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CONTROLLABILITY AND POLE ASSIGNMENT FOR DISCRETE TIME LINEAR SYSTEMS DEFINED OVER ARBITRARY FIELDS*

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1. Introduction. The theory of controllability and observability due to Kalman is certainly one of the most important conceptual contributions to linear systems theory. An account of the development of the ideas of controllability and observability as well as its implications on feedback control theory and realization theory may be found in the recent book of Kalman, Falb and Arbib [1].

It has been known for some time that for a linear continuous, finite-dimensional autonomous system with a scalar control variable, complete controllability is equivalent to being able to assign arbitrary poles to the closed loop transfer matrix by a suitable choice of state variable feedback gain matrix. This result was generalized to the vector control case by Wonham [2] and Simon and Mitter [3]. In [3] constructive recursive algorithms to achieve pole assignment were also presented. The objective of this note is to generalize this result to cover discrete-time, finite-dimensional, autonomous linear systems defined over arbitrary fields. The result can thus be applied to the feedback control of linear sequential machines [4]. By duality arguments the problem of state determination is also solved.

2. Notation and system definition. Let

T = time set = Z = (ordered Abelian group of) integers;
U = input values = F^m = vector space of m-tuples over the field F;
X = state space = F^n;
Y = output space = F^p;
Ω = input space of functions $t \mapsto u(t)$; that is, arbitrary sequences $u(-1), u(0), u(1), \cdots$, with $u(t) \in U$.

We shall be concerned with the discrete-time, autonomous, linear dynamical system Σ defined over a field F,

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{2.1}$$

with $t \in Z$, $x(t) \in F^n$, $u(t) \in F^m$, $y(t) \in F^p$ and where

$$A : F^n \to F^n, \quad B : F^m \to F^n, \quad C : F^n \to F^p \tag{2.2}$$

are F-homomorphisms.

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We shall usually not make a distinction between (A, B) and (A, C) as a pair of F-homomorphisms or as a pair of matrices representing these homomorphisms with respect to a given basis.

With respect to the system (2.1) we make the assumption:

(i) the pair (A, B) is *completely reachable*, that is, the rank of the $n \times nm$ matrix

$$H(A, B) = [B, AB, \cdots, A^{n-1}B] \tag{2.3}$$

is n.

(ii) the pair (A, C) is *completely observable*, that is, the rank of the $n \times np$ matrix

$$K(A, C) = [C^T, A^TC^T, \cdots, (A^T)^{n-1}C^T] \tag{2.4}$$

is n.

3. Statement of main theorem.[1] The principal result of this paper is the following theorem.

THEOREM 3.1. *For the linear autonomous system* (2.1), (A, B) *is a completely reachable pair if and only if for every monic polynomial* g *of degree* n, *there exists an* $m \times n$ *matrix* K *over* F *such that the characteristic polynomial of* $A + BK$ is *precisely* g (*up to a factor of* ± 1).

The proof of the result proceeds via several propositions and is presented in the next section.

4. Proof of main theorem. The proof will be divided into three parts: necessity for the case when B is a column vector, necessity for a general B and sufficiency.

PROPOSITION 4.1 (case $m = 1$). *In* (2.1) *let* $B = b = n \times 1$ *matrix. If* (A, B) *is a completely reachable pair, then there exists a* $1 \times n$ *matrix* k *such that the characteristic polynomial of* $A + bk$ *has an arbitrary preassigned form* (*of degree* n).

The proof of this proposition essentially consists of transforming A to rational canonical form and is identical to the proof given for the field of real numbers (see, for example, [5, Theorems 7 and 9]).

We now consider the case where B is an $n \times m$ matrix.

PROPOSITION 4.2. *If* (A, B) *is a completely reachable pair, then there exists a matrix* K *and a vector* b *such that* $(A + BK, b)$ *is a completely reachable pair and* b *is in the column space of* B.

Proof. The proof presented is essentially the same as independently given by Heymann [6] and hence only an outline of the proof will be given.

Let b_j be the jth column of B and let E_j be the cyclic subspace of the coordinate space $E = F^n$ generated by b_j. Since (A, B) is a completely reachable pair $E = E_1 + \cdots + E_m$. In general, the E_i are not independent, that is, $E_i \cap E_j \neq \varnothing$ for $i \neq j$. However, it is easy to see that there are subspaces S_i and a finite integer t, $0 < 1 \leqq m$, such that $E = S_1 + \cdots + S_t$ and $S_i \cap S_j = \varnothing$ for $i \neq j$, that is, E is

[1] It was pointed out by the reviewer that a similar result has been obtained by R. E. Kalman in the unpublished notes: *Lectures on Controllability and Observability*, CIME Seminar, Italy, February 1969.

a direct sum of the subspaces S_i. A basis for E can now be obtained by combining the bases for the subspaces.

By rearranging the columns of B (hence the coordinates of the control) it can be assumed that the first t columns of B are used. Hence the basis is

$$b_1, \cdots, A^{k_1-1}b_1, \cdots, b_t, \cdots, A^{k_t-1}b_t \quad \text{and} \quad \sum_{i=1}^{t} k_i = n.$$

Let $R = [b_1, \cdots, A^{k_1-1}b_1, \cdots, b_t, \cdots, A^{k_t-1}b_t]$ be the matrix whose columns are the above basis vectors. Clearly R is invertible.

Define an $m \times n$ matrix $S = [s_1 \cdots s_n]$, where each column is an m-tuple defined as follows:

$$s_{r_j} = \varepsilon_{j+1}^{(m)} \quad \text{if} \quad r_j = \sum_{i=1}^{j} k_i \quad \text{and} \quad j = 1, \cdots, t-1; \quad s_j = 0 \quad \text{otherwise},$$

where $\varepsilon_i^{(m)}$ is the ith standard basis vector of F^m.

Finally, let $P = SR^{-1}$. Clearly $PA^{k_i-1}b_j = \varepsilon_{j+1}^{(m)}$, $j = 1, \cdots, t-1$, and $PA^ib_j = 0$ for all other powers of A.

Let $\underline{A} = A + BP$. Then the controllability matrix of the pair $(\underline{A}, b_1)$ is $\underline{H} = [b_1\underline{A}b_1 \cdots \underline{A}^{n-1}b_1]$ and it has rank n. Clearly b_1 is in the column space of B. The necessity part of the theorem now follows from Proposition 4.1.

We now prove sufficiency.

PROPOSITION 4.3. *Given an arbitrary monic polynomial g of degree n, if there exists an $m \times n$ matrix K such that the characteristic polynomial of $A + BK$ is precisely g, then (A, B) is a completely reachable pair.*

Proof. We first assume that the field F has a sufficient number of scalars $a_1, \cdots, a_n$ such that $\det(A - a_iI) \neq 0$, $i = 1, 2, \cdots, n$. From the above assumption and by hypothesis there is a K such that $(A + BK)v_i = a_iv_i$ and $v_i \neq 0$. Since $a_iI - A$ is invertible, we have

$$(a_iI - A)^{-1}BKv_i = v_i, \qquad i = 1, 2, \cdots, n. \tag{4.1}$$

Now for each a_i there are scalars $b_j(a_i)$ such that

$$(a_iI - A)^{-1} = \sum_{j=1}^{n} b_j(a_i)A^{j-1}, \qquad i = 1, 2, \cdots, n. \tag{4.2}$$

Hence from (4.1) and (4.2), we obtain

$$\sum_{j=1}^{n} A^{j-1}B(b_j(a_i)Kv_i) = v_i, \qquad i = 1, 2, \cdots, n. \tag{4.3}$$

Let $H = [B, AB, \cdots, A^{n-1}B]$ and $y = (y_1 \cdots y_n)^T \in F^{nm}$. Then

$$Hy = \sum_{j=1}^{n} A^{j-1}By_j.$$

If we set $y_j = b_j(a_i)Kv_j$, then (4.3) becomes

$$Hy_i^* = v_i, \qquad i = 1, 2, \cdots, n, \tag{4.4}$$

where $y_i^* = (b_1(a_i)Kv_i \cdots b_n(a_i)Kv_i)^T$.

Since the eigenvalues of $A + BK$ are distinct, the eigenvectors $v_1, \cdots, v_n$ are linearly independent and form a basis for F^n. Hence, by using (4.4), any $v \in F^n$ can be written as $v = H(\sum_i c_i y_i^*)$. Therefore the range of H is F^n and hence (A, B) is a completely reachable pair.

Now if F does not contain enough distinct scalars, apply Proposition A.3 of the Appendix to $f = \det(A - xI)$ and $g = \det(A + BK - xI)$. Then over some extension field $F' \supset F$, g has n distinct roots none of which are roots of f. Now from the proof of Proposition 4.3, H considered as a linear transformation of $(F')^{nm} \to (F')^n$ has rank n. But H is a matrix over $F \subset F'$; hence it has rank n over F also.

For finite fields (containing at least 2 elements) the following stronger result can be proved.

THEOREM 4.4. *The following statements are equivalent*:

(i) (A, B) *is a completely reachable pair*;

(ii) *Given a monic polynomial* g *of degree* n, *there exists a matrix* K *such that the characteristic polynomial of* $A + BK$ *is precisely* g;

(iii) $B \neq 0$, *and given an irreducible polynomial* p *of degree* n, *there exists a matrix* K *such that the characteristic polynomial of* $A + BK$ *is* p.

Proof. The theorem will be proved by showing that the statements (i) and (iii) are equivalent.

(i) $\Rightarrow$ (iii) from Theorem 3.1.

We now prove the reverse implication. For $n = 1$, the result is obvious. For $n > 1$, by Proposition A.1 we can construct an irreducible polynomial of degree n.

Let $\mathcal{R}$ denote the range of $H(A, B)$. Define the map

$$\bar{A}: F^n/\mathcal{R} \to F^n/\mathcal{R}$$

by

$$\bar{A}\bar{x}_i = \overline{(A + BK)x_i}, \qquad i = 1, 2, \cdots, n,$$

where $F^n/\mathcal{R}$ is the quotient space, $\{x_1, \cdots, x_n\}$ is a basis for F^n and $\bar{x}$ denotes the coset of x in the quotient space $F^n/\mathcal{R}$ and K is an $m \times n$ matrix. This is a well-defined map since $\mathcal{R}$ is an A-invariant subspace of F^n.

Let $p(x) = \sum_{i=0}^n p_i x^i$ be the characteristic polynomial of $A + BK$. Then by the Cayley–Hamilton theorem $p(A + BK) = 0$. It is easily verified that $\bar{A}$ is an endomorphism of $F^n/\mathcal{R} \to F^n/\mathcal{R}$. Using an induction on k we can show

$$\bar{A}^k\bar{x} = \overline{(A + BC)^k x}$$

and we may verify that $p(\bar{A}) = 0$ (that is, the zero map on $F^n/\mathcal{R}$).

Let m be the minimal polynomial of $\bar{A}$. Then m divides p since $p(\bar{A}) = 0$. Since by hypothesis p is irreducible, either $m = 1$ or $m = \pm p$.

Since $B \neq 0$, $\deg m < n = \deg p$, so $m = 1$. But $m(\bar{A}) = 0$; this means that the identity map on $F^n/\mathcal{R}$ is equal to the zero map and hence $F^n/\mathcal{R} = \mathcal{R}$. Therefore $\mathcal{R} = F^n$ and (A, B) is a completely reachable pair.

5. An example. As an example, consider the following three-state circuit over the field Z_3.

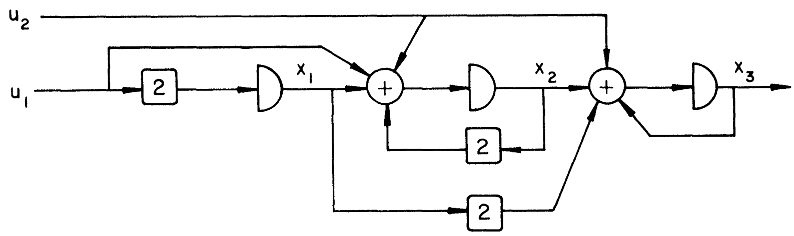

FIG. 1

By inspection of the diagram,

$$x_1(n+1) = 2u_1(n),$$
$$x_2(n+1) = x_1(n) + 2x_2(n) + u_1(n) + u_2(n),$$
$$x_3(n+1) = 2x_1(n) + x_2(n) + x_3(n) + u_2(n).$$

Letting $x^T(n) = [x_1(n) \quad x_2(n) \quad x_3(n)]$ and $u^T(n) = [u_1(n) \quad u_2(n)]$,

$$x(n+1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 1 & 1 \end{pmatrix} x(n) + \begin{pmatrix} 2 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} u(n) = Ax(n) + Bu(n).$$

By direct calculation,

$$H = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 2 & 0 & 1 \end{pmatrix},$$

which has rank 3. Following the construction in Proposition 4.2,

$$R = [b_1 \quad Ab_1 \quad A^2b_1] = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}.$$

Then $R^{-1} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 2 \\ 2 & 2 & 2 \end{pmatrix}$; also, $S = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore $P = SR^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 2 & 2 \end{pmatrix}$.

Again by direct calculation, $\underline{A} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ and $\underline{H} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$, which is the same as R in this case, so $\underline{H}$ has rank 3. Letting $p_1 = (p_{11}p_{12}p_{13})$, the characteristic polynomial of $\underline{A} + b_1p_1$ is $-x^3 + (2p_{11} + p_{12} + 1)x^2 + (2p_{13} + p_{11})x + (p_{12} + p_{13})$. To see that these coefficients may be chosen arbitrarily, it suffices to note that the following determinant is nonzero:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{pmatrix} = 1 \neq 0.$$

6. Observability and state reconstruction. Since the pair (A, C) has been assumed to be a completely observable pair, it follows from Theorem 3.1 that for every monic polynomial g of degree n, there exists a $p \times n$ matrix $-D^T$ such that the characteristic polynomial of $A^T - C^T D^T$ is precisely g. Hence the characteristic polynomial of $A - DC$ can be made arbitrary.

Now consider an observer [7]:

$$\hat{x}(t+1) = X\hat{x}(t) + Dy(t) + Bu(t), \qquad t = 0, 1, 2, \cdots, \tag{6.1}$$

where X and D are $n \times n$ and $n \times p$ matrices respectively. Let D be chosen such that $A^T - C^T D^T$ has arbitrary characteristic polynomial and let $X = A - DC$.

Then

$$\begin{aligned} \hat{x}(t+1) - x(t+1) &= (A - DC)[\hat{x}(t) - x(t)], \\ \hat{x}(0) - x(0) &= \text{given}. \end{aligned} \tag{6.2}$$

From (6.2) it follows that

$$\hat{x}(n) - x(n) = (A - DC)^n[\hat{x}(0) - x(0)].$$

Since the characteristic polynomial of $A - DC$ can be made arbitrary, the matrix $A - DC$ can in particular be made nilpotent and hence the observer reconstructs the initial state in at most n steps.

Appendix. In this Appendix some results on finite fields which are used in the proof of Proposition 4.3 are presented.

The following results are needed. The proof of the first two propositions are consequences of well-known results on finite fields (see Lang [8, Chap. VII, § 5]).

PROPOSITION A.1 (see [9, p. 128]). *If F is a finite field consisting of at least two elements, then the polynomial ring $P(F)$ contains irreducible polynomials of every degree $\geqq 2$.*

PROPOSITION A.2. *For every irreducible polynomial over a finite field F, there is an extension field F' such that the given polynomial has n distinct roots in F', where n is the degree of the polynomial.*

PROPOSITION A.3. *Let F be a finite field and f a given polynomial of degree n over F. Then there is a polynomial g of degree n over F and some extension field $F' \supset F$ such that g has n distinct roots in F', none of which are roots of f.*

Proof. First, consider the case when f has at least one root in F. Then $f = f'f''$, where f' is a product of linear factors and f'' has no roots in F. Also, $\deg f' \geqq 1$, so $\deg f'' < n$.

By Proposition A.1, there is an irreducible polynomial g over F of degree n. Then $\gcd(f'', g) = 1$ since $\deg f'' < \deg g$. Clearly $\gcd(f', g) = 1$, so $\gcd(f, g) = 1$.

By Proposition A.2, there is an extension field $F' \supset F$ such that g has n distinct roots in F'. But $\gcd(f, g) = 1$ in F' also, so no root of g is a root of f in F'.

Next, consider the case when f has no roots in F. Now, etiher F has n distinct scalars or not. If it has, let $g = (a_1 - x)(a_2 - x) \cdots (a_n - x)$, where the a_i are distinct scalars in F. Then no root of g is a root of f, and F is the desired extension field.

If F does not have n distinct scalars, consider the prime factorization of f: $f = p_1 p_2 \cdots p_k$, where each p_i is an irreducible polynomial of degree at least 2, say $\deg p_i = m_i$, and $\sum_{i=1}^{k} m_i = n$. (Note that the p_i may not be distinct.)

If $k > 1$, then each $m_i < n$. Pick a polynomial g over F irreducible of degree n. Then $\gcd(f, g) = 1$. Again. let $F' \supset F$ be an extension of F containing n distinct roots of g. Since $\gcd(f, g) = 1$, no root of g is a root of f.

If $k = 1$, then $f = p_1$ and $m_1 = n$. Let F have p distinct scalars, and choose $g' = (a_1 - x) \cdots (a_p - x)$, where the $a_i \in F$ are distinct. Also choose g'' over F irreducible of degree $n - p$. Then $\gcd(f, g'') = 1$ and $\gcd(f, g') = 1$; hence, $\gcd(f, g'g'') = 1$.

Let $F' \supset F$ be an extension field in which g'' has $n - p$ distinct roots. Therefore $g = g'g''$ has n distinct roots in F'. Also, since $\gcd(f, g) = 1$, no root of g is a root of f.

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Note added in proof. It was recently pointed out to me by R. W. Brockett that V. M. Popov proved the result on pole assignment earlier in his paper: *Hyperstability and optimality of automatic systems with several control functions*, Rev. Roumaine Sci. Techn.-Electrotechnct. Energ., 9 (1964), pp. 629–690.

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