THE GENERALIZED POLE ASSIGNMENT PROBLEM+

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Abstract

For some linear, strictly proper system given by its transfer function, two dynamic output feedback problems can be posed. The first one is that of using dynamic-output feedback to assign the closed-loop characteristic polynomial and the second that of assigning the closed-loop invariant factors. We are concerned with these problems and their inter-relationships. The formulation is done in the frequency domain and the investigation carried out from an algebraic point of view, in terms of linear equations over rings of polynomials. Using the notion of genericity, we express several necessary and sufficient conditions.

1. Introduction

Two of the central results of linear system theory are the following:

(A) Let A and B be matrices of dimension $n\times n$ and $n\times \ell$ respectively. The pair (A,B) is controllable if and only if for every symmetric set Λ of n complex numbers, there is a matrix C such that A + BC has Λ for its set of eigenvalues;

(B) Let A and B be matrices of dimension $n\times n$ and $n\times \ell$ respectively with (A,B) being a controllable pair. The input-state transfer function P is given by $P=(sI-A)^{-1}B$. If state feedback $u=Cx+v$ is used, the closed-loop transfer function G is given by $G = P(I+CP)^{-1}$. Let $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell \geq 0$ be the controllability indices of P. Let ϕ_i be given polynomials such that $\phi_i | \phi_{i-1}$ with $\sum_{i=1}^{\ell} \theta(\phi_i) = n$.

Then, there exists a constant C such that the invariant polynomials of G are the ϕ_i if and only if

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} \lambda_i \qquad k=1,2,\dots \ell, \text{ with equality at } k=\ell.$$

Subsequently there has been considerable work to generalize (A) to the case where static output feedback is allowed. For the most recent results on this topic, see Willems and Hesselink [14] and Brockett and Byrnes [4]. Some recent work involving dynamic output feedback can be found in [2,3,7,14,15].

Generalization of problem (B) to the output feedback case has been investigated by Rosenbrock and Hayton [13]. They consider a transfer function given in Rosenbrock's system matrix form and present several interesting results. We consider the same problem in the following form.

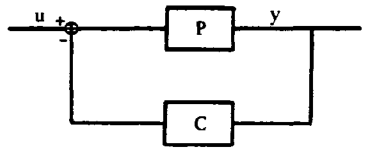

Figure 1

The $m\times\ell$ ($m\geq\ell$) matrix P is the input-output matrix of a strictly proper plant and C ($\ell\times m$) that of some proper dynamic compensator. Both P and C have elements in R(s), the field of rational functions in the indeterminate s over the reals R. The closed-loop transfer function is:

$$G = P(I+CP)^{-1} .$$

The condition $m>\ell$ is not restrictive because the situation $m\leq\ell$ can be treated in a similar manner and dual results obtained. The transfer function P is assumed to be given. We are interested in the following two problems.

(The Characteristic Polynomial Problem)

Let ϕ be some polynomial in R[s]. What are necessary and sufficient conditions for the existence of a proper compensator C, so that if χ is the characteristic polynomial of the closed-loop system, then χ is a factor of ϕ? A variant of this is the investigation of the situation in which χ is equal to ϕ .

(The Invariant Factor Problem)

Let Φ be an $\ell\times\ell$ matrix with elements in R[s]. What are the necessary and sufficient conditions for the existence of a proper compensator C, so that if Ψ is the closed-loop invariant factor matrix, Ψ is equivalent to Φ? A variant of this is to let $\Phi = (\phi_i)$ be in Smith form and to require that $[\Psi = (\psi_i)]\psi_i$ divides ϕ_i for $1\leq i\leq\ell$, or more specifically, that $\psi_i = \phi_i$.

It is clear that, from a mathematical standpoint, the invariant factors of a transfer function determine the deeper structure of a system. If $P=C(sI\ A)^{-1}B$ with (A,B,C) minimal, then A can be written in companion form as:

$$\bar{A} = \begin{bmatrix} C_1 & & & \\ & C_2 & & 0 \\ & 0 & \ddots & \\ & & & C_\ell \end{bmatrix} , \quad \bar{A} = T^{-1}AT$$

where $\psi_i = \det(sI-C_i)$ are the invariant factors of P. It is true that there does exist a relationship between the degrees of the invariant factors and the controllability or observability indices of a certain

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class of systems [9].

The invariant factors are closely related with the transmission zeros of a plant as defined by Desoer and Schulman. Let P be an $m \times \ell$ plant with Smith-McMillan form given by M_P :

$$M_P = \begin{bmatrix} \frac{\varepsilon_1}{\phi_1} & & & \\ & \frac{\varepsilon_2}{\phi_2} & 0 & \\ & 0 & & \frac{\varepsilon_\ell}{\phi_\ell} \\ & & 0 & \end{bmatrix}$$

The ϕ_i are the invariant factors of P and the transmission zeros of P are associated with the zeros of the polynomials ε_i. Suppose that $\varepsilon_i \neq 0$. Then, [8], $z \in C$ is a zero of P of order m iff $\varepsilon_\ell(\cdot)$. has a zero of order m at z. The significance of this order, roughly speaking, is that the system completely blocks the transmission of some input of the form $\sum_{k=0}^{\sigma} g_k t^k \exp(zt)$ for $\sigma = 0,1, \ldots m-1$. For $\sigma = m$, there is an input of this form for which the output is non-zero and proportional to exp(zt). Therefore, if two systems P and $\bar{P}$ have the same characteristic polynomial $\chi = \bar{\chi}$ ($\bar{x} = \bar{\phi}_1 \ldots \bar{\phi}_\ell$, $x = \phi_1 \ldots \phi_\ell$) but different invariant factors (and zeros), the transmission-blocking properties of the two systems would be different.

This paper is divided into five sections. In section 2, we formulate the problem in an algebraic manner using the notion of matrix fraction representation. This, in a very natural way, will suggest a method of solution and in doing so, demonstrate the importance of the equation $XD + YN = \Phi$, where X,Y,D,N and Φ are all matrices with elements in R[s]. In section 3, we will study this equation as it pertains to our problem and will construct what we shall call 'acceptable' solutions. In sections 4 and 5, we discuss the characteristic polynomial problem and the invariant factor problem. From this it will be seen that the results are unsatisfactory in two ways. On the one hand, they are only sufficient conditions, and on the other, they apply in 'almost all' cases. In section 6 we show that by introducing the notion of genericity, more complete results can be formulated. Remaining questions are under continued investigation. Even though we do not specifically address ourselves to specific algorithms for solution, the procedures used are constructive and can be programmed on a digital computer.

2. Formulation and Method of Solution

Assume that we have the feedback system shown in Fig. 1 with P being a strictly proper $m \times \ell$ ($m \geq \ell$) input-output transfer function and C some $\ell \times m$ proper dynamic compensator. Both P and C have elements in R[s]. The closed-loop transfer function G is given by

$$G = P(I + CP)^{-1} ,$$

where we assume that $(I + CP)^{-1}$ exists. Since P is a rational matrix, it can be factored [Desoer-Vidyasagar] as follows:

$$P = BA^{-1} = D^{-1}N$$

where B,A,D, N are polynomial matrices. We use the following notation:

$$\begin{aligned} P &= B_{RP}A_{RP}^{-1} && \text{some right representation of P} \\ &= A_{LP}^{-1}B_{LP} && \text{some left representation of P} \\ &= N_{RP}D_{RP}^{-1} && \text{some right coprime representation of P} \\ &= D_{LP}^{-1}N_{LP} && \text{some left coprime representation of P.} \end{aligned}$$

The closed-loop transfer function G can then be expressed in the following ways:

$$\begin{aligned} G &= P(I + CP)^{-1} . \\ &= B_{RP}(A_{LC}A_{RP} + B_{LC}B_{RP})^{-1} A_{LC} . \\ &= N_{RP}(A_{LC}D_{RP} + B_{LC}N_{RP})^{-1} A_{LC} . \\ &= N_{RP}(D_{LC}D_{RP} + N_{LC}N_{RP})^{-1} D_{LC} = N_{RP}\Omega^{-1}D_{LC} . \\ &= \tilde{N}_{RP} \tilde{\Omega}^{-1}\tilde{D}_{LC} \quad \text{(least order)}, \end{aligned}$$

where $\tilde{N}_{RP},\tilde{\Omega}$ are right coprime and $\tilde{D}_{LC},\tilde{\Omega}$ left coprime. From [5,7] we have that χ, the characteristic polynomial of the closed-loop system, can be written as $\chi = \alpha \det \Omega$, a non-zero constant. If coprime representations for both the plant and compensator are not used, then

$$\chi = \frac{\alpha \det(A_{LC}A_{RP} + B_{LC}B_{RP})}{\det K \cdot \det L} ,$$

where $\alpha \neq 0$ is a constant, K a greatest common left divisor of A_{LC},B_{LC} and L a greatest common right divisor of B_{RP},A_{RP}.

If M_G is the Smith-McMillan form of G,

$$M_G = \begin{bmatrix} \frac{\varepsilon_1}{\psi_1} & & & \\ & \frac{\varepsilon_2}{\psi_2} & & \\ & & \ddots & \\ & & & \frac{\varepsilon_\ell}{\psi_\ell} \\ & 0 & & \end{bmatrix} \qquad \begin{matrix} \psi_i \text{ monic,} \\ \psi_i | \psi_{i-1} , \\ 1 \leq i \leq \ell \end{matrix}$$

we call $\Psi = \begin{bmatrix} \psi_\ell & & 0 \\ & \ddots & \\ 0 & & \psi_1 \end{bmatrix}$ the invariant factor matrix of G, ψ_i being the invariant factors. As shown in [9], we have that $\tilde{\Omega}$ and Ψ are equivalent.

One way to proceed is to utilize the form in which the closed-loop transfer function has been expressed:

$$\begin{aligned} G &= N_{RP}(A_{LC}D_{RP} + B_{LC}N_{RP})^{-1} A_{LC} \\ &= N_{RP}(D_{LC}D_{RP} + N_{LC}N_{RP})^{-1}D_{LC} \\ &= \tilde{N}_{RP} \tilde{\Omega}^{-1}\tilde{D}_{LC} . \end{aligned}$$

Suppose we are investigating the characteristic polynomial problem. Let Φ be some $\ell \times \ell$ matrix with $\det \Phi = \phi$, where ϕ is some given polynomial. If a polynomial solution X,Y to $XD_{RP} + Y N_{RP} = \Phi$ exists with $X^{-1}Y$ existing and proper, then $C = X^{-1}Y$ is a proper compensator making the closed-loop characteristic polynomial χ equal to $\chi = \phi/q$ where $q = \det K$, K a greatest common left divisor of X and Y. If, in addition, X,Y are left coprime, then $\chi = \alpha\phi$. In a similar manner, suppose we are looking at the invariant factor problem. Let $\Phi = (\phi_i)$ be an $\ell \times \ell$ matrix in Smith form. If a polynomial solution X,Y to $XD_{RP} + YN_{RP} = \Phi$ exists with $X^{-1}Y$ existing and proper, X,Φ being left coprime and N_{RP},Φ being right coprime, then $C = X^{-1}Y$ is a proper compensator which makes the closed-loop invariant factors ψ_i equal to ϕ_i.

It is clear from the above that equation $XD + YN = \Phi$ plays a very important role in our investigation. We devote the next section to the study of this equation. Before doing this we also formulate a third problem, the Denominator Matrix problem.

(The Denominator Matrix Problem)

Let $P = N_{RP}D_{RP}^{-1}$ be an $m \times \ell$ strictly proper transfer function described by the right coprime representation N_{RP},D_{RP}. Let Φ be an $\ell \times \ell$ matrix. What are necessary and sufficient conditions for the existence of a polynomial solution X,Y of $X D_{RP} + Y N_{RP} = \Phi$ for which $X^{-1}Y$ exists and is proper? A variant of this would be to require also that N_{RP},Φ are right coprime and X,Φ left coprime.

Remark. The issue of coprimeness has not been explicitly dealt with by Rosenbrock and Hayton [13].

3. The Equation $X D_{RP} + Y N_{RP} = \Phi$

The importance of this equation in the problems at hand has been shown in the last section. It is nothing else but a set of linear equations over the field of rational functions R(s),

$$[X,Y]\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix} = \Phi \tag{3.1}$$

(i.e., $Z F = \Phi$). As such all (rational) solutions can be written as $Z = Z_1 + Z_0$ where Z_1 is a particular solution and Z_0 is such that $Z_0F = 0$. We, though, are interested only in polynomial solutions, and as can be shown [9, 11]:

Proposition 3.1. Let U,V be a polynomial solution to $UD_{RP} + VN_{RP} = I$. Then all polynomial solutions (X,Y) of $X D_{RP} + Y N_{RP} = \Phi$ can be expressed as:

$$X = \Phi U - NN_{LP}$$
$$Y = \Phi V + ND_{LP}$$

where N is a polynomial matrix.

Now from [12], we know that since N_{RP},D_{RP} are right coprime, a polynomial solution X,Y always exists for any Φ. This is an algebraic condition. Emre, in a recent paper [10], gives a nice system theoretic interpretation of this, using module theory and the realization techniques suggested by Fuhrmann. He also suggests an alternate description of all polynomial solutions, which has a system theoretic flavor.

As we have noted, we are interested in solutions of $X D_{RP} + Y N_{RP} = \Phi$, which are polynomial but which in addition have the property that (a) $\det X \neq 0$ and (b) $X^{-1}Y$ is proper. We call such solutions acceptable.

Satisfying the first requirement is easy, as we see from [9].

Proposition 3.2. Let Φ be an $\ell \times \ell$ matrix with $\det\Phi \neq 0$. Then there exists a polynomial solution X,Y to equation (3.1) for which $\det X \neq 0$.

This next result describes how both requirements are satisfied simultaneously, which consequently plays a crucial role in our investigation [9,13].

Theorem 3.3. Let $P = N_{RP}D_{RP}^{-1}$ be a strictly proper $m \times \ell$ rational transfer function with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ being column-proper with column degrees (controllability indices) $\lambda_1 \geq \lambda_2 \geq \ldots \geq 0$. Let Φ be an $\ell \times \ell$ non-singular matrix with $q = \theta(\det \Phi) - \theta(\det D_{RP}) \geq 0$. Let X,Y be a polynomial solution of $X D_{RP} + Y N_{RP} = \Phi$. Then $X^{-1} Y$ exists and is proper iff there exists a unimodular matrix M and indices $d_i \geq 0$, satisfying $\sum_{i=1}^{\ell} d_i = q$ such that

$$\operatorname{diag}(s^{-d_i})\, M[Y, \Phi \operatorname{diag}(s^{-\alpha_i})] \text{ is proper.} \tag{3.2}$$

Remark. The above theorem is clarified if we look at what happens in the single-input, single-output (siso) situation. Let x,y be a solution to $xd_p + yn_p = \phi$, n_p,d_p coprime, $\theta(n_p) < \theta(d_p) = \alpha$.

Necessity: If $x^{-1}y$ is proper, we must have

$$\theta(x) + \alpha = \theta(\phi) = \alpha + q \text{ and } q = \theta(x) \geq \theta(y) \quad .$$

Therefore, we must have $s^{-q}[y,\phi s^{-\alpha}]$ proper.

Sufficiency: If $s^{-q}[y,\phi\, s^{-\alpha}]$ is proper, we have that $\theta(y) \leq q$ and since $\theta(\phi) = q + \alpha$, we must have $\theta(x) = q$, otherwise $\theta(xd_p + yn_p) \neq \theta(\phi)$. This means that $x^{-1}y$ exists and is proper.

After looking at Theorem 3.1, it is quite natural to attempt the construction of acceptable solutions by making sure that requirement (3.2) is satisfied. We know that all solutions to (3.1) are given by

$$X = \Phi U - NN_{LP}$$
$$Y = \Phi V + ND_{LP}$$

where $UD_{RP} + VN_{RP} = I$. The question is how to choose N. Let us look at the siso situation for a moment. Then $y = \phi v + n d_p$, or written differently, $\phi v = -nd_p + y$. We know that whatever ϕv is there exists an n such that $\theta(y) < \theta(d_p)$. This is nothing else but division of ϕv by d_p. This as shown in [15] and holds in the matrix case where the column degrees of Y are strictly less than the column degrees of D_{LP}. If we then let $[D_{LP},N_{LP}]$ be row-proper, the row degrees are the observability indices of P and we can construct a unique Y with the $\theta(Y) \leq \mu_1 - 1$, μ_1 the largest observability index. Therefore, $\operatorname{diag}(s^{-(\mu_1-1)})Y$ is proper. To fulfill requirement 3.2, care must be taken in choosing a Φ that makes $\operatorname{diag}(s^{(\mu_1-1)})\Phi\operatorname{diag}(s^{-\alpha_i})$ proper as well.

Theorem 3.1 provides a test for determining whether a specific polynomial solution is actually an acceptable one. It would be greatly desirable, though, if a particular solution could serve as a representative for all solutions. In the following situation, this can be done.

Let P be a strictly proper transfer function with all observability indices equal to μ. Let $[D_{LP},N_{LP}]$ be row proper, $D_{LP} = I s^{\mu} + \ldots + D_0$.

Let U,V be such that $UD_{RP} + VN_{RP} = I$ with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column proper with controllability indices $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$. Using right division, there exist unique $-\bar{N}$ and $\bar{Y}$ such that

$$\Phi V = -\bar{N} D_{LP} + \bar{Y} \qquad \theta(\bar{Y}) < \mu .$$

Let $\bar{X} = \Phi U + \bar{N} N_{LP}$.

<u>Proposition 3.3</u>. Let $P = N_{RP}D_{RP}^{-1} = D_{LP}^{-1} N_{LP}$ be as above. Let Φ be a diagonal matrix with $\theta(\phi_i) = \lambda_i + \gamma$, $\gamma \geq 0$. Then $X D_{RP} + Y N_{RP} = \Phi$ has an acceptable solution iff $\bar{X},\bar{Y}$ is an acceptable solution.

<u>Remark</u>: The above does include the siso case.

4. <u>The Characteristic Polynomial Problem</u>

We are now in a position to give a partial answer to the Characteristic Polynomial Problem as stated in section 1. I say 'partial' because it is only a sufficient condition.

<u>Theorem 4.1</u>. Let P be an $m \times \ell$ strictly proper transfer function and $N_{RP}D_{RP}^{-1}$ a right coprime representation where $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ is column-proper with column degrees $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell \geq 0$ (controllability indices). Let $D_{LP}N_{LP}$ be a left coprime representation, with $[D_{LP} N_{LP}]$ row-proper. Let ϕ be a polynomial of degree $t = \sum_{i=1}^{\ell} \alpha_i + \ell(\mu_1 - 1)$ (μ_1 the largest observability index). Then there exists a proper compensator C such that the characteristic polynomial χ of the closed-loop system is given by

$$\chi = \frac{\phi}{q_{XY}}$$

where q_{XY} is a polynomial with $0 \leq \theta(q_{XY}) \leq \ell(\mu_1 - 1)$.

The proof is constructive [9], with the compensator being given by $C = X^{-1}Y$ where X,Y is an acceptable solution to some equation of the form $X D_{RP} + Y N_{RP} = \Phi$. The polynomial q_{XY} is nothing else but $\det K_{XY}$, where K_{XY} is a greatest common left divisor of X and Y.

<u>Remark</u>. It is clear that this Theorem can be used for purposes of stabilization. If ϕ is chosen to be a stable polynomial, then so will be χ, the closed-loop characteristic polynomial. We also note that the compensator may or may not be stable. We will investigate this issue later in this section.

<u>Remark</u>. From a closer examination of the procedure we can see that, in general, one does not have prior knowledge of what q_{XY} is. The larger the degree of q_{XY} implies a smaller increase in the overall dynamics of the system. One can therefore use this in the design of compensators.

<u>Remark</u>. Using a different output feedback configuration, Brasch and Pearson [3] show that the characteristic polynomial of the closed-loop system can be assigned by only increasing the system dynamics by μ_1-1. These results can be obtained using the approach outlined in this paper [9]. Even though more dynamics are added in our approach, it may be that the computations are less cumbersome. This issue warrants further investigation.

<u>Remark</u>. The approach taken in [15] suggests that compensation involves input as well as output dynamics, and it also differs from the present approach in that it requires stable, pole-zero cancellation and the presence of 'hidden' modes.

On the one hand, it is quite worthwhile to investigate compensation schemes that require as little added dynamics as possible. An equally worthwhile task is to investigate whether, by adding more dynamics than the least required, one can achieve other design objectives as well [2]. The following Lemma and Example deal with this issue.

<u>Lemma 4.1</u> Let ϕ be a polynomial with $\theta(\phi) = 2n - 1 + k$, $k \geq 1$. Let x_1, y_1 be an acceptable solution of $xd_p + yn_p = \phi$, [n_p/d_p strictly proper, $\theta(d_p) = n$]. All acceptable solutions are of the form

$$x_2 = x_1 - mn_p$$
$$y_2 = y_1 + mn_p$$

where $\theta(m) \leq k - 1$.

<u>Example.</u>

Let $p = \frac{1}{s^2 - 1}$, and suppose that we want to construct a proper and stable compensator which makes the characteristic polynomial of the closed-loop system equal to the stable polynomial $\phi = s^4 + s^3 + 3s^2 + s + 1$.

The compensator $C_1 = \frac{5s^2 + 2s}{s^2 + s - 1} = \frac{y_1}{x_1}$ does satisfy the requirements except that it is unstable. Now, all acceptable solutions are given by

$$x_2 = x_1 - mn_p$$
$$y_2 = y_1 + md_p$$

where m is a constant. Let $m = -2$. Then,

$$x_2 = s^2 + s + 1$$
$$y_2 = 3s^2 + 2s + 2 \quad .$$

Clearly, $C_2 = 3s^2 + 2s + 2/s^2 + s + 1$ meets all the requirements.

<u>Remark.</u> This idea can certainly be extended to the multiple-input, multiple-output situation.

5. <u>The Invariant Factor Problem</u>

Let P be an $m \times \ell$ strictly proper transfer function and $\bar{\Phi} = (\phi_i)$ an $\ell \times \ell$ diagonal matrix in Smith form. If $P = N_{RP}D_{RP}^{-1}$ is some right coprime representation for which there exists an acceptable solution to the equation $X D_{RP} + Y N_{RP} = \Phi$, where Φ and $\bar{\Phi}$ are equivalent, with (a) X and Φ left coprime and (b) N_{RP},Φ right coprime, then $C = X^{-1}Y$ is a proper compensator making the invariant factor matrix $\Psi = (\psi_i)$ of G equal to $\bar{\Phi}$. If conditions (a) and (b) are not met, then [6] we have $\psi_i | \phi_i$. We also know [13] that if P with controllability indices $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq 0$ and observability indices $\mu_1 \geq \mu_2 \geq \ldots \mu_m \geq 0$ and $\bar{\Phi} = (\phi_i)$, also satisfies

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} (\lambda_i + \mu_1 - 1) \qquad k = 1,2,\ldots \text{ with equality at } k = \ell,$$

then there exists a matrix Φ equivalent to $\bar{\Phi}$ such that $\lim_{s \to \infty} [\operatorname{diag}(s^{-(\mu_1 - 1)}) \Phi \operatorname{diag}(s^{-\alpha_i})] = I$.

<u>Theorem 5.1</u>. Let $P, \bar{\Phi}$, ϕ be as above, with the ϕ_i satisfying

$$\sum_{i=1}^{k} \theta(\phi_i) \geq \sum_{i=1}^{k} (\lambda_i + \mu_1 - 1) \quad k = 1,2,\ldots\ell \text{ with equality at } k = \ell.$$

Then there exists a proper compensator $C = X^{-1}Y$ with

$X\,D_{RP} + Y\,N_{RP} = \Phi$ and such that if Ψ is the closed-loop invariant factor matrix, then $\psi_i | \phi_i$.

We will only have $\psi_i = \phi_i$ if X and Φ are left coprime and N_{RP} and Φ are right coprime.

Remark. In earlier work Rosenbrock [12] gave a necessary and sufficient condition in the case of state feedback. That result can be obtained using the theory developed in this approach [9]. In that situation, the invariant factors are assigned exactly for all cases. Here, as we see, the conditions are merely sufficient and apply to 'some' cases; this warrants further clarification.

Remark. The fact that in the state feedback the system trasfer function can take the form $P = (sI-A)^{-1}B$ simplifies the problem, and using the procedure suggested here, Rosenbrock's earlier result can be proved.

The results we have discussed so far are unsatisfactory in two respects. We have seen that degree constraints on ϕ or Φ are not enough to ensure that the closed-loop transfer function G will have the desired characteristics. If X, Φ , and left coprime and N_{RP},Φ are right coprime, this will be true. However, they are only sufficient conditions. It is therefore quite natural to ask whether degree constraints are solely sufficient in 'almost all' cases and whether these are necessary in 'almost all' situations as well. We will show in the next section that in some cases this is indeed true.

6. Generic Results

Let q be some positive integer. We define the Zariski topology on R^q this way. Let $\underline{u}$ be an ideal in $R[x_1, \ldots x_q]$. All points $x = (x_1, \ldots x_q)$ x_2 in $\mathbb{C}$ such that $f(x)=0^q$ for all f in $\underline{u}$ form the variety of $\underline{u}$. If closed sets in $\mathbb{C}^q$ are defined to be the varieties of $\mathbb{C}^q$ [16], then $\mathbb{C}^q$ becomes a topological space with the Zariski topology. Let R^q have the subspace topology.

Definition. A set $S \subset R^q$ is called 'generic' if it contains a non-empty Zariski open set of R^q. Roughly speaking, a set is generic if it contains almost all of R^q, (its complement is contained in a set of Lebesque measure zero). The way in which we use the notion of genericity is to first take a set of R^q and then define a property which is valid for all points in $S \subset R^q$. We then attempt to show that S is generic. This means, in effect, that the property is valid on almost all of R^q. We now give explicit definitions.

Definition. An $m \times \ell$ strictly proper transfer function P of order n, given by $P = N_{RP}D_{RP}^{-1}$ has the generic characteristic polynomial assignability property if the monic polynomials $\phi \varepsilon R^{n+q}$ for which there exists a proper compensator C making the closed-loop characteristic polynomial equal to ϕ is a generic subset of R^{n+q} .

Definition. An $m \times \ell$ strictly proper transfer function P of order n given by $P = N_{RP}D_{RP}^{-1}$ has the generic denominator matrix assignability property if the $\ell \times \ell$ matrices $\phi \varepsilon R^t$ for which there exists an acceptable solution X,Y to $X\,D_{RP} + Y\,N_{RP} = \Phi$ with N_{RP}, Φ right coprime and X,Φ left coprime is a generic subset of R^{n+t}.

In what follows, we find that looking at the equation $X\,D_{RP} + Y\,N_{RP} = \Phi$ as an operator is greatly advantageous. If

$$X = X_{k-1}s^{k-1} + X_{k-2}s^{k-2} + \ldots + X_o$$
$$Y = Y_{k-1}s^{k-1} + \ldots. + Y_o$$
$$D_{RP} = D_t s^t + \ldots + D_o$$
$$N_{RP} = N_t s^t + \ldots + N_o$$

then

$$[X_{k-1}\ Y_{k-1}\ \ldots\ X_oY_o]\ S_k(D_{RP},N_{RP}) = [\Phi_{k=t-1}\Phi_{k+t-2}\ldots\Phi_o]$$

where

$$\underset{\text{block rows}}{\overset{}{S_k(D_{RP}\ N_{RP})}} = \begin{bmatrix} D_t & D_{t-1} & & D_o & 0 & 0 & \ldots. & 0 \\ N_t & \ldots. & & N_o & 0 & 0 & \ldots. & 0 \\ 0 & D_t & \ldots & & D_o & & \vdots & \\ 0 & N_t & \ldots & & N_o & & & \\ & & \ddots & & & & & \\ & & & D_t & \ldots & & & D_o \\ & & & N_t & \ldots & & & N_o \end{bmatrix}$$

2 k block rows

This [1] we immediately recognize as the generalized Sylvester Resultant of D_{RP} and N_{RP} of order k [it is a $k(m+\ell) \times \ell(t+k)$ matrix with real entries.]

The following two Lemata taken from [1] give the rank of $S_k(D,N)$ for some transfer function ND^{-1} in terms of the dual dynamical indices (observability indices if ND^{-1} proper) of ND^{-1} and relate coprimeness of N,D with the rank of some $S_{k'}(D,N)$. These are generalizations of siso results.

Lemma 6.1. Let ND^{-1} $m \times \ell$ be proper with λ_i observability indices of ND^{-1}. Then

$$\text{rank } S_k(D,N) = (\ell + m)k - \sum_{i:\mu_i < k} (k - \mu_i) .$$

Lemma 6.2. Let ND^{-1} $m \times \ell$ be proper and q the least integer for which rank $S_{q+1}(D,N)$ - rank $S_k(D,N) \leq \ell$. Then, for $n > q$, N,D are right coprime iff $S_n(D,N) = \ell n + \theta(\det D)$.

A consequence of viewing the equation $X\,D_{RP} + Y\,N_{RP} = \Phi$ as an operator is:

Proposition 6.3. Let $P = N_{RP}D_{RP}^{-1}$ be an $m \times \ell$ strictly proper transfer function with controllability indices $\lambda_1 = \lambda_2 = \ldots = \lambda_\ell = \lambda$ and observability indices $\mu_1 = \mu_2 = \ldots = \mu_m = \mu$ and D_{RP},N_{RP} of the form $D_{RP} = Is^\lambda + D_{\lambda-1}s^{\lambda-1} + \ldots + D_0$ and $N_{RP} = N_{\lambda-1}s^{\lambda-1} + \ldots + N_0$. Let R^t be the set of $\ell \times \ell$ matrices of the form

$$\Phi = I\,s^{\lambda+q} + \Phi_{\lambda+q-1}\,s^{\lambda+q-1} + \ldots + \Phi_o .$$

Let $Q = \{(X,Y) \mid X = Is^q + X_{q-1}s^{q-1} + X_o,\ Y = Y_q s^q + \ldots + Y_o\}$

A necessary and sufficient condition for the existence of a solution to $X\,D_{RP} + Y\,N_{RP} = \Phi$ in the class Q for generic Φ is $q \geq \mu - 1$.

Proof:

(necessity).

Equation $X\,D_{RP} + Y\,N_{RP} = \Phi$ with the conditions imposed can be written as

$$[I\ Y_q\ \ X_{q-1}\ \ Y_{q-1}\ \ldots\ X_oY_o]S_{q+1} = [I\ \Phi_{q+\lambda-1}\ldots\Phi_o\ ].$$

S_{q+1} can be thought of as a function

$$S_{q+1}: R^{(\ell+m)(q+1)} \longrightarrow R^{(\lambda+q+1)\ell} .$$

From Lemma 6.1 we have that S_k [a $k(\ell+m) \times (\lambda+k)\ell$ matrix] has rank

$$\text{rank } S_k = (\ell+m)k - \sum_{i:\mu_i < k} (k - \mu_i)$$

which, under the special circumstances, becomes:

$$\begin{aligned} \text{rank } S_k &= (\ell+m)k & \text{if } 1 \leq k \leq \mu \\ &= (\ell+m)k - m(k-\mu) & \text{if } \mu < k \quad . \end{aligned}$$

By observing dimensions, we see that:

a) $S_1, S_2, \ldots S_{\mu-1}$ are not onto

b) S_μ is both one-one and onto

c) $S_{\mu+1}, S_{\mu+2} \ldots$ are onto.

Assume now that $q < \mu-1$ and that $X D_{RP} + Y N_{RP} = \Phi$ has a solution in Q for generic Φ. Show a contradiction. If we think of (X,Y) as an element in $R^{\ell(\ell+m)q+\ell m}$ and Φ as an element in $R^{\ell(\lambda+q)\ell}$, we see that the Φ that can be reached from elements in Q are a set of dimenions less than $\ell(q+\lambda)\ell$, which implies that the set of Φ which can be reached does not contain a non-empty Zariski open set. This is a contradiction; therefore, $q \geq \mu-1$.

(Sufficiency).

Suppose that $q \geq \mu-1$ (or equivalently, $q=\mu-1+k$, $k \geq 0$). We want to show that the set $\Phi \varepsilon R^t$, $[t = \ell(\lambda+\mu+k)\ell - \ell^2]$ for which a solution in Q exists, is a generic subset of R^t. We already know that $S_{\mu+k}$ is an $(\ell+m)(\mu+k) \times (\lambda+\mu+k)\ell$ matrix with

$$\text{rank } S_{\mu+k} = (\ell+m)\mu + \ell k \quad .$$

This means that the operator

$$S_{\mu+k}: R^{(\ell+m)(\mu+k)} \to R^{(\ell+m)\mu+\ell k}$$

is onto. We want to show that $S_{\mu+k}(Q) = R^t$. For $\Phi \varepsilon R^t$ there exists some (X,Y) of the form

$$X = X_{\mu+k-1} S^{\mu+k-1} + \ldots X_o$$

$$Y = Y_{\mu+k-1} S^{\mu+k-1} + \ldots Y_o$$

such that

$$[X_{\mu+k-1}, Y_{\mu+k-1} \quad X_o Y_o] S_{\mu+k} = [I \ \Phi_{\lambda+\mu+k-1} \ldots \Phi_o].$$

For this we must have $X_{\mu+k-1} = I$, which implies that $(X,Y) \varepsilon Q$. This completes the proof.

We are now in a position to give two results concerning the generic characteristic polynomial assignability property.

Theorem 6.4. Let $P = n_p d_p^{-1}$ be a siso, strictly proper transfer function of order n $[\theta(d_p)=n]$, d_p monic. A necessary and sufficient condition for generic characteristic polynomial assignability is $q \geq n-1$.

Proof.

Since $\phi \varepsilon R^{n+q}$ is to be the characteristic polynomial of the closed system, the compensator accomplishing this must be of order q. From Proposition 6.3 we then have that a necessary condition is $q \geq \mu - 1$.

For sufficiency, assume that $q \geq \mu-1$. Let $t=n+q$ and define

$$S = \left\{ (\phi_o, \ldots \phi_{t-1}) \varepsilon R^t \;\middle|\; \begin{array}{l}\text{For which there exists}\\ \text{an acceptable solu-}\\ \text{tion } x,y \quad xd_p + yn_p = \phi\\ \text{and } x,y \text{ coprime.}\end{array} \right\}$$

We need to show that S contains a non-empty Zariski open set (i.e., it is generic). Since $q \geq \mu-1$, Theorem 3.3 can be used to show that the solution $\overline{x},\overline{y}$ which is formed by letting $-\overline{n}$ (in $\overline{y} = \phi v + \overline{n} d_p$, $\overline{x} = \phi u - \overline{n} n_p$) be the unique quotient of the division $d_p/\phi v$, is an acceptable solution. Let $g = \text{Res}(\overline{x},\overline{y})$ (i.e., the resultant of x and y). Since $\theta(x) = q$, we must have that x,y are coprime iff $g \neq 0$.

Let

$$V_g = \{(\phi_o, \ldots \phi_{t-1}) \varepsilon R^t \mid g(\phi_o, \ldots \phi_{t-1}) = 0\}.$$

It is clear that $S \supseteq \overline{V}_g$. We need to show that $\overline{V}_g \neq 0$.

Let f be in R[s], with $\theta(f) = q$ and fd_p monic. Define $\phi = fd_p + n_p$.

$$\to \quad \phi v = (fv-u)d_p + 1.$$

Since for this particular ϕ, the corresponding $\overline{y}$ is equal to 1, we must have $\overline{x},\overline{y}$ being coprime. Therefore, $S(\supseteq \overline{V}_g)$ contains a non-empty Zariski open set making it generic. This completes the proof.

In a similar manner, we can also show [9]:

Theorem 6.5. Let $P = N_{RP} d_{RP}^{-1} = (n_{i1}/d_{i1})$ be an $m \times 1$ strictly proper transfer function with d_{i1}, d_{ji}, $i \neq j$ coprime. A sufficient condition for generic characteristic polynomial assignability is $q \geq \mu_1 - 1$ (μ_i the largest observability index of P). In the event that all observability indices are equal to μ, then this condition is necessary as well.

Remark. In proving these results, we make use of the generalized Sylvester resultants. The results are confined to the case when the denominator matrix ($X D_{RP} + Y N_{RP} = \Phi$) is just a polynomial. For the general $m \times \ell$ case, a closer examination of the structure of the resultant matrices is needed.

Remark. Results similar to these proved in a different way can also be found in a recent paper of Willems and Hesselink [14].

The generalized Sylvester resultants can be used more effectively to treat the generic denominator matrix assignability problem. As expected for the single-input, single-output case, we have

Theorem 6.6. Let $P = n_p d_p^{-1}$ be a siso strictly proper transfer function of order n. Let ϕ be a monic polynomial with $\theta(\phi) = n + q$ ($\phi \varepsilon R^{n+q}$). A necessary and sufficient condition for generic denominator assignability is $q \geq n-1$.

The proof proceeds in a similar manner as that of Theorem 6.4. The multiple-input, multiple-output situation is much more challenging. For this, we interpret Lemma 6.2 in the following way: The matrices N,D are right coprime iff at least one $\ell n + \theta(\det D) \times \ell n + \theta(\det D)$ minor of $S_n(D,N)$ is not zero. Denote these minors by $m_i(D,N)$. By symmetry, the argument can also be made for left coprimeness. We can now state [9].

Proposition 6.7. Let $P = N_{RP} D_{RP}^{-1}$ be an $m \times \ell$ strictly proper transfer function with $\begin{bmatrix} D_{RP} \\ N_{RP} \end{bmatrix}$ column proper, and column degrees $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$. Let $D_{LP}^{-1} N_{LP}$ be such that $\theta(D_{LP}) = \mu_1$, the largest observability index. Let R^t denote the set of $\ell \times \ell$ diagonal matrices $\Phi = (\phi_i)$, ϕ_i monic with $\theta(\phi_i) = \lambda_i + t_i$, $t = \sum_{i=1}^{\ell} (\lambda_i + t_i)$. Let m_i (Φ, N_{RP}), $n_j(\Phi, \overline{X})$ be the appropriate minors for Φ, N_{RP}, and Φ, $\overline{X}$, respectively, ($\overline{X},\overline{Y}$ obtained by right division $\Phi V = -\overline{N} D_{LP} + \overline{Y}$).

If $t_i \geq \mu_1 - 1$ and at least one $m_i(\Phi, N_{RP}) \neq 0$ and at least one $n_i(\Phi,\overline{X}) \neq 0$, then P has the denominator matrix assignability property.

Remark. For this result as well we see that degree constraints are not enough and that 'undesirable' additional conditions are present.

On the other hand, this is merely a sufficient condition. For the special case of diagonal systems, we have [9].

Proposition 6.8. Let P be an $m \times \ell$ strictly proper transfer function of the form

$$P = \begin{bmatrix} \frac{n_1}{d_1} & & 0 \\ & \ddots & \\ 0 & & \frac{n_\ell}{d_\ell} \\ & 0 & \end{bmatrix} = \underbrace{\begin{bmatrix} n_1 & & 0 \\ & \ddots & \\ 0 & & n_\ell \\ & 0 & \end{bmatrix}}_{N_{RP}} \underbrace{\begin{bmatrix} d_1 & & 0 \\ & \ddots & \\ 0 & & d_\ell \end{bmatrix}^{-1}}_{D_{RP}}$$

with n_i, d_i coprime, d_i monic (this means that the controllability indices λ_i are equal to $\theta(\phi_i) 1 \le i \le \ell$ and the observability indices μ_i are equal to $\theta(d_i)$ $1 \le i \le \ell$ with $\mu_{\ell+1} = \dots = \mu_m = 0$.)

Let R^t denote the set of $\ell \times \ell$ diagonal matrices $\Phi = (\phi_i)$, ϕ_i monic with $\theta(\phi_i) = \lambda_i + t_i$, $t = \sum_{i=1}^{\ell} (\lambda_i + t_i)$. A sufficient condition for generic denominator matrix assignability is $t_i \ge \mu_i - 1$. In the event that $m = \ell$ and $\lambda_1 = \lambda_2 \dots = \lambda_\ell = \lambda$ and $\mu_1 = \mu_2 = \dots = \mu_\ell = \lambda = \mu$, then $t_i \ge \mu - 1$ is a necessary condition as well.

Remark. Under the assumptions of Proposition 6.7, we have that a sufficient condition for generic denominator matrix assignability is

a) $t_i \ge \mu_1 - 1$

b) at least one $m_i(\Phi, N_{RP}) \ne 0$ and at least one $n_i(\Phi, \bar{X}) \ne 0$.

It is desirable to eliminate condtion (b). To accomplish this it has to be shown that for some $\Phi_0 \in R^t$ we have $m_i(\Phi_0, N_{RP}) \ne 0$ and $n_j(\Phi_0, \bar{X}_{\Phi_0}) \ne 0$. Proposition 6.8 suggests a way in which this may be achieved. Instead of looking at some specific system and some space of Φ, look at the space $T \times \Phi$, where T is an appropriate space of systems (which includes diagonal systems). Then attempt to show that for some t_0 (a diagonal system) and some Φ_0, $m_i(N_{RP}, t_0 \Phi_0) \ne 0$ and $n_j(\Phi_0, X_{t_0 \Phi_0}) \ne 0$. This way we will, in effect, have proved that 'almost all' systems in T have the generic denominator matrix assignability property, if $t_i \ge \mu_1 - 1$.

Theorem 6.9

Let N, D be $\ell \times \ell$ matrices and define W, Z, S as follows:

$$W = \{(N,D) \in R^{2\lambda\ell^2} \mid D = Is^\lambda + D_{\lambda-1}s^{\lambda-1} + \dots + D_0, \quad N = N_{\lambda-1}s^{\lambda-1} + \dots + N_0\}$$

$$Z = \{\Phi \in R^{(\lambda+q)\ell} \mid \Phi \text{ diag}, \Phi = Is^{\lambda+q} + \Phi_{\lambda+q-1}s^{\lambda+q-1} + \dots + \Phi_0\}$$

$$S = \left\{(N,D,\Phi) \in R^{2\lambda\ell^2} \times R^{(\lambda+q)\ell} \,\middle|\, \begin{array}{l}\text{For which there exists an} \\ \text{acceptable solution } X,Y \\ \text{of } XD + YN = \Phi \text{, with} \\ N,\Phi \text{ right coprime,} \\ X,\Phi \text{ left coprime.}\end{array}\right\}$$

A necessary condition for S to be a generic subset of $R^{2\lambda\ell^2} \times R^{(\lambda+q)\ell}$ is $q \ge \lambda - 1$.

Proof:

Suppose that S is generic (i.e., it contains a non-empty Zariski open set) and let $q < \lambda - 1$. Show a contradiction.

Let M be the subset of $R^{2\lambda\ell^2} \times R^{(\lambda+q)\ell}$ for which (N,D) are right coprime and ND^{-1} has observability indices equal to $\mu (=\lambda)$. (If N,D are right coprime, the controllability indices of ND^{-1} are all equal to λ). We have that M is generic because of the following:

The set $F \subseteq R^{2\lambda\ell^2} \times R^{(\lambda+q)\ell}$ for which rank $S_i(D,N) = 2i\ell$ $\quad 1 \le \ell \le \lambda$

and rank $S_{\lambda+1} = 2\lambda\ell + \ell$

is generic. This means that for every $(N,D,\Phi) \in F$ we have that:

1) N,D are right coprime (Lemma 6.2)
2) Since ND^{-1} is proper, the observability indices of ND^{-1} are all equal to λ (Lemma 6.1).

This implies that $(F \subset M)$ M is generic.

Since we have assumed S to be generic, we must have that $S \cap M$ is non-empty. Let $(N_1, D_1, \Phi_1) \in S \cap M$. This means that for N_1, D_1 and almost all $\Phi \in Z$ we have that an acceptable solution X,Y of $XD + YN = \Phi$ exists. Since X,Y is acceptable, we must have (Corollary 2, p. 548, Rosenbrock-Hayton), $\theta(Y) \le q$. This means that $(X,Y) \in Q$ of Proposition 6.3. But then $q \ge \lambda - 1$, which contradicts our assumption that $q < \lambda - 1$. Therefore, $q \ge \lambda - 1$. This completes the proof of Theorem 6.9.

Proposition 6.10 With W,Z,S,M as in Theorem 6.9, a sufficient condition for S to be generic is $q = \lambda - 1$.

Proof:

Let $q = \lambda - 1$. From above we already have that M is generic. For any (N,D,Φ) in M we have that $S_\mu(D,N)$ is one-one and onto, therefore invertible. This means that for any $\Phi \in Z$ there exists a unique (X,Y) such that $XD + YN = \Phi$, and X,Y is an acceptable solution. It is clear that N,Φ are right coprime for almost all (N,D,Φ). The question then remains as to whether X,Φ are left coprime. (i.e., Φ, X' right coprime).

From Proposition 6.8 we already know that there exists some diagonal system $N D^{-1}$ and some diagonal $\Phi \in Z$ for which X and Φ are coprime [call the point $(N,D,F) \in W \times Z, \alpha$]. This means (Lemma 6.1) that

$$\text{rank } S_i(\Phi_\alpha, X'_\alpha) = i \cdot 2\ell \qquad 1 < i < 2\lambda - 1$$

and

$$\text{rank } S_{2\lambda}(\Phi_\alpha, X'_\alpha) = 2\lambda \cdot 2\ell - \ell .$$

This implies that the above also hold for generic α. Using Lemma 6.2 we then have that X,Φ are left coprime for generic α. This means that S is a generic subset of $R^{2\lambda\ell^2} \times R^{(\lambda+\lambda-1)\ell}$.

Remark. In Theorem 6.9 we see that $q \ge \lambda - 1$ is a necessary condition so that for almost all systems of order, $\lambda\ell$ and equal observability indices λ, there exists an acceptable solution X,Y of $XD + YN = \Phi$ with N,Φ right coprime, X Φ left coprime for almost all Φ in Z. In Proposition 6.10 we have that $q = \lambda - 1$ is a sufficient condition. We conjecture that $q \ge \lambda - 1$ is actually a sufficient condition, thus completing Theorem 6.9.

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