation for $t = 0$ and we exploit the fact that $w(l) = w(l+1) = \cdots = w(L-1) = 0$ and that p_l, p_l are not zero-divisors in R, then we argue that $w(L) = 0$. We can repeat the same kind of argument, and using induction we see that $w(t) = 0$ for all $t \geq l$. In the same way we can prove that $w(t) = 0$ for all $t \leq l$ and so $w = 0$. By prop. 6.3 of [1] we have that $\bar{\mathcal{B}}$ and so also $\mathcal{B}$ are Noetherian and so are finitely generated over R.

■

If R is a domain, the equivalent characterization of the previous proposition become more simple as shown in the following proposition whose proof is a direct consequence of the previous proposition.

Proposition 4 *Let R be a Noetherian domain and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then Σ is finitely generated if and only if $Ann(\mathcal{B}) \neq \{0\}$.*

Let R be a domain and F its field of fractions. Suppose moreover that $W := R^q$, for some $q \in \mathbb{N}$. If $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$ is a linear shift-invariant system, define the system $\Sigma_e = (\mathbb{Z}, F^q, \mathcal{B}_e)$ as follows:

$$\mathcal{B}_e = \{\bar{a}w \in (F^q)^{\mathbb{Z}} \ : \ \bar{a} \in F, \ w \in \mathcal{B}\},$$

and let $\Sigma_{ce} = (\mathbb{Z}, F^q, \mathcal{B}_{ce})$ be the system such that $\mathcal{B}_{ce} = CP(\mathcal{B}_e)$ is the completion of $\mathcal{B}_e$, i.e. $CP(\mathcal{B}_e)$ is the smallest complete behaviour containing $\mathcal{B}_e$. More explicitly if $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is a linear shift-invariant system we define

$$CP(\mathcal{B}_e) = \{w \in W^{\mathbb{Z}} : w_{|I} \in \mathcal{B}_{|I}, \text{ for all finite } I \subseteq \mathbb{Z}\}.$$

We will call Σ_e and Σ_{ce} field extension and complete field extension of Σ respectively. It is clear that Σ_e and Σ_{ce} are linear shift-invariant systems on the field F and that Σ_{ce} is also complete and can be studied using all the techniques that are available for these kind of dynamical systems (see [15]). Therefore it is useful to connect properties of the system Σ with the properties its field extensions. For finitely generated dynamical systems this is provided by the following theorem, that seems to have some connections with the classical Rouchaleau-Kalman-Wyman theorem (see [13]).

Theorem 1 *Let R be a Noetherian domain. $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$ be linear shift-invariant system, $\Sigma_e = (\mathbb{Z}, F^q, \mathcal{B}_e)$ be its field extension and $\Sigma_{ce} = (\mathbb{Z}, F^q, \mathcal{B}_{ce})$ be its complete field extension. Then the following facts are equivalent:*

1. *Σ is finitely generated.*
2. *Σ_e is finitely generated.*
3. *Σ_{ce} is finitely generated.*

If one of the previous equivalent conditions holds, then $\Sigma_e = \Sigma_{ce}$.

Proof ($1 \Rightarrow 2$) If $\mathcal{B}$ is generated as an R module by $w_1, \ldots, w_n$, then also $\mathcal{B}_e$ is generated by $w_1, \ldots, w_n$ as an F-vector space.
($2 \Rightarrow 3$) By proposition 2, since $\mathcal{B}_e$ is finitely generated, then it is strongly complete and so $\mathcal{B}_{ce} = CP(\mathcal{B}_e) = \mathcal{B}_e$.
($3 \Rightarrow 1$) If $\mathcal{B}_{ce}$ is finitely generated, then, by the previous proposition, $Ann(\mathcal{B}_{ce}) = \{\bar{p} \in F[z, z^{-1}] : \bar{p}\bar{w} = 0, \ \forall \bar{w} \in \mathcal{B}_{ce}\} \neq \{0\}$. Let $\bar{p}$ be a nonzero element of $Ann(\mathcal{B}_{ce})$. Then there exists $a \in R$ such that $p = a\bar{p} \in R[z, z^{-1}]$ and so $p \in Ann(\mathcal{B})$. By proposition 4 we argue that Σ is finitely generated. ∎

If we suppose that the linear shift-invariant system $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$ is L-complete, then there exists a efficient way for checking if Σ is finitely generated when the ring R is a domain, using the previous theorem. First note that in general if Σ is L-complete, then Σ_{ce} is $L+1$-complete. Actually, since Σ_{ce} is complete, then, as shown by Willems, it is strongly complete and so l-complete for some $l \in \mathbb{N}$. If $l \leq L+1$, then Σ_{ce} is also $L+1$-complete. Suppose that $l > L+1$ and consider a trajectory $\bar{w}$ in $(F^q)^{\mathbb{Z}}$ such that $\bar{w}_{|[t,t+L]} \in \mathcal{B}_{ce|[t,t+L]} = \mathcal{B}_{e|[t,t+L]}$ for all $t \in \mathbb{Z}$. Then for all t there exists $r_t \in R$ such that $r_t\bar{w}_{|[t,t+L]} \in \mathcal{B}_{|[t,t+L]}$. Let $\bar{r}_t := r_t r_{t+1} \cdots r_{t+l-L}$. Then by L completeness of Σ it can be seen that $\bar{r}_t\bar{w}_{|[t,t+l]} \in \mathcal{B}_{|[t,t+l]}$ and so $\bar{w}_{|[t,t+l]} \in \mathcal{B}_{ce|[t,t+l]}$. By the l-completeness we have that $\bar{w} \in \mathcal{B}_{ce}$.

Now, since Σ and Σ_{ce} are $L+1$-complete, then they are completely described by the finite generated submodule $\mathcal{M} := \mathcal{B}_{|[0,L+1]}$ of $R^{(L+2)q}$ and the subspace $\mathcal{V} := \mathcal{B}_{e|[0,L+1]}$ of $F^{(L+2)q}$, respectively. It is easy to see that

$$\mathcal{V} = \{am : a \in F, \ m \in \mathcal{M}\}.$$

Therefore a basis of $\mathcal{V}$ is easily computable from a set of generators of $\mathcal{M}$ and from this basis we can obtain a kernel representation of Σ_{ce}, i.e. we can compute a polynomial matrix $N \in F[z, z^{-1}]^{g \times q}$ such that

$$\mathcal{B}_{ce} = \ker N := \{w \in (F^q)^{\mathbb{Z}} : Nw = 0\}.$$

Since (see [15]) Σ_{ce} is finitely generated if and only if N is full column rank, then in this way we have a test also for Σ. Actually, by theorem 1, Σ is finitely generated if and only if N is a full column rank polynomial matrix.

4 Controllable systems

Another class of systems with interesting properties are the controllable systems. The notion of controllability that we will consider is not connected with a state space realization, but is a property of the system itself. This property has been first introduced by Willems in [15]. Other notions of controllability have been introduced also in [7, 11] and they are not always equivalent. In this section we will list various definitions of controllability and we will show how they are connected each other. Moreover we will see some interesting properties of controllable systems and we will introduce the notion of controllable subsystem.

Definition 3 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over a ring R. Then*

1. *Σ is zero-controllable if for all $w \in \mathcal{B}$, there exist $k \in \mathbb{N}$ and $w' \in \mathcal{B}$ such that*

$$w'_{|(-\infty,0]} = w_{|(-\infty,0]},\ w'_{|[k,+\infty)} = 0.$$

2. *Σ is symmetrically controllable if for all $w \in \mathcal{B}$, there exist $h, k \in \mathbb{N}$ and $w', w'' \in \mathcal{B}$ such that*

$$\begin{cases} w'_{|(-\infty,0]} = w_{|(-\infty,0]}, & w'_{|[k,+\infty)} = 0, \\ w''_{|(-\infty,-h]} = 0, & w''_{|[0,+\infty)} = w_{|[0,+\infty)}. \end{cases}$$

3. *Σ is strongly zero-controllable if there exists $k \in \mathbb{N}$ such that for all $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that*

$$w'_{|(-\infty,0]} = w_{|(-\infty,0]},\ w'_{|[k,+\infty)} = 0.$$

4. *Σ is controllable if for all $w_1, w_2 \in \mathcal{B}$, there exists $k \in \mathbb{N}$ and $w \in \mathcal{B}$ such that*

$$w_{|(-\infty,0]} = w_{1|(-\infty,0]},\ w_{|[k,+\infty)} = (z^{-k}w_2)_{|[k,+\infty)}.$$

5. *Σ is strongly controllable if there exists $k \in \mathbb{N}$ such that for all $w_1, w_2 \in \mathcal{B}$, there exists $w \in \mathcal{B}$ such that*

$$w_{|(-\infty,0]} = w_{1|(-\infty,0]},\ w_{|[k,+\infty)} = (z^{-k}w_2)_{|[k,+\infty)}.$$

It can be seen that strong zero-controllability and strong controllability are equivalent and so they will not be distinguished. Moreover strong controllability implies controllability that implies symmetric controllability that finally implies zero-controllability. This is summarized in the following scheme.

$$
\begin{array}{c}
\text{Strong zero-controllability} \\
\Updownarrow (i) \\
\text{Strong controllability} \\
\Downarrow (ii) \\
\text{controllability} \\
\Downarrow (iii) \\
\text{Symmetric controllability} \\
\Downarrow (iv) \\
\text{Zero-controllability}
\end{array}
$$

We now give the proofs of the above mentioned implications:

(i) It is trivial to show that strong controllability implies strong zero-controllability. Suppose conversely that the linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is strongly zero-controllable and let $w_1, w_2 \in \mathcal{B}$. If $\tilde{w} := w_1 - z^{-k}w_2 \in \mathcal{B}$, then there exists $w' \in \mathcal{B}$ such that

$$w'_{|(-\infty,0]} = \tilde{w}_{|(-\infty,0]},\ w'_{|[k,+\infty)} = 0.$$

Consequently, if $w := w' + z^{-k}w_2 \in \mathcal{B}$, then we have that

$$w_{|(-\infty,0]} = w_{1|(-\infty,0]},\ w_{|[k,+\infty)} = (z^{-k}w_2)_{|[k,+\infty)}.$$

(ii) Trivial.

(iii) Suppose that Σ is controllable and let $w \in \mathcal{B}$. If in the definition of controllability we take $w_1 := w$ and $w_2 := 0$, then we have that there exist $k \in \mathbb{N}$ and $w' \in \mathcal{B}$ such that

$$w'_{|(-\infty,0]} = w_{1|(-\infty,0]} = w_{|(-\infty,0]},\ w'_{|(k,+\infty)} = (z^{-k}w_2)_{|(k,+\infty)} = 0.$$

On the other hand, if in the definition of controllability we take $w_1 := 0$ and $w_2 := w$, then we have that there exist $k \in \mathbb{N}$ and $\bar{w} \in \mathcal{B}$ such that

$$\bar{w}_{|[0,+\infty)} = 0,\ \bar{w}_{|[k,+\infty)} = (z^{-k}w_1)_{|[k,+\infty)} = (z^{-k}w)_{|[k,+\infty)}$$

and so by the shift-invariance we have that $w'' := z^k\bar{w} \in \mathcal{B}$ and

$$w''_{|(-\infty,-k]} = 0,\ w''_{|[0,+\infty)} = w_{|[0,+\infty)}.$$

(iv) Trivial

Note that both controllabilities are the original definitions given by Willems in [15], while strong zero-controllability and zero-controllability have been introduced by Forney and Trott in [7]. The symmetric version of zero-controllability, that could be called zero-reachability, can be introduced and connected with the other notions of controllabilities in a obvious way. In the following proposition we show that if R is Noetherian, then zero-controllability, symmetric controllability and strong controllability coincide. If moreover the system is strongly complete, then all the controllabilities are equivalent.

Proposition 5 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over a ring R. Then*

1. *If R is Noetherian, then Σ is symmetric controllable if and only if Σ is strongly controllable.*
2. *If R is Noetherian and Σ is strongly complete, the all the notions of controllability of Σ are equivalent.*

Proof Suppose that Σ is symmetric controllable. Let $\hat{\mathcal{B}}$ is the set of trajectories in $\mathcal{B}$ with finite support. Note that $\hat{\mathcal{B}}$ is a Noetherian module over $R[z, z^{-1}]$ (see problem 10 pag. 85 in [1]). Let $w_1, \dots, w_n$ a set of generators for $\hat{\mathcal{B}}$ and suppose that their supports are included in $[-N, N]$. Take now a $w \in \mathcal{B}$. We want to show that there exists $w' \in \mathcal{B}$ such that $w'_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $w'_{|[2N,+\infty)} = 0$. By symmetric controllability it is easy to see that there exists $\hat{w} \in \hat{\mathcal{B}}$ and $w_1, w_2 \in \mathcal{B}$ such that $w_{1|(-\infty,0]} = 0$, $w_{2|[0,+\infty)} = 0$ and $w = \hat{w} + w_1 + w_2$. Then

$$\hat{w} = \sum_{j=1}^{n} \sum a_{ij} z^i w_j = \sum_{j=1}^{n} \sum_{|i|/leN} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{i<-N} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{i>N} a_{ij} z^i w_j,$$

where $a_{ij} \in R$. If we let

$$\hat{w}' := \sum_{j=1}^{n} \sum_{|i| \leq N} a_{ij} z^i w_j \in \hat{\mathcal{B}},$$

$$w'_1 := w_1 + \sum_{j=1}^{n} \sum_{i>N} a_{ij} z^i w_j \in \mathcal{B}$$

and

$$w'_1 := w_1 + \sum_{j=1}^{n} \sum_{i<-N} a_{ij} z^i w_j \in \mathcal{B},$$

then we have that $w'_{1|(-\infty,0]} = 0$, $w'_{2|[0,+\infty)} = 0$ and $w = \hat{w}' + w'_1 + w'_2$. Define finally $w' := \hat{w}' + w'_2 = w - w'_1$. Then it is easy to verify that $w'_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $w'_{|[2N,+\infty)} = 0$.

In order to prove the second assertion it is sufficient to show that if Σ is zero-controllable, then it is strongly controllable. Suppose that Σ is zero-controllable and let for all $n \in \mathbb{N}$

$$\mathcal{M}_n := \{w_{|[-L,0]} : w \in \mathcal{B},\ w_{|[n,+\infty)} = 0\}.$$

Then we have that $\mathcal{M}_1 \subseteq \mathcal{M}_2 \subseteq \mathcal{M}_3 \subseteq \cdots$ and so, since R is Noetherian, there exists $N \in \mathbb{N}$ such that $\mathcal{M}_N = \mathcal{M}_{N+1} = \mathcal{M}_{N+2} = \cdots$. Take now any $w \in \mathcal{B}$. Then, since Σ is zero-controllable then $w_{|[-L,0]} \in \mathcal{M}_n$ for some n and so $w_{|[-L,0]} \in \mathcal{M}_N$. Therefore any trajectory in $\mathcal{B}$ can be controlled to zero in at most N steps and so, by L-completeness, Σ is strongly controllable. ■

In the following example we show that for systems that are not strongly complete, controllability does not imply strong controllability.

Example Let $R = \mathbb{Z}$ and $\Sigma = (\mathbb{Z}, R, \mathcal{B})$, where

$$\mathcal{B} := \{pw : p \in R[z, z^{-1}]\}$$

and where w is any irrational trajectory in $R^{\mathbb{Z}}$ (i.e. a trajectory such that pw has infinite support, for every polynomial $p \in R[z, z^{-1}]$) such that $w_{|(0,+\infty)} = 0$. It is clear that Σ is a linear shift-invariant zero-controllable system. However it is not difficult to verify that it is not symmetrically controllable, since the only trajectories in $\mathcal{B}$ with finite support is the zero trajectory.

Suppose now that R is a domain and $W = R^q$ for some $q \in \mathbb{N}$. It is easy to see that all the controllability properties of Σ are inherited by both its field extension system Σ_e and by its complete field extension system Σ_{ce}. The converse is not true even if R is Noetherian domain as shown by the following example. Therefore a version of theorem 1 for the controllability properties does not hold true.

Example Let $R = \mathbb{Z}$ and $\Sigma = (\mathbb{Z}, \mathbb{Z}, \mathcal{B})$, where

$$\mathcal{B} := \{w \in \mathbb{Z}^{\mathbb{Z}} : 2w(t+1) - 3w(t) \text{ is a multiple of } 5\} = \{w \in \mathbb{Z}^{\mathbb{Z}} : (2z-3)w \in (5)^{\mathbb{Z}}\},$$

where (5) is the ideal in $\mathbb{Z}$ of the integers that are multiple of 5 and $(5)^{\mathbb{Z}}$ is the set of sequences whose elements are in (5). It is clear that Σ is a linear shift-invariant strongly complete system over the ring $\mathbb{Z}$. It is not difficult to verify that

$$w \in \mathcal{B} \quad \Leftrightarrow \quad \exists\, v \in \mathbb{Z}^{\mathbb{Z}} : \left\{ \begin{array}{lcl} w(t) & = & 9^t w(0) + 5v(t) \\ w(t) & = & 4^t w(0) + 5v(t). \end{array} \right.$$

Consequently Σ is not zero-controllable, while it is easy to see that $\mathcal{B}_e = \{\bar{w} \in \mathbb{Q}^{\mathbb{Z}} : \exists n \in \mathbb{Z},\ n\bar{w} \in \mathcal{B}\}$ and $\mathcal{B}_{ce} = \mathbb{Q}^{\mathbb{Z}}$ and so both of them are strongly controllable.

5 Image representation for controllable systems

As shown in [15] for linear shift-invariant complete and controllable systems over fields there exists a useful representation that is called image representation. According this representation, we have that the behaviour coincides with the image of a suitable linear operator that is called shift operator. We will see now that this kind of representation holds true also for systems over Noetherian rings.

Let V and W be two finitely generated modules over R and let $\mathrm{Hom}(V,W)$ be the set of all the R-homomorphisms from V to W. In the usual way we can define the set $\mathrm{Hom}(V,W)[z,z^{-1}]$ of all polynomials with coefficients in $\mathrm{Hom}(V,W)$. This is not a ring but only an $R[z,z^{-1}]$-module. Given an $M \in \mathrm{Hom}(V,W)[z,z^{-1}]$

$$M = \sum_{i=l}^{L} M_i z^i,$$

where $M_i \in \mathrm{Hom}(V,W)$, we can associate an $R[z,z^{-1}]$-homomorphism Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ in the following way:
If $v \in V^{\mathbb{Z}}$, then for all $t \in \mathbb{Z}$ we define

$$\Psi_M(v)(t) := \sum_{i=l}^{L} M_i v(t+i).$$

The homomorphisms defined in this way are called shift operators.

The family of shift operators can be characterized in terms of continuity w.r. to a suitable topology defined on the signal spaces. More precisely consider the discrete topology on V and W and the product topology on the signal spaces $V^{\mathbb{Z}}$ and $W^{\mathbb{Z}}$. Such a topology is called pointwise convergence topology since a sequence $\{w_n\}_{n=1}^{\infty} \subseteq W^{\mathbb{Z}}$ converges to $w \in W^{\mathbb{Z}}$ if and only if the sequence $\{w_n(t)\}_{n=1}^{\infty} \subseteq W$ converges to $w(t) \in W$ in the discrete topology for all $t \in \mathbb{Z}$ and so if and only if $w_n(t)$ is eventually equal to $w(t)$ for all $t \in \mathbb{Z}$.

It is not difficult to prove that closed subsets of $W^{\mathbb{Z}}$ corresponds to complete behaviours. Note moreover that $W^{\mathbb{Z}}$ with this topology satisfies the first axiom of countability (see [8, pag. 92]) and therefore (see [3, pag. 218]) a subset $\mathcal{B}$ of $W^{\mathbb{Z}}$ is closed if and only if the fact that $\{w_n\}_{n=1}^{\infty} \subseteq \mathcal{B}$ converges to $w \in W^{\mathbb{Z}}$ implies that $w \in \mathcal{B}$. Moreover a map $\Phi : V^{\mathbb{Z}} \to W^{\mathbb{Z}}$ is continuous if and only if for every sequence $\{w_n\}_{n=1}^{\infty} \subseteq V^{\mathbb{Z}}$ converging to w we have that $\{\Phi(w_n)\}_{n=1}^{\infty} \subseteq W^{\mathbb{Z}}$ converges to $\Phi(w)$. Now we can give the characterization of the shift operators in terms of continuity w.r. to the pointwise topology.

Proposition 6 *Let Φ be an operator from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ and consider in $V^{\mathbb{Z}}$ and $W^{\mathbb{Z}}$ the pointwise convergence topology. Then Φ is a continuous $R[z,z^{-1}]$-homomorphism from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ if and only if Φ is a shift operator.*

Proof Suppose that Φ is continuous. Consider for all $v \in V$ the signal $\delta_v \in V^{\mathbb{Z}}$ such that $\delta_v(0) = v$ and $\delta_v(t) = 0$ for all $t \neq 0$. Since the sequence $\{z^n \delta_v\}_{n=1}^{\infty}$ converges to zero, then, by continuity, the sequence $\{z^n \Phi(\delta_v)\}_{n=1}^{\infty}$ converges to zero. Analogously, since the sequence $\{z^{-n}\delta_v\}_{n=1}^{\infty}$ converges to zero, then the sequence $\{z^{-n}\Phi(\delta_v)\}_{n=1}^{\infty}$ converges to zero. We can argue that $\Phi(\delta_v)$ has finite support. Since Φ is an R-homomorphism, then for all $i \in \mathbb{Z}$ there exists $M_i \in \mathrm{Hom}(V, W)$ such that $\Phi(\delta_v)(i) = M_i v$ for all $v \in V$. Consider the polynomial $M := \sum M_i z^i$ in $\mathrm{Hom}(V, W)[z, z^{-1}]$.

We want to show that Φ coincides with the shift operator Ψ_M and consequently we have to show that for all $v \in (R^l)^{\mathbb{Z}}$ we have that $\Phi(v) = \Psi_M(v)$. If v has finite support, then this is true. For any $v \in (R^l)^{\mathbb{Z}}$ consider the sequence $\{v_n\}_{n=1}^{\infty}$ defined as follows:

$$v_{n|[-n,n]} = v_{|[-n,n]}, \; v_{n|(-\infty,-n)} = 0 \text{ and } v_{n|(n,+\infty)} = 0.$$

It is clear that $\{v_n\}$ converges to v and so, by continuity of Φ, $\{\Phi(v_n)\}$ converges to $\Phi(v)$. On the other hand, since v_n has finite support, then $\Phi(v_n) = \Psi_M(v_n)$ and so, for every $t \in \mathbb{Z}$, there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $\Psi_M(v_n)(t) = \Phi(v)(t)$. It is clear that, if n is big enough, then $\Psi_M(v_n)(t) = \Psi_M(v)(t)$.
Conversely it is easy to see that Ψ_M, where $M \in \mathrm{Hom}(V, W)[z, z^{-1}]$, is a continuous $R[z, z^{-1}]$ homomorphism from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$. Suppose that the sequence $\{v_n\}_{n=1}^{\infty} \subseteq (R^l)^{\mathbb{Z}}$ converges to v. Actually, we have that

$$\Psi_M(v_n)(t) = \sum_{i=l}^{L} M_i v_n(t+i)$$

and so, since there exists $N \in \mathbb{N}$ such that for all $n > N$ we have $v_n(t+i) = v(t+i)$, $\forall i = l, l+1, \ldots, L$, then for all $n > N$ we have

$$\Psi_M(v_n)(t) = \sum_{i=l}^{L} M_i v_n(t+i) = \sum_{i=l}^{L} M_i v(t+i) = \Psi_M(v)(t).$$

Consequently the sequence $\{\Psi_M(v_n)\}_{n=1}^{\infty} \subseteq (R^l)^{\mathbb{Z}}$ converges to $\Psi_M(v)$ and we have the continuity of Ψ_M. ∎

As shown by Willems, complete controllable linear shift-invariant systems over fields admit an image representation. More precisely, it can be shown that the behaviour of a controllable system coincides with the image of a suitable shift operator. From the coding theory point of view, the shift operator can be seen as the encoder, i.e. the algorithm that maps the signal in the corresponding coword. An analogous result holds true for systems over Noetherian rings as shown in the following theorem.

Theorem 2 *Let R be a Noetherian ring and let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant complete system. Then Σ is strongly controllable if and only if $\mathcal{B} = \operatorname{im} \Psi_M$ for some shift operator Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$, where V is a suitable finitely generated R-module.*

Proof It is trivial to prove that if $\mathcal{B} = \operatorname{im} \Psi_M$, then Σ is strongly controllable. Suppose conversely that Σ is strongly controllable. Then there exists $k \in \mathbb{N}$ such that for all $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that

$$w'_{|(-\infty,0]} = w_{|(-\infty,0]}, \; w'_{|[k,+\infty)} = 0.$$

Let

$$\mathcal{B}_{[0,k)} := \{w \in \mathcal{B} : w(t) = 0, \forall t \notin [0,k)\}.$$

Since R is Noetherian, then $\mathcal{B}_{[0,k)}$ is finitely generated. Let $u_1, \ldots, u_l$ be a family of generators. Fix $V := R^l$ and let $e_1, \ldots, e_l$ be the canonical basis in R^l. Let moreover M_t be the unique homomorphism in $\operatorname{Hom}(R^l, W)$ such that for all $i = 1, \ldots, l$ we have $M_t(e_i) = u_i(-t)$ and define

$$M := \sum_{t=-k+1}^{0} M_t z^t$$

as an element in $\operatorname{Hom}(R^l, W)[z, z^{-1}]$. We want to show that $\mathcal{B} = \operatorname{im} \Psi_M$.
Let $w \in \operatorname{im} \Psi_M$. Then $w = \Psi_M(v)$ for some $v \in (R^l)^{\mathbb{Z}}$. Let $v_h \in (R^l)^{\mathbb{Z}}$ be defined as follows:

$$v_h(t) = \begin{cases} v(t) & \text{if } |t| \leq h \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that, since the support of M is included in $(-k, 0]$, then we have that $w_h := \Psi_M(v_h)$ coincides with w in the interval $[-h+k, h]$. Since $w_h \in \mathcal{B}$ and since Σ is complete, then we have $w \in \mathcal{B}$.
Let $w \in \mathcal{B}$. First we show that $w = w_p + w_f$, where $w_p, w_f \in \mathcal{B}$ and $w_p(t) = 0$ for all $t \geq k$ and $w_f(t) = 0$ for all $t \leq 0$. Actually, by strong controllability, there exists $w_p \in \mathcal{B}$ such that $w_{p|(-\infty,0]} = w_{|(-\infty,0]}$, $w_{p|[k,+\infty)} = 0$ and so, if we define $w_f := w - w_p$ we have that w_p, w_f satisfy the conditions we required. We want to show now that $w_p, w_f \in \operatorname{im} \Psi_M$. Consider the sequence $w_0, w_1, w_2, \ldots$ such that $w_{i|(-\infty,i]} = 0$ constructed recursively in the following way:
Let $w_0 := w_f$. If we suppose we have found w_i such that $w_{i|(-\infty,i]} = 0$, then, by strong controllability, there exists $\hat{w}_i$ such that $\hat{w}_{i|(-\infty,i+1]} = w_{i|(-\infty,i+1]}$, $\hat{w}_{i|[k+i+1,+\infty)} = 0$. Define $w_{i+1} := w_i - \hat{w}_i$. It is clear that $w_{i+1|(-\infty.i+1]} = 0$. Moreover $z^{i+1}\hat{w}_i \in \mathcal{B}_{[0,k)}$ and so

$$z^{i+1}\hat{w}_i = \sum_{j=1}^{l} a_{i+1,j} u_j$$

Consequently we have that

$$(z^{i+1}\hat{w}_i)(t) = \sum_{j=1}^{l} a_{i+1,j}u_j(t) = \sum_{j=1}^{l} a_{i+1,j}M_{-t}e_j = M_{-t}\begin{bmatrix} a_{i+1,1} \\ \vdots \\ a_{i+1,l} \end{bmatrix}$$

and so $\hat{w}_i = \Psi_M(a_{j+1})$ where $a_{j+1} \in (R^l)^{\mathbb{Z}}$ such that

$$a_{j+1}(t) := \begin{cases} \begin{bmatrix} a_{i+1,1} \\ \vdots \\ a_{i+1,l} \end{bmatrix} & \text{if } t = i+1 \\ 0 & \text{otherwise} \end{cases}$$

Therefore, if $v_i := \sum_{j=0}^{i} a_{j+1}$, then $w_f - w_i = \Psi_M(v_i)$. We have that w_i converges to zero and so $\Psi_M(v_i)$ converges to w_f. It is clear that also the sequence v_i converges to a limit signal that we call v_f, and, by continuity of Ψ_M we have that $w_f = \Psi_M(v_f)$ and so $w_f \in \text{im } \Psi_M$. In a similar way it can be shown that $w_p \in \text{im } \Psi_M$ and so also $w = w_f + w_p \in \text{im } \Psi_M$. ∎

The following corollary follows immediately from the previous theorem and from proposition 5.

Corollary 1 *Let R be a Noetherian ring and let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant strongly complete system. Then Σ is zero-controllable if and only if $\mathcal{B} =$ im Ψ_M for some shift operator Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$, where V is a suitable finitely generated R-module.*

If $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ is complete and strongly controllable, then $\mathcal{B}$ is the image of a shift operator Ψ_M from $V^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$. From the coding theory point of view, the behaviour $\mathcal{B}$ is seen as the code and so the map Ψ_M can be seen as an encoder. It is clear that if we want that Ψ_M represents really an encoder, this map should be one to one. It is not true in general that the behaviour of complete and strongly controllable system coincides with the image of an one to one map shift operator Ψ_M. Actually, in this case the $R[z, z^{-1}]$ module $\mathcal{B}$ would be homomorphic to $V^{\mathbb{Z}}$ and this is not always possible. Some aditional requirements stronger than strong controllability are necessary (see [9, 5]).

As shown in the proof of the previous theorem it is possible to choose $V = R^l$ for some $l \in \mathbb{N}$. In this case the shift operators can be described in a slightly more concrete way. Actually, if we fix a set of generators $\bar{w}_1, \dots, \bar{w}_q$ for W, then there exists a canonical way to associate to a homomorphism in $\text{Hom}(R^l, W)$ a matrix in $R^{l \times q}$ and so $\text{Hom}(R^l, W)[z, z^{-1}]$ and $R[z, z^{-1}]^{l \times q}$ can be considered isomorphic $R[z, z^{-1}]$-modules. More concretely we can associate to a matrix $M \in R[z, z^{-1}]^{q \times l}$,

an operator Ψ_M from $(R^l)^{\mathbb{Z}}$ to $W^{\mathbb{Z}}$ in the following way:
If $M = [m_{ij}]$ and $v = (v_1, \dots, v_l)^T \in (R^l)^{\mathbb{Z}}$, then for all $t \in \mathbb{Z}$ we define

$$\Psi_M(v)(t) := u_1(t)\bar{w}_1 + \cdots + u_q(t)\bar{w}_q,$$

where u_1 are signals in $R^{\mathbb{Z}}$ so defined $u_i := \sum_{j=1}^{l} m_{ij}v_j$. Note that $v_j \in R^{\mathbb{Z}}$ and that $m_{ij}v_j$ is well defined. In this way for every $t \in \mathbb{Z}$ we have that

$$\Psi_M(v)(t) = \sum_{i=l}^{L} M_i v(t+i). \tag{2}$$

6 Controllable subsystems

Given a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$, it is possible to define the concept of zero-controllable subsystem $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ as the biggest linear shift-invariant zero-controllable subsystem of Σ. More precisely $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ is the zero-controllable subsystem of Σ if

1. $\mathcal{B}_c \subseteq \mathcal{B}$.
2. $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ is a linear shift-invariant zero-controllable system.
3. For any linear shift-invariant zero-controllable system $\Sigma' = (\mathbb{Z}, W, \mathcal{B}')$, such that $\mathcal{B}' \subseteq \mathcal{B}$, we have that $\mathcal{B}' \subseteq \mathcal{B}_c$.

The existence of a system with these properties is provided by the observation that if $\Sigma_i = (\mathbb{Z}, W, \mathcal{B}_1)$, $i \in I$, is a family of linear shift-invariant zero-controllable systems such that $\mathcal{B}_i \subseteq \mathcal{B}$, then $(\mathbb{Z}, W, \mathcal{B})$, where

$$\mathcal{B} := \sum_{i \in I} \mathcal{B}_i,$$

is a linear shift-invariant zero-controllable system.

In the same way we can define the symmetrically controllable subsystem $\Sigma_{sc} = (\mathbb{Z}, W, \mathcal{B}_{sc})$ of a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ and in the same way we can prove its existence. Note first that if R is Noetherian, then the symmetrically controllable subsystem is strongly controllable and so it is the biggest strongly controllable subsystem of Σ, which does not exists in general. Note moreover that if Σ is zero-controllable, then the zero-controllable subsystem Σ_c coincides with Σ and if Σ is symmetrically controllable, then the symmetrically controllable subsystem Σ_{sc} coincides with Σ.

Given a linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ we define now two interesting subsystems that are connected with the controllable subsystems. First define the linear shift-invariant subsystem $\Sigma_p = (\mathbb{Z}, W, \mathcal{B}_p)$ as follows:

$$\mathcal{B}_p := \{w \in \mathcal{B} : w_{|[k,+\infty)} = 0,\ \exists k \in \mathbb{Z}\}.$$

Define moreover the subsystem $\hat{\Sigma} = (\mathbb{Z}, W, \hat{\mathcal{B}})$ of Σ as as the system with behaviour

$$\hat{\mathcal{B}} := \{\hat{w} \in \mathcal{B} : \hat{w} \text{ has finite support}\}.$$

The relations between these subsystems and the controllable subsystems are very interesting when Σ is strongly complete as shown in the following proposition. First, we introduce the concept of L-completion. If $\mathcal{B}$ is any behaviour of a linear shift-invariant system, then we define $CP_L(\mathcal{B})$ as the smallest L-complete behaviour containing $\mathcal{B}$. More explicitly we have that

$$CP_L(\mathcal{B}) = \{w \in W^{\mathbb{Z}} : w_{|[t,t+L]} \in \mathcal{B}_{|[t,t+L]}\}.$$

Proposition 7 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant L-complete system, $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ its zero-controllable subsystem and $\Sigma_{sc} = (\mathbb{Z}, W, \mathcal{B}_{sc})$ its symmetrically controllable subsystem. Then*

1. $\mathcal{B}_c = CP_L(\mathcal{B}_p)$.
2. $\mathcal{B}_{sc} = CP_L(\hat{\mathcal{B}})$.

Moreover, if R is Noetherian, then $\Sigma_c = \Sigma_{sc}$.

Proof First if we show that $CP_L(\mathcal{B}_p)$ is zero-controllable, then we would argue that $\mathcal{B}_c \supseteq CP_L(\mathcal{B}_p)$. Actually, if $w \in CP_L(\mathcal{B}_p)$, then there exists $\hat{w} \in \mathcal{B}_p$ such that $w_{|[-L,0]} = \hat{w}_{|[-L,0]}$. Let $\bar{w}$ be a trajectory in $W^{\mathbb{Z}}$ such that $\bar{w}_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $\bar{w}_{|[-L,+\infty)} = w_{|[-L,+\infty)}$. By L-completeness of $CP_L(\mathcal{B}_p)$ we have that $\bar{w} \in CP_L(\mathcal{B}_p)$. Finally if we show that $\mathcal{B}_{c|[0,L]} \subseteq \mathcal{B}_{f|[0,L]}$, then we would argue that $\mathcal{B}_c \subseteq CP_L(\mathcal{B}_c) \subseteq CP_L(\mathcal{B}_f)$. Actually, suppose that $w \in \mathcal{B}_c$. Then there exists $w' \in \mathcal{B}_c$ such that $w'_{|(-\infty,L]} = w_{|(-\infty,L]}$ and $w'_{|[L+h,+\infty)} = 0$. It is easy to see that $w' \in \mathcal{B}_p$ and that $w'_{|[0,L]} = w_{|[0,L]}$.

The second part of the proof works in the same way as the previous one. First if we show that $CP_L(\hat{\mathcal{B}})$ is symmetrically controllable, then we would argue that $\mathcal{B}_{sc} \supseteq CP_L(\hat{\mathcal{B}})$. Actually, if $w \in CP_L(\hat{\mathcal{B}})$, then there exists $\hat{w} \in \hat{\mathcal{B}}$ such that $w_{|[-L,0]} = \hat{w}_{|[-L,0]}$. Let $\bar{w}$ be a trajectory in $W^{\mathbb{Z}}$ such that $\bar{w}_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $\bar{w}_{|[-L,+\infty)} = w_{|[-L,+\infty)}$. By L-completeness of $CP_L(\hat{\mathcal{B}})$ we have that $\bar{w} \in CP_L(\hat{\mathcal{B}})$. The symmetric can be shown similarly.
Finally if we show that $\mathcal{B}_{sc|[0,L]} \subseteq \hat{\mathcal{B}}_{|[0,L]}$, then we would argue that $\mathcal{B}_{sc} \subseteq CP_L(\mathcal{B}_{sc}) \subseteq CP_L(\hat{\mathcal{B}})$. Actually, suppose that $w \in \mathcal{B}_{sc}$. Then there exists $w' \in \mathcal{B}_{sc}$ such that $w'_{|(-\infty,L]} = w_{|(-\infty,L]}$ and $w'_{|[L+h,+\infty)} = 0$. Moreover there exists $w'' \in \mathcal{B}_{sc}$ such that $w''_{|[0,+\infty)} = w'_{|[0,+\infty)}$ and $w''_{|(-\infty,-k]} = 0$. It is easy to see that $w'' \in \hat{\mathcal{B}}$ and that $w''_{|[0,L]} = w_{|[0,L]}$.

The last part of the proof is the direct consequence of of proposition 5. ∎

A linear shift-invariant strongly complete system $\Sigma = (\mathbb{Z}, F^q, \mathcal{B})$ over a field F can be decomposed in the direct sum of a controllable subsystem and an autonomous subsystem. This kind of decomposition can not be directly extended to systems over rings. In the following proposition shows what kind of extension can be done.

Proposition 8 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant strongly complete system. Let moreover $\Sigma_c = (\mathbb{Z}, W, \mathcal{B}_c)$ the zero-controllable subsystem of Σ. Then the module $\mathcal{B}/\mathcal{B}_c$ is finitely generated over R.*

Proof Suppose that Σ is L-complete. Note that in this case by the previous proposition $\mathcal{B}_c = CP_L(\hat{\mathcal{B}}) = CP_L(\mathcal{B}_p)$. By symmetry it is possible to show also that $\mathcal{B}_c = CP_L(\mathcal{B}_f)$, where $\mathcal{B}_f := \{w \in \mathcal{B} : w_{|(-\infty,k]} = 0,\ \exists k \in \mathbb{Z}\}$. If we take $m \in \mathcal{B}_{|[0,L]}$, then there exists $w \in \mathcal{B}$ such that $w_{|[0,L]} = m$. If $w' \in \mathcal{B}$ is such that $w'_{|[0,L]} = m$, then $\delta := w - w' \in \mathcal{B}_c$. Actually $\delta_{|[0,L]} = 0$ and so, by L-completeness the signal δ_1 such that $\delta_{1|(-\infty,O]} = \delta_{|(-\infty,O]}$ and $\delta_{1|(0,+\infty)} = 0$ is in $\mathcal{B}$. Since $\delta_1 \in \mathcal{B}_f$ and $\delta_2 := \delta - \delta_1 \in \mathcal{B}_p$, then $\delta_1, \delta_2 \in \mathcal{B}_c$ and so $\delta \in \mathcal{B}_c$. Consider the projection map

$$\Phi : \mathcal{B}_{|[0,L]} \to \mathcal{B}/\mathcal{B}_c$$

defined as follows: if we take $m \in \mathcal{B}_{|[0,L]}$, then there exists $w \in \mathcal{B}$ such that $w_{|[0,L]} = m$. Define $\Phi(m) := w + \mathcal{B}_c$. This is a good definition as seen above. It is easy to see that the map Φ is onto and so we have that $\mathcal{B}/\mathcal{B}_c$ is isomorphic to $\mathcal{B}_{|[0,L]}/\ker\Phi$ that is Noetherian. ∎

The module $\mathcal{B}/\mathcal{B}_c$ can be seen in a way as the behaviour of the autonomous subsystem of Σ. Actually, in the hypotheses of the previous proposition we can define the linear shift-invariant system $\Sigma_a = (\mathbb{Z}, \bar{W}, \mathcal{B}_a)$ where $\bar{W}$ is the finitely generated R-module $\mathcal{B}/\mathcal{B}_c$ and where

$$\mathcal{B}_a := \{z^t\bar{w} : t \in \mathbb{Z},\ \bar{w} \in \mathcal{B}/\mathcal{B}_c\}.$$

Obviously we have that Σ_a is a finitely generated and so an autonomous system. It can be considered the autonomous subsystem even if it is not properly a subsystem of Σ.

7 Finitely generated state space module and realizable systems

In this section we will study the state space module of a linear shift-invariant system over a Noetherian ring. The concept of state space have been introduced in the behavioural approach by Willems in [15] in its most generality. For systems over groups it is possible to define a canonical state space as a quotient group. We will follow this approach.

Definition 4 *Let $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system over a ring R. Then the state space module $\mathcal{X}$ of Σ is the R-module defined as follows*

$$\mathcal{X} := \mathcal{B}/(\mathcal{B}_- + \mathcal{B}_+),$$

where $\mathcal{B}_-$ is the subset of all the trajectories w in $\mathcal{B}$ supported in $(-\infty, 0)$ and analogously $\mathcal{B}_+$ is the subset of all the trajectories w in $\mathcal{B}$ supported in $[0, +\infty)$.

As shown by Willems in [15] we have that the system $\Sigma_s = (\mathbb{Z}, W \times \mathcal{X}, \mathcal{B}_s)$, with

$$\mathcal{B}_s = \{(w, x) \in (W \times \mathcal{X})^{\mathbb{Z}} : w \in \mathcal{B},\ x(t) := z^t w + \mathcal{B}_- + \mathcal{B}_+,\ \forall t \in \mathbb{Z}\}$$

constitutes a minimal state space representation (or state realization) of Σ in the sense that, up to isomorphisms, it is the smallest system of that form such that $\mathcal{B} = \{w : (w, x) \in \mathcal{B}_s\}$ and satisfying the axiom of state. Willems showed moreover that when Σ is complete, then Σ_s is 2-complete. In other words he showed that in this case Σ_s is completely determined by its evolution low, i.e.

$$(w, x) \in \mathcal{B}_s \Leftrightarrow (x(t), x(t+1), w(t)) \in \mathcal{M}, \quad \forall t \in \mathbb{Z}$$

where

$$\begin{aligned} \mathcal{M} &= \{(x(0), x(1), w(0)) \in \mathcal{X} \times \mathcal{X} \times W : (x, w) \in \mathcal{B}_s\} = \\ &= \{(w + \mathcal{B}_- + \mathcal{B}_+, z^{-1}w + \mathcal{B}_- + \mathcal{B}_+, w(0)) : w \in \mathcal{B}\}. \end{aligned}$$

In coding theory words, the evolution law fixed by the module $\mathcal{M}$ gives the trellis diagram describing the code associated to the system Σ. Note that these considerations are really useful in practice only if the state space module $\mathcal{X}$ is finitely generated. In this case we say that the system Σ is realizable. Only when Σ is realizable, in the state space representation Σ_s the signal alphabet $W \times \mathcal{X}$ is a finitely generated module and so is the module $\mathcal{M}$. Consequently the evolution law can be expressed in a constructive way. More precisely, if $m_1, \ldots, m_l$ is a set of generators of $\mathcal{M}$, then the evolution law can be expressed in the following way:
$(w, x) \in \mathcal{B}_s$ if and only if the equation

$$(x(t), x(t+1), w(t)) = a_1 m_1 + \cdots + a_l m_l$$

admits solutions $a_1, \ldots, a_l \in R$ for all $t \in \mathbb{Z}$.

In the following we will analyze some condition ensuring the realizability of a linear shift-invariant system over a Noetherian ring. We start showing that strongly complete and symmetrically controllable systems are realizable.

Proposition 9 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Then the following facts hold:*

1. *If Σ is strongly complete system, then Σ is realizable.*

2. If Σ is a symmetrically controllable system, then Σ is realizable.

Proof Let $\mathcal{X}$ be the state space module of Σ. Suppose that Σ is L-complete. Consider the the R-homomorphism

$$\Phi : \mathcal{B}_{|[0,L]} \to \mathcal{X}$$

defined as follows: if we take $m \in \mathcal{B}_{|[0,L]}$, then there exists $w \in \mathcal{B}$ such that $w_{|[0,L]} = m$. We show that, if $w' \in \mathcal{B}$ is such that $w'_{|[0,L]} = m$, then $\delta := w - w' \in \mathcal{B}_- + \mathcal{B}_+$. Actually $\delta_{|[0,L]} = 0$ and so, by L-completeness the signal w_1 such that $w_{1|(-\infty,0]} = \delta_{|(-\infty,0]}$ and $w_{1|(0,+\infty)} = 0$ is in $\mathcal{B}$. It is clear that $w_1 \in \mathcal{B}_-$ and $w_2 := \delta - w_1 \in \mathcal{B}_+$. Therefore we can argue that if we define $\Phi(m) := w + \mathcal{B}_- + \mathcal{B}_+$, then this is a good definition. It is easy to see that the map Φ is onto and so $\mathcal{X} \cong \mathcal{B}_{|[0,L]}/\ker\Phi$ that is Noetherian.

Suppose now that Σ is symmetrically controllable. Then it is easy to see that

$$\mathcal{B} = \hat{\mathcal{B}} + \mathcal{B}_- + \mathcal{B}_+,$$

where $\hat{\mathcal{B}}$ is the set of trajectories in $\mathcal{B}$ with finite support. Consequently we have that

$$\mathcal{X} \cong \hat{\mathcal{X}},$$

where $\hat{\mathcal{X}} := \hat{\mathcal{B}}/\hat{\mathcal{B}}_- + \hat{\mathcal{B}}_+$, where $\hat{\mathcal{B}}_-$ is the set of trajectories in $\mathcal{B}_-$ with finite support and similarly $\hat{\mathcal{B}}_+$ is the set of trajectories in $\mathcal{B}_+$ with finite support. Note that $\hat{\mathcal{B}}$ is a Noetherian module over $R[z, z^{-1}]$. Let $w_1, \dots, w_n$ a set of generators for $\hat{\mathcal{B}}$ and suppose that their support are included in $[-N, N]$. We want to show that

$$\{z^i w_j : i = -N, -N+1, \dots, -1, 0, 1, \dots, N-1, N;\ j = 1, 2, \dots, n\}$$

constitutes a set of generators for $\hat{\mathcal{X}}$. Take $w \in \hat{\mathcal{B}}$. Then

$$w = \sum_{j=1}^{n} \sum a_{ij} z^i w_j = \sum_{j=1}^{n} \sum_{|i| \le N} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{|i|>N} a_{ij} z^i w_j.$$

It is clear that the second summand is in $\hat{\mathcal{B}}_- + \hat{\mathcal{B}}_+$ and so we have the thesis. ■

Note that in the proof of the first part of the previous proposition we need something weaker than the strongly completeness. Actually we need only that the system has finite memory. A linear shift-invariant system $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ has finite memory if there exists $L \in \mathbb{N}$ such that if $w \in \mathcal{B}$ and $w_{[0,L)} = 0$, then $\bar{w}$ such that $\bar{w}_{(-\infty,0]} = 0$ and $\bar{w}_{(0,+\infty)} = w_{(0,+\infty)}$ is contained in $\mathcal{B}$. It is not difficult to prove that [15] a system is strongly complete if and only if it is complete and has finite memory.

The last result we present shows that there exists a strict relation between the realizability and the autonomous subsystem, as it has been defined in the previous section.

Proposition 10 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant system. Let moreover $\Sigma_{sc} = (\mathbb{Z}, W, \mathcal{B}_{sc})$ be the symmetrically controllable subsystem of Σ. Then Σ is realizable if and only if $\mathcal{B}/\mathcal{B}_{sc}$ is finitely generated.*

Proof ($\Leftarrow$) Let $\mathcal{X}$ be the state space module of Σ. From the previous proposition we have that the state space module $\mathcal{X}_{sc} := \mathcal{B}_{sc}/\mathcal{B}_{sc} \cap \mathcal{B}_{\pm}$, where $\mathcal{B}_{\pm} := \mathcal{B}_{-} + \mathcal{B}_{+}$, is Noetherian and so since $\mathcal{B}/\mathcal{B}_{sc}$ is Noetherian, we can argue that $\mathcal{B}/\mathcal{B}_{sc} \cap \mathcal{B}_{\pm}$. Consequently we have that also $\mathcal{X} = \mathcal{B}/\mathcal{B}_{\pm}$ is Noetherian.
($\Rightarrow$) If $\mathcal{X} = \mathcal{B}/\mathcal{B}_{\pm}$, where $\mathcal{B}_{\pm} := \mathcal{B}_{-} + \mathcal{B}_{+}$, is Noetherian, then $\mathcal{B}_f + \mathcal{B}_p/\mathcal{B}_{\pm}$ is Noetherian, where $\mathcal{B}_f$ denotes the submodule $\{w \in \mathcal{B} : w_{|(-\infty,h]} = 0,\ \exists h \in \mathbb{Z}\}$ and $\mathcal{B}_p$ denotes the submodule $\{w \in \mathcal{B} : w_{|[k,+\infty)} = 0,\ \exists k \in \mathbb{Z}\}$. We can argue that both the submodules $\mathcal{B}_f/\mathcal{B}_f \cap \mathcal{B}_{\pm}$ and $\mathcal{B}_p/\mathcal{B}_p \cap \mathcal{B}_{\pm}$ are Noetherian.
Let $w_1, \ldots, w_n \in \mathcal{B}_f$ such that $w_{i|(-\infty,0)} = 0$ and $w_1(0), \ldots, w_n(0)$ is a set of generators for the R-module $\{w(0) \in W : w \in \mathcal{B},\ w_{|(-\infty,0)} = 0\}$. Then it is clear that

$$\mathcal{B}_f = \mathcal{B}(w_1) + \cdots + \mathcal{B}(w_n) + \mathcal{B}_{+},$$

where $\mathcal{B}(w_i) := \{pw_i : p \in R[z, z^{-1}]\}$. Consider the following increasing sequence of modules

$$\langle w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_{\pm} \subseteq \langle w_i, zw_i \rangle + \mathcal{B}_f \cap \mathcal{B}_{\pm} \subseteq \langle w_i, zw_i, z^2 w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_{\pm} \subseteq \cdots,$$

where with $\langle w_i, zw_i, \ldots, z^n w_i \rangle$ we mean the R-module generated by $w_i, zw_i, \ldots, z^n w_i$. Since $\mathcal{B}_f/\mathcal{B}_f \cap \mathcal{B}_{\pm}$ is Noetherian, then there exists $N \in \mathbb{N}$ such that

$$\langle w_i, zw_i, \ldots, z^{N-1} w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_{\pm} = \langle w_i, zw_i, \ldots, z^N w_i \rangle + \mathcal{B}_f \cap \mathcal{B}_{\pm}$$

and so there exist $p_j \in R$, such that

$$z^N w_i = \sum_{j=1}^{N-1} p_j z^j w + w_{+} + w_{-},$$

where $w_{-} \in \mathcal{B}_{-}$ and $w_{+} \in \mathcal{B}_{+}$. It is clear that $\bar{w}_i := z^{-N} w_{-}$ has finite support and moreover $\bar{w}_i(0) = w_i(0)$. If we do the same with all w_i we obtain a family $\bar{w}_1, \ldots, \bar{w}_n \in \mathcal{B}$ with finite support satisfying

$$\mathcal{B}_f = \mathcal{B}(\bar{w}_1) + \cdots + \mathcal{B}(\bar{w}_n) + \mathcal{B}_{+}.$$

In a similar way we can show that there exists a family $\hat{w}_1, \ldots, \hat{w}_m \in \mathcal{B}$ with finite support satisfying

$$\mathcal{B}_p = \mathcal{B}(\hat{w}_1) + \cdots + \mathcal{B}(\hat{w}_m) + \mathcal{B}_{-}.$$

We can argue that

$$\mathcal{B}_f + \mathcal{B}_p = \hat{\mathcal{B}} + \mathcal{B}_{\pm},$$

where with $\hat{\mathcal{B}}$ we mean the set of trajectories in $\mathcal{B}$ with finite support. Note that $\hat{\mathcal{B}}$ is a finitely generated module over $R[z, z^{-1}]$. Let $w_1, \dots, w_n$ be a set of generators for $\hat{\mathcal{B}}$. It is not restrictive to suppose that their support are included in $[-N, N]$. We want to show that

$$\mathcal{B}_f + \mathcal{B}_p = \mathcal{B}_{[-2N,2N]} + \mathcal{B}_{\pm},$$

where with $\mathcal{B}_{[-2N,2N]}$ we mean the set of trajectory in $\mathcal{B}$ with support in $[-2N, 2N]$. Actually, if $w \in \mathcal{B}_f + \mathcal{B}_p$, then we have that $w = \hat{w} + w_- + w_+$, where $\hat{w} \in \hat{\mathcal{B}}$, $w_- \in \mathcal{B}_-$ and $w_+ \in \mathcal{B}_+$. Then

$$\hat{w} = \sum_{j=1}^{n} \sum a_{ij} z^i w_j = \sum_{j=1}^{n} \sum_{|i| \leq N} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{i<-N} a_{ij} z^i w_j + \sum_{j=1}^{n} \sum_{i>N} a_{ij} z^i w_j.$$

If we let

$$\hat{w}' := + \sum_{j=1}^{n} \sum_{|i| \leq N} a_{ij} z^i w_j \in \mathcal{B}_{[-2N,2N]},$$

$$w'_- := w_- + \sum_{j=1}^{n} \sum_{i<-N} a_{ij} z^i w_j \in \mathcal{B}_-$$

and

$$w'_+ := w_+ + \sum_{j=1}^{n} \sum_{i>N} a_{ij} z^i w_j \in \mathcal{B}_+,$$

then we have that $w = \hat{w}' + w'_- + w'_+$.

We show finally that $\mathcal{B}_f + \mathcal{B}_p$ is the behaviour of a strongly controllable system. Take $w \in \mathcal{B}_f + \mathcal{B}_p$. By the previous decomposition we can argue that $\bar{w} := z^{2N} w = \hat{w} + w_- + w_+$ with $\hat{w} \in \mathcal{B}_{[-2N,2N]}$, $w_- \in \mathcal{B}_-$ and $w_+ \in \mathcal{B}_+$. Then it is easy to see that $w' = z^{-2N}(\hat{w} + w_+)$ is such that that $w'_{|(-\infty,0]} = w_{|(-\infty,0]}$ and $w'_{|[4N,+\infty)} = 0$. Since $\mathcal{B}_f + \mathcal{B}_p$ is strongly controllable, we have that $\mathcal{B}_{\pm} \subseteq \mathcal{B}_f + \mathcal{B}_p \subseteq \mathcal{B}_{sc}$ and so $\mathcal{B}/\mathcal{B}_{sc}$ is Noetherian. ■

From the proof of the previous proposition we can obtain the complete characterization of the zero-controllable systems with finitely generated state space module. Actually we show that for realizable systems all the controllability concepts coincide.

Corollary 2 *Let R be a Noetherian ring and $\Sigma = (\mathbb{Z}, W, \mathcal{B})$ be a linear shift-invariant realizable system. Then Σ is zero-controllable if and only if Σ is strongly controllable.*

Proof ($\Leftarrow$) It is the given by proposition 9.
($\Rightarrow$) As we have seen in the proof of the previous proposition, if the state space module $\mathcal{X}$ of Σ is finitely generated, then $\mathcal{B}_p + \mathcal{B}_f$ is strongly controllable. Finally it is easy to verify that, if Σ is zero-controllable, then $\mathcal{B}_p + \mathcal{B}_f = \mathcal{B}$. ■

It could be reasonable to ask whether a result similar to the Rouchaleau-Kalman-Wyman theorem [2, 13, 12] holds true also in the behavioural approach. It is easy to see that if $\Sigma = (\mathbb{Z}, R^q, \mathcal{B})$ is a linear shift-invariant system and $\Sigma_e = (\mathbb{Z}, F^q, \mathcal{B}_e)$ is its field extension system, then Σ realizable implies Σ_e realizable. The converse is not true in general as shown in the following example.

Example Let $\Sigma = (\mathbb{Z}, \mathbb{Z}, \mathcal{B})$, where $\mathcal{B}$ is the $R[z, z^{-1}]$ submodule of $\mathbb{Z}^{\mathbb{Z}}$ generated be the trajectories w_1, w_2 such that

$$w_1(t) = \begin{cases} 2 & \text{if } t = 0 \\ 0 & \text{if } t \neq 0 \end{cases}$$

and w_2 is any irrational trajectory (i.e. a trajectory such that pw has infinite support, for every polynomial $p \in R[z, z^{-1}]$) with $w_2(0) = 1$ and $w_{2|(-\infty,0)} = 0$. For instance we can take the trajectory w_2 defined below

$$\bar{w_2}(t) = \begin{cases} 1 & \text{if } t = 2^k \ k \in \mathbb{N} \\ 0 & \text{othewise.} \end{cases}$$

Consequently we have that if $w \in \mathcal{B}$ has finite support, then $w = qw_1$ for some $q \in R[z, z^{-1}]$. We show how that the state space module $\mathcal{X} = \mathcal{B}/(\mathcal{B}_- + \mathcal{B}_+)$ of Σ is not Noetherian. Consider the increasing sequence of modules

$$\langle w_2 \rangle + \mathcal{B}_- + \mathcal{B}_+ \subsetneq \langle w_2, zw_2 \rangle + \mathcal{B}_- + \mathcal{B}_+ \subsetneq \langle w_2, zw_2, z^2 w_2 \rangle + \mathcal{B}_- + \mathcal{B}_+ \subsetneq \cdots$$

This sequence is strictly increasing. Actually, suppose that

$$\langle w_2, zw_2, \ldots, z^{N-1} w_2 \rangle + \mathcal{B}_- + \mathcal{B}_+ = \langle w_2, zw_2, \ldots, z^{N-1} w_2, z^N w_2 \rangle + \mathcal{B}_- + \mathcal{B}_+.$$

Then

$$z^N w_2 = \sum_{i=1}^{N-1} p_i z^i w_2 + w_- + w_+,$$

where $p_i \in R$, $w_- \in \mathcal{B}_-$ and $w_+ \in \mathcal{B}_+$. If we evaluate the the signals in the previous equation in $-N$ we obtain that $1 = (z^N w_2)(-N) = w_-(-N)$. Taking into account that w_- has finite support, we have that $w_-(-N)$ is even and this gives a contradiction. Therefore Σ is not realizable.

On the other hand it is not difficult to verify that the field extension $\Sigma_e = (\mathbb{Z}, \mathbb{Q}, \mathcal{B}_e)$ is realizable and that its state space module $\mathcal{X}_e = \{0\}$.

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