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MODELLING FOR STOCHASTIC SYSTEMS AND QUANTUM FIELDS[1]

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1. Introduction

In this paper I would like to give a very partial account of integration theory in Hilbert space and related questions of absolute continuity which may be important in problems of stochastic realization theory, linear and non-linear filtering, detection theory and quantum communication theory. This theory is largely the creation of I.E. Segal and his former students, notably Gross and Nelson. The need for such a theory arose for the purpose of putting quantum field theory on a rigorous mathematical basis. The theory has a distinct algebraic character and I believe is particularly suited to the needs of stochastic system theory. An account of this algebraic approach may be found in SEGAL-KUNZE [1], SEGAL ([1] and the bibliography cited there). This theory is different from the work of the Russian school (cf. GELFAND-VILENKIN) in the sense that essentially Hilbert space techniques are used and in general one works with "weak" processes as opposed to "strict" processes. In this theory non-linear functions of processes can be handled and in particular certain non-linear functionals of white noise can be given mathematical meaning. The other approach to some of these questions is due to GROSS (cf. GROSS [1], [2] and the bibliography cited therein) where a countably additive "extension" on a separable Banach space of the finitely-additive Gaussian measure on a Hilbert space is obtained. These ideas have recently been modified and developed by Balakrishnan (cf. for example BALAKRISHNAN [1], [2]) in a series of papers related to detection and filtering theory.

2. Segal-Gross Theory of Weak Processes

It is a known fact that there is no analog of Lebesgue measure (i.e. a countably additive measure which is translation and notation invariant) on an infinite dimensional Hilbert space. In fact no such measure exists even when invariance is relaxed to quasi-invariance. Such an "invariant" measure however exists if we do not insist on countable additivity.

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Let V be a real topological vector space, V^* its topological dual and let $<.,.>$ denote the pairing of V and V^*. A <u>tame set</u> is a set C of the form

$$(2.1)\quad C = \{x\varepsilon V \mid (<x,y_1>, \ldots <x,y_n>) \in A\}$$

where A is a Borel set in R^n and $y_j \varepsilon V^*$, $j=1,2,\ldots n$.

If K is a finite dimensional subspace of V^* containing $y_1,\ldots,y_n$ then C is said to be <u>based</u> on K. The collection S_K of tame sets based on K is a σ-ring. Let $R = \bigcup_K S_K$ be the ring generated by $(S_K)_{K\varepsilon V^*}$.

A non-negative set function μ defined on R is a <u>tame set measure</u> if

1) $\mu(V) = 1$

2) $\forall$ finite dimensional subspace $K\subset V^*$, is countably additive when restricted to the σ-ring S_K.

An equivalent concept is that of a <u>weak distribution</u>. Let $M = (\Omega,A,P)$ be a probability space. A <u>weak distribution</u> on V is an equivalence class of linear maps $F : V^* \to RV(\Omega,A,P)$. Two such maps are equivalent if for any finite set $y_1,\ldots,y_n \in V^*$, the joint distribution of $F_j(y_1),\ldots,F_j(y_n)$ in E^n is the same for $j = 1,2$.

A <u>tame function</u> F on V is one of the form $F(x) = f(<x,y_1>,\ldots,<x,y_n>)$ for some Baire function on R^n and for some finite sequence $y_1,\ldots,y_n$ in V^*. If K is some finite dimensional subspace containing $y_1,\ldots,y_n$ then F is said to be based on K.

Let H be a real Hilbert space. For C a tame subset of V based on K, define

$$(2.2)\quad \mu(C) = (2\pi)^{-\frac{n}{2}} \int_A \exp\left(-\frac{||x||^2}{2}\right) dx$$

where A is a Borel set in K and $n = \dim K$. It is possible to enlarge H and obtain a countably additive measure on a larger space which is in a sense an extension of μ.

Let $f(x) = f(Px)$ be a tame function for some finite dimensional projection P. Let ϕ be the restriction of f to F = Range (P). Then ϕ is Borel measurable on F and

$$(2.3)\quad \int_H f(x)d\mu(x) = (2\pi)^{-\frac{n}{2}} \int_F \phi(x)\mathrm{esp}\left(-\frac{||x||^2}{2}\right)dx$$

where $n = \dim(F)$.

Let A = algebra of bounded complex-valued functions on H together with their uniform limits. The integral defined in (2.3) can be extended to all of A as a continuous linear functional which

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we denote by E(f). A is a commutative C^*-algebra with unit in the supnorm and hence $A \simeq C(x)$ for some compact Hausdorff space X. Let $f \mapsto \tilde{f}$ be this isomorphism. Moreover E is a continuous positive linear function on A and hence by the Riesz representation theorem

$E(f) = \int_X \tilde{f}\, dm$, where m in this case is probability measure. The ismorphism $f \mapsto \tilde{f}$ can be extended to tame functions in $L^1(H,\mu)$ by density such that $(fg)^\sim = \tilde{f}\tilde{g}$. Hence if $f = \text{char.fn.}(A)$, A a tame set, $\tilde{f}$ is characteristic function of some measurable set $\tilde{A}$ and $\mu(A) = m(\tilde{A})$. Using the Gelfand Transform, we can see that for $f \in A$, $\tilde{f}$ is an extension of f from H to all of X. Now A and in fact H is such that $m(H) \neq 0$.

Let H be identified with H^*. The continuous linear functionals on H are in $L^2(H,\mu)$. To the linear functional determined by y there corresponds a measurable F(y)(.) on X. $F : H \to L^2(X,m)$ is norm-preserving. It can be shown (1) that the map F completely determines the extension $f \mapsto \tilde{f}$, (2) the map $f \mapsto \tilde{f}$ can be extended to all tame functions and (3) $\tilde{f} = \phi(F(y_1),\ldots,F(y_n))$ where $f(x) = \phi((x,y_1),\ldots,(x,y_n))$. The functions F(y) on X are normally distributed with variance $||y||^2$ and $y_1,\ldots,y_n$ are orthogonal, then $F(y_1),\ldots,F(y_n)$ are independent. More concrete realizations of H and the measure space (X,m) can be obtained, for example

a) where $H = \ell^2$, $X = R^\infty$, m the product measure corresponding to Gauss measure on each coordinate, b) $H = H^1(0,1)$, $X = C(0,1)$ with m = Wiener measure. However it can be easily proved that these various extensions are all measure theoretically isomorphic.

3. Abstract Wiener Spaces and Absolute Continuity

The discussion above could be formalized using the ideas of Abstract Wiener Space due to Gross.

Let H be a Hilbert space and let μ be the tame measure given by (2.2). A measurable norm on H is a norm $||\cdot||$ such that $\forall \varepsilon > 0$, $\exists$ finite dimensional projection P_0 such that $\forall$ finite dimensional $P \perp P_0$,

$$\mu(\{x\varepsilon H \,\big|\, ||Px||>\varepsilon\})>\varepsilon.$$

Let E = completion of H with respect to $||\cdot||$. E is a Banach space. The canonical embedding $i : H \to E$ is compact. Identifying H and H^*, we obtain by duality the embeddings

$$E^* \xhookrightarrow[i^*]{} H^* = H \xhookrightarrow[i]{} E$$

E^* can be identified with its image in H and H with its image in E. The measure μ has a countably additive extension p on the borel fields of E. The triple (i,H,E) is called Wiener space and p Wiener measure on E.

Now, E^* would be interpreted as functions on E belonging to $L^2(E,p)$ and their L^2-norm equals the H^* norm. Hence the closure of E^* in $L^2(E,p)$ can be identified with H^*. If $e \in H^*$, we denote by $\tilde{e}$ the corresponding random variable on E. Let P be a finite dimensional orthogonal projection on H such that

$$Px = \sum_{i=1}^{n} \langle e_i,x\rangle e_i,\ (e_i)$$

orthonormal in H, then $\tilde{P} = \sum_{i=1}^{n} \tilde{e}_i \circ e_i$ defines a random variable on E with values in H.

A function f on H with values in a Banach space F determines a random variable $\tilde{f}$ on E with values in F if for any sequence of finite dimensional projection $P_n \to I$ strongly in H the sequence of random variables $f\circ\tilde{P}_n \to \tilde{f}$ in measure p on E.

Some Preliminaries

Let H be a real Hilbert space and let $L(H)$ denote the space of bounded linear operators on H. Let I_1 denote the Banach space of nuclear operators on H under the norm $||K||_1 = \mathrm{tr}[(K^*K)^{\frac{1}{2}}]$. I_1 is a *-ideal in $L(H)$. Let I_2 denote the Banach space of Hilbert Schmidt operators on H with norm $||K||_2 = [\mathrm{tr}(K^*K)]^{\frac{1}{2}}$. I_2 is also a *-ideal in $L(H)$.

If $K \in I_1$, the Fredholm determinant of (I+K) is defined by $\det(I+K) = \prod_{i=1}^{\infty}(1+\lambda_i)$ where the λ_i are the eigenvalues of K counted with their multiplicities. If $K \in I_2$, the Carleman Fredholm determinant of I+K is defined by $\delta(I+K) = \prod_{i=1}^{\infty}(1+\lambda_i)e^{-\lambda_i}$. det(I+K) is an analytic function on I_1 and $\delta(I+K)$ is an analytic function on I_2. ▨

Preliminary Lemmes

The following lemmes follow from the work of Gross (cf. GROSS[1]).

Lemma 3.1 Let $K \in L(H)$. Then K determines a random variable $\tilde{K}$ on E with values in E. ▨

Lemma 3.2 Let P_n be a family of orthogonal projections converging to I strongly. Let $K \in I_2$. Then $(K\circ\tilde{P}_n)$ is a Cauchy sequence in $L^2(E,p;H)$ and $\mathrm{Prob}(\tilde{K} \in H) = 1$. ▨

Suppose $K \in I_2$. Then in general <Kx,x> and Tr(K) need not exist. However <Kx,x> - Tr(K) can be given a meaning as a real random variable on E via stochastic extension.

In fact, for certain non-linear operators $K : E \to H$ $\langle Kx,x\rangle - \mathrm{tr}K_x$ can be identified as a random variable. In the above K is continuous and its H-derivative (defined below) K_x is Hilbert-Schmidt.

Let $U \subset E$ be open. A function $f : U \to F$, F Banach is H-continuous at $x \in U$ if the function $g(h) = f(x+h)$ defined on $(U\setminus\{x\}) \cap H$ is continuous at the origin in the induced (Hilbert) topology. f is H-differentiable at x if g is Fréchet differentiable at the origin in H. It can then be shown (cf. RAMER).

Proposition: Let $U \subset E$ be open and let $K : U \to E$ be such that (i) $K(U) \subset H$, (ii) the H-derivative at x, $K_x : U \to L(H,E)$ is continuous and Hilbert-Schmidt. Let $(e_n)_{n\in N}$ be an orthonormal basis in H such that $e_i \in E^*$, $\forall i$. Let $P_n = \sum_{i=1}^{n} e_i \otimes e_i : E \to E^*$. Then

(i) $\{a_n(x)\} = \{\langle P_n Kx,x\rangle - tr(P_n K)_x\}_{n\in N}$ is a Cauchy sequence in $L^2(E,p)$.
(ii) There exists a subsequence (n_k) such that $\{a_{n_k}(x)\}$ converges almost everywhere on U to a random variable on U. Denoting this random variable by $\langle Kx,x\rangle - trK_x$, if $(m_k)_{k\in N}$ is any other sequence for which $(a_{m_k}(x))$ is Cauchy a.e., then $a_{m_k}(x) \to \langle Kx,x\rangle - trK_x$ a.e.
(iii) $\langle Kx,x\rangle - trK_x$ does not depend on the choice of basis $(e_i)_{i\in N}$ in H. ▨

Absolute Continuity and Computation of the Radon-Nikodym Derivative

__Case I__ (__Translation__)

__Theorem 3.1__ (Segal)
Let (i,H,E) be an abstract Wiener Space and let p be standard Wiener measure on E. Let $e \in E$ and let $T_e : E \to E : x \mapsto x + e$. Then the transformed measure pT_e and p are mutually absolutely continuous if and only if $e \in H$. The R - N derivative of pT_e with respect to p is the random variable $\exp\left(-\tilde{e} - \frac{|e|^2}{2}\right)$. ▨

__Remark__:
If $E = C(0,1;\mu_W)$ where μ_W denotes Wiener measure then $H = H^1(0,1;\mu_G)$, the Sobolev space with Gaussian measure. ▨

__Case II__ (Linear Transformation)

__Theorem 3.2__ (Segal-Feldman)
Let (i,H,E) be an abstract Wiener Space and p standard Wiener measure. Let q be a Gaussian measure on E^* with covariance Q. Then p and q are either mutually singular or mutually absolutely continous. They are mutually absolutely continuous if and only if there exists a $K \in I_2$, symmetric, such that the quadratic form $Q(x)$ on E^* is of the form $Q(x) = \langle (I+K)x,x\rangle$. The R - N derivative of q with respect to p is the random variable on E given by

$$\prod_{i=1}^{\infty} (\lambda_i+1)^{-\frac{1}{2}} \exp\left(\tfrac{1}{2}\lambda_i(\lambda_i+1)^{-1}\tilde{e}_i^{\,2}\right).$$ ▨

It is possible to use Theorem 3.2 to prove

__Theorem 3.3__
Let (i,H,E) be abstract Wiener Space and let p be the standard Wiener measure on E. Let $T = I + K$ be an invertible linear transformation on E with $K \in L(E,H)$. Then $K|_H \in I_2$ and $(K|_H)^{\sim} = K$. Then the R - N derivative of the transformed measure pT with respect to p is given by

$$|\delta(T)|\exp[-(\langle Kx,x\rangle - trK) - \tfrac{1}{2}|Kx|^2] \text{ a.e.}$$ ▨

The affine case could now be proved using Theorem 3.1. There is a non-linear version of Theorem 3.3.

__Theorem 3.4__ (Ramer)
Let (i,H,E) be an abstract Wiener Space and p be standard Wiener measure on E. Let $U \subset E$ be open and let $T : I + K : U \to E$ be a continuous non-linear transformation such that
(i) T is a homeomorphism of U onto an open subset of E.
(ii) $K(U) \subset H$ and $K : U \to H$ is continuous.
(iii) For each $x \in U$, the H-derivative of K at x, K_x exists, is Hilbert-Schmidt and $K_x : U \to I_2$ is continuous and $I_H + K_x$ is invertible.

Then p and the transformed measure pT are mutually absolutely continuous as measures on U. The R - N derivative of pT with respect to p is given by

$$|\delta(T_x)|\exp[-(\langle Kx,x\rangle - trK_x) - \tfrac{1}{2}|Kx|^2] \text{a.e } x\in U.$$ ▨

__Remarks__:
(i) As mentioned earlier $\langle Kx,x\rangle - tr(K_x)$ is a random variable. It is intriguing to see the appearance of the term $tr(K_x)$ which bears a striking resemblance to the Wong-Zakai correction term relating the Ito and Stratanovich integral.
(ii) Consider the Kalman Filtering problem

$$dx_t = Fx_t dt + Gdw_t$$

$dy_t = Hx_t dt + d\eta_t$, where w_t and η_t are standard Wiener processes assumed to be independent.

Then by passing to the Innovations Representation

$$dy_t = H\hat{x}_t dt + d\nu_t$$

where $\hat{x}_t = E(x_t|F_t^y)$ and ν_t is the Innovations process (which is a standard Wiener process) and noting that

$$\hat{x}_t = \int_0^t K(t,s)d\nu_s, \text{ with}$$

$K(\cdot,\cdot) \in L^2([0,t]x[0,t]; L(R^p;R^n))$, we are in the situation of Theorem 3.3. A "causal" representation for the R - N derivative could be obtained by invoking the Krein Factorization Theorem in conjunction with Theorem 3.3. (cf. HITSUDA where the reverse process is followed).

4. The Free Quantum Field and Kalman Filtering

In the previous section we have indicated how starting from a Hilbert space H with Gauss measure of unit variance n on it we can construct a Banach space E and a measure μ which is countably additive on the borel sets of E such μ is an extension of n in a certain precise sense. Integration of functions on H and questions of absolute continuity can be answered by passing to the Banach space by an appropriate stochastic extension. There is a purely Hilbert space point of view due to Segal which may turn out to be more important for the needs of System Theory. Due to lack of space we do not give a detailed exposition of this theory here. This theory is

needed to show the equivalence of various representations of the free quantum field[1] viz.
(i) The particle representation which involves the symmetric tensor products of a complex Hilbert space H with itself, (ii) the wave representation (functional integration) in the space $L^2(H')$ of a real part of H and (iii) the complex-wave representation in which a space K of entire anti-holomorphic functions on H are involved. The intertwining operators between the various representations requires absolute continuity considerations and the use of the Fourier-Wiener Transform. Mathematically, the field is diagonalized in the functional integration representation whereas the particle numbers are diagonalized in the tensor product representation. In the complex wave representation the creation operators achieve a kind of diagonalization.

Brockett (cf. BROCKETT) has recently shown that the group with 4 generators H, P, Q, E with the commutation relations

$$[H,P] = -Q \ , \quad [H,Q] = P \ , [P,Q] = E \quad \text{with}$$

the rest zero plays an important role in Kalman Filtering theory. This group has been called the Harmonic Oscillator Group (cf. STREATER). The group generated by P, Q, E, the Heisenberg group, is a subgroup of the oscillator group. The oscillator group is not nilpotent but soluable. Streater has obtained all the continuous unitary irreducible representations of the Harmonic Oscillator group. He shows that if complex Lie algebras are allowed then one can obtain the Bargmann-Segal representation of the harmonic oscillator by holomorphic functions using the technique of Kirilov. In this representation the creation operator $C(z)$ is multiplication by z and the annhilation operator $C^*(z)$ is $\frac{\partial}{\partial z}$.

Segal (cf. SEGAL [2]) has explicitly given the intertwining operators between the holomorphic and real representations. It would thus appear that the Zakai equations for the unnormalized conditional density corresponding to the Kalman filtering problem defines a "field" which is analogous to the "free quantum field".

Definition:

A concrete free Boson field over a given complex Hilbert space H, denoted as $\Phi(H)$ is a quadruple (K,W,Γ,V) consisting of

(i) a complex Hilbert space K,
(ii) A continuous mapping $z \mapsto W(z) : H \to U(K)$, the space of unitary operators on K satisfying the Weyl relations

$$W(z)W(z') = \exp(\tfrac{i}{2} \operatorname{Im}\langle z,z'\rangle W(z+z'), \ \forall z,z' \varepsilon H,$$

(iii) A continuous representation Γ from $U(H) \to U(K)$ satisfying

$$\Gamma(U)W(z)\Gamma(U)^{-1} = W(Uz) \ , \forall U \ \varepsilon \ U(H), \ z \ \varepsilon \ H$$

(iv) A unit vector $v \ \varepsilon \ K$ having the properties that $\Gamma(U)v = v \ \forall U \ \varepsilon \ U(H)$ and $W(z)v$, $z \ \varepsilon \ H$ span K topologically
(v) Γ is positive in the sense that if A is any positive self-adjoint operator on H, then $d\Gamma(A)$ is positive where for any positive self-adjoint A in H $d\Gamma(A)$ is the self-adjoint generator of the one-parameter unitary group $[U(e^{itA})|t\varepsilon R]$. ∎

Let H' be a real Hilbert space and let g denote the centred Gaussian weak distribution on H' with variance 1. We define a positive linear functional E (expectation) on the algebra $A(H')$ of all bounded tame functions on H'. Let $L^2(H',g)$ be the completion of $A(H')$ with respect to the inner product $\langle f,f'\rangle = E(ff')$. Let θ denote the canonical homomorphism of $A(H')$ into $L^2(H',g)$.

If H is a complex Hilbert space, it has also the structure of a real Hilbert space with inner product equal to the real part of the complex inner product in H. In this way, we can define g on H and hence $L^2(H,g)$. From the work of Segal, we know $L^2(H,g)$ can be regarded as the completion of the algebra $P'(H)$ of functions of the form

$$f(x) = p(\operatorname{Re}\langle x,e_1\rangle,\ldots,\operatorname{Re}\langle x,e_n\rangle)$$

p a real polynomial and the e_j are orthonormal.

In addition to P', one can also consider the algebra of functions

$$f(x) = p(\langle x,e_1\rangle,\ldots,\langle x,e_n\rangle) \ , \quad p \text{ a}$$

polynomial function on $\mathbf{C}^n$ and complex conjugates of the above. Let $P(H)$ denote the last mentioned algebra.

In the complex-wave representation the representation space K is the closure of $P(H)$ in $L^2(H,g)$. Segal has shown that the elements of K can be identified as functions well defined at every point of H and which satisfy an L^2-boundedness condition. We do not go into the details of the construction of W and Γ of the Weyl System here. It can be shown that there exists a unique (upto unitary equivalence) Weyl System. It is however worthwhile stating explicitly the form of the "creation" and "annhilation" operators.

Definition:

For any representation $\Phi = (K,W,\Gamma,v)$ of the free Boson field over the given Hilbert space H and for given vector $z \ \varepsilon \ H$, the creation operator for z denoted by $C(z)$ is defined as the operator $\frac{1}{\sqrt{2}}(dW(z) - i dW(iz))$, where $dW(z)$ denotes the self adjoint generator of the one-parameter group $\{W(tz)|t\varepsilon R\}$. The annhilation operator for the vector z, denoted by $C^*(z)$ is defined as the operator $\frac{1}{\sqrt{2}}(dW(z) + i dW(iz)$

Theorem (Segal)

The operators $C(z)$ and $C^*(z)$ are closed, densely defined and mutually adjoint. In the complex-wave (anti-holomorphic) representation, $C(z)$ has domain consisting of all $F\varepsilon K$ such that $\langle z,\cdot\rangle F(\cdot) \ \varepsilon \ K$. $C(z)$ is the mapping $F(\cdot) \mapsto \frac{1}{\sqrt{2}\, i} \langle z,\cdot\rangle F(\cdot)$. $C^*(z)$ has domain consisting of all $F \ \varepsilon \ K$ such that $F_z \varepsilon K$ where $F_z = \lim_{\varepsilon\to 0} \frac{F(u+\varepsilon z) - F(u)}{\varepsilon}$. $C^*(z)$ is the mapping

1. The free quantum field is an infinite assembly of non-interacting harmonic oscillators.

$F \mapsto -\sqrt{2}\, i\, F_z.$ ∎

The Particle Representation

Let H' be a real Hilbert space and let H be its complexification. Let $H^{n\circledS}$ be the n-fold symmetric tensor product of H with itself. We give $H^{n\circledS}$ the inner product

$$\langle \mathrm{Sym}\, g_1 \otimes \cdots \otimes g_n, \mathrm{Sym}\, f_1 \otimes \cdots \otimes f_n \rangle =$$

$$\sum_\pi \langle g_{\pi(1)}, f_1 \rangle \cdots \langle g_{\pi(n)}, f_n \rangle$$

where π is a permutation of (1,2,...n) and Sym is the symmetrization operator

$$\mathrm{Sym}\, f_1 \otimes \cdots \otimes f_n = \frac{1}{n!} \sum_\pi f_{\pi(1)} \otimes \cdots \otimes f_{\pi(n)}$$

Let F be the weak centred Gaussian distribution of unit variance on H'. Associated with H' is a probability space $(\Omega, \mathcal{B}, \mu)$, where $\mathcal{B}$ is generated by F(f), $f \in H$ and if $f_1, \ldots, f_n$ are orthonormal in H' and ϕ is a Baire function on R^n, then

$$\int_\Omega \phi(F(f_1), \ldots F(f_n))\, d\mu =$$

$$\frac{1}{(2\pi)^{n/2}} \int_{R^n} \phi(x) e^{-\frac{x^2}{2}} dx$$

Let $L^2(H')$ denote $L^2(\Omega, \mathcal{B}, \mu)$

Let $S^-(H')_n$ be the closed linear span in $L^2(H')$ of all elements of the form $F(f_1)\ldots F(f_m)$ $m \le n$ and let $S(H')_n$ be the orthogonal complement of $S^-(H')_{n-1}$ in $S^-(H')_n$. For $f_1, \ldots, f_n$ in H', Define $: F(f_1)\ldots F(f_n):$ to be the orthogonal projection of $F(f_1)\ldots F(f_n)$ on $S(H')_n$.

Then it is easy to see that

$$: F(f_1)\ldots F(f_n) : \mapsto \mathrm{Sym}\, f_1 \otimes \cdots \otimes f_n$$

extends uniquely to a unitary mapping from $S(H')_n$ onto $H^{n\circledS}$. We identify $S(H')_n$ with $H^{n\circledS}$ via this mapping. Segal showed that $S(H')_n$ span $L^2(H')$.

Hence $L^2(H') = \sum_{n=0}^{\infty} H^{n\circledS}$. This is <u>Fock space</u>.

Let $\Gamma(H)$ denote $\sum_{n=0}^{\infty} H^{n\circledS}$. $\Gamma(H)$ is intrinsically attached to the structure of H as a real Hilbert space. Hence if $U : H' \to K'$ is an orthogonal mapping of one real Hilbert space into another it induces a unitary mapping $\Gamma(U) : \Gamma(H) \to \Gamma(K)$. On $S(H)_n$, $\Gamma(U)$ is $U \otimes \ldots \otimes U$ (n-factors). The ideas of Fock space are important in filtering theory and are related to Wiener's homogeneous chaos. For a recent application see MARCUS-MITTER-OCONE.

5. Conclusions

The mathematics used in quantum field theory may have applications to modelling of stochastic systems and filtering theory. In this paper I have concentrated on ideas surrounding the free quantum field. I believe ideas of non-linear quantum field theory, for example, those developed in SEGAL [3] have applications in non-linear filtering theory. But this we have to leave for the future.

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