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A Unified Framework for Hybrid Control[*]

Michael S. Branicky[†‡] Vivek S. Borkar[§] Sanjoy K. Mitter[†¶]

Abstract

We propose a very general framework for hybrid control problems that encompasses several types of hybrid phenomena considered in the literature. A specific control problem is studied in this framework, leading to an existence result for optimal controls. The "value function" associated with this problem is expected to satisfy a set of "generalized quasi-variational inequalities."

1 Introduction

Hybrid control systems are those that involve both continuous dynamics and controls, as well as discrete phenomena. Some examples include computer disk drives [13], transmissions and stepper motors [10], constrained robotic systems [2], and intelligent vehicle/highway systems [19]. More generally, such systems arise whenever one mixes logical decision-making with the generation of continuous control laws, such as in modern flight control systems.

In this paper, our focus is on the case where the continuous dynamics is modeled by a differential equation

$$\dot{x}(t) = \xi(t), \qquad t \geq 0. \tag{1}$$

Here, $x(t)$ is the *continuous component* of the state taking values in some subset of a Euclidean space. $\xi(t)$ is a *controlled vector field* that generally depends on $x(t)$, the *continuous component* $u(t)$ of the control policy, and the aforementioned discrete phenomena. The discrete phenomena generally considered are of four types: (1) autonomous switching, (2) autonomous jumps, (3) controlled switching, and (4) controlled jumps.

In this paper, we study a model that subsumes all these phenomena and study an associated control problem. The paper is organized as follows. The next section details the above discrete phenomena. Section 3 reviews models of hybrid systems from the control and dynamical systems literature. Section 4 abstracts the phenomena found in hybrid systems to a unified framework that encompasses all the other reviewed models. Section 5 defines an optimal control problem in this framework. The existence of an optimal control for this problem is established in Section 6. Section 7 gives a formal derivation of the associated *generalized quasi-variational inequalities*. In Section 8 we solve some example problems using the developed theory. Section 9 concludes with a list of some open issues.

Finally, we collect some notation used throughout. First, we make use of the abbreviations ODEs (ordinary differential equations), FA (finite automata/on), and DEDS (discrete event dynamical systems; see [15]). $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Z}$, $\mathbb{N}$ denote the reals, nonnegative reals, integers, and nonnegative integers, respectively. For $x \in \mathbb{R}$, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x, and, in an abuse of common notation, $\lceil x \rceil$ denotes the least integer greater than x. $\underline{N}$ denotes the set $\{1, 2, \ldots, N\}$. $f(t^+), f(t^-)$ denote the right- and left-hand limits of f at t, respectively.

This paper is a summary of [8], which will be published in full elsewhere. For lack of space, all proofs have been deleted from the present paper. See [8] for proofs and more details. Closely related issues occur in the study of piecewise deterministic processes [12].

2 Hybrid Phenomena

In this section, we briefly examine the discrete phenomenon that arise in the study of hybrid systems.

Autonomous Switching. Here the vector field $\xi(\cdot)$ changes discontinuously when the state $x(\cdot)$ hits certain "boundaries" [17, 18]. The simplest example of this is when it changes depending on a "clock" that may be modeled as a supplementary state variable [10]. An example of autonomous switching is the following:

Example 2.1 *Consider a control system with hysteresis:*

$$\dot{x} = f(x, u) \equiv H(x) + u,$$

where the multi-valued function H is shown in Figure 1.

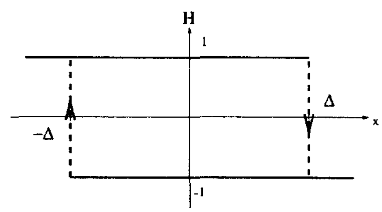

Figure 1: Hysteresis function, H.

Note that this system is not just a differential equation whose right-hand side is piecewise continuous. There is "memory" in the system, which affects the value of the vector field. Indeed, such a system naturally has a finite automaton associated with the hysteresis function H, as pictured in Figure 2.

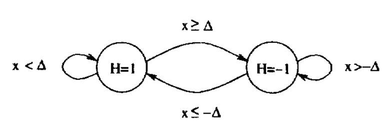

Figure 2: Finite automaton associated with H.

Autonomous Jumps. Here $x(\cdot)$ jumps discontinuously on hitting prescribed regions of the state space [2, 3]. The simplest examples possessing this phenomenon are those involving collisions.

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†Lab. for Information and Decision Systems and Center for Intelligent Control Systems, Dept. of Electrical Engineering and Computer Science, Massachusetts Institute of Technology.

‡Graduate Student. LIDS, MIT, 77 Mass. Ave., 35-415, Cambridge, MA 02139-4307. E-mail: branicky@lids.mit.edu.

§Professor. Dept. of Electrical Engineering, Indian Institute of Science, Bangalore 560012, India.

¶Professor. LIDS, MIT, 77 Mass. Ave., 35-304, Cambridge, MA 02139-4307.

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Example 2.2 *Consider the case of the vertical and horizontal motion of a ball of mass m in a room R under gravity with constant g. Take* $R = [0,a] \times [0,b]$. *In this case, the dynamics are given by*

$$\dot{x} = v_x, \quad \dot{y} = v_y, \quad \dot{v}_x = 0, \quad \dot{v}_y = -mg.$$

Further, upon hitting the boundaries $\{(x,y) \mid y = 0 \text{ or } y = b\}$ *we instantly set* v_y *to* $-\rho v_y$, *where* $\rho \in [0,1]$ *is the coefficient of restitution. Likewise, upon hitting* $\{(x,y) \mid x = 0 \text{ or } x = a\}$ v_x *is set to* $-\rho v_x$.

Controlled Switching. Here $\xi(\cdot)$ changes abruptly in response to a control command with an associated cost. This can be interpreted as switching between different vector fields [21]. Controlled switching arises, for instance, when one is allowed to pick among a number of vector fields: $\dot{x} = f_i(x)$, $i \in \underline{N}$.

Example 2.3 *A simplified model of a manual transmission is given by* [10]

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = [-a(x_2) + u]/(1+v),$$

where x_1 *is the ground speed,* x_2 *is the engine RPM,* $u \in [0,1]$ *is the throttle position, and* $v \in \{1,2,3,4\}$ *is the gear shift position. The function a is positive for positive argument.*

Controlled Jumps. Here $x(\cdot)$ changes discontinuously in response to a control command, with an associated cost [4].

Example 2.4 *In a simple inventory management model* [4], *there are a "discrete" set of restocking times* $\theta_1 < \theta_2 < \cdots$ *and associated order amounts* $\alpha_1, \alpha_2, \ldots$. *The equations governing the stock at any given moment are* $\dot{y}(t) = -\mu(t) + \sum_i \delta(t-\theta_1)\alpha_i$, *where* μ *represents degradation or utilization dynamics and* δ *is the Dirac delta function.*

3 Review of Models of Hybrid Systems

This section briefly summarizes five models of hybrid systems developed from the dynamical systems and control point of view [18, 2, 17, 1, 10]. For sure, there are many others and no review is attempted here [14].

3.1 Tavernini's Model

Tavernini discusses so-called *differential automata* in [18]. He was motivated to study such systems as a means of modeling hysteretic phenomena such as backlash and friction (cf. Example 2.1).

A *differential automaton*, A, is a triple (S, f, ν) where S is the state space of A, $S = \mathbb{R}^n \times Q$, $Q \simeq \underline{N}$ is the *discrete state space* of A, and $\mathbb{R}^n$ is the *continuous state space* of A; f is a finite family $f(\cdot, q) : \mathbb{R}^n \to \mathbb{R}^n$, $q \in Q$, of vector fields, the *continuous dynamics* of A; and $\nu : S \to Q$ is the *discrete transition function* of A.

Let $\nu_q \equiv \nu(\cdot, q)$, $q \in Q$. Define $I(q) = \nu_q(\mathbb{R}^n)\backslash\{q\}$, that is, the set of discrete states "reachable in one step" from q. We require that for each $q \in Q$ and each $p \in I(q)$ there exist closed sets

$$M_{q,p} \equiv \nu_q^{-1}(p).$$

The sets $\partial M_{q,p}$ are called the *switching boundaries* of the automaton A. Define $M_q = \bigcup_{p \in I(q)} M_{q,p}$ and define $C(q) \equiv \mathbb{R}^n \backslash M_q$. The equations of motion are

$$\dot{x}(t) = f(x(t), q(t)), \qquad q(t) = \nu(x(t), q(t^-)),$$

with initial condition $[x(0),\ q(0)]^T \in \bigcup_{q \in Q} C(q) \times \{q\}$. The notation t^- indicates that the discrete state is piecewise continuous from the right. Thus, starting at $[x_0, i]$, the continuous state trajectory $x(\cdot)$ evolves according to $\dot{x} = f(x, i)$. If $x(\cdot)$ hits some $\partial M_{i,j}$ at time t_1, then the state becomes $[x(t_1), j]$, from which the process continues. See Figure 3.

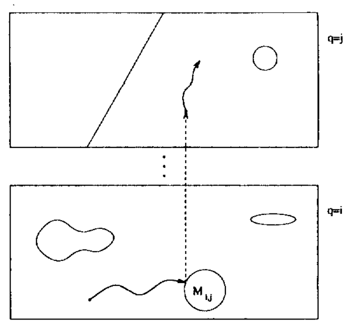

Figure 3: Example dynamics of Tavernini's model.

The switching boundaries are not allowed to be arbitrary; several key assumptions are placed on them in [18].

3.2 Back-Guckenheimer-Myers Model

The framework proposed by Back, Guckenheimer, and Myers in [2] is similar in spirit to the Tavernini model. The model is more general, however, in allowing "jumps" in the continuous state space and setting of parameters when a switching boundary is hit. This is done through *transition functions* defined on the switching boundaries. Also, the model allows a more general state space.

More specifically, the model consists of a state space

$$S = \bigcup_{q \in Q} S_q, \qquad Q \simeq \{1, \ldots, N\},$$

where each S_q is a connected, open set of $\mathbb{R}^n$. Notice that the sets S_q are not required to be disjoint.

The continuous dynamics are given by vector fields $f_q : S_q \to \mathbb{R}^n$. Also, one has open sets U_q such that $\overline{U}_q \subset S_q$ and ∂U_q is piecewise smooth. For $q \in Q$, the transition functions

$$G_q : S_q \to S \times Q$$

govern the jumps that take place when the state in S_q hits ∂U_q. They must satisfy $\pi_1(G_q(x)) \in \overline{U}_{\pi_2(G_q(x))}$, where π_k is the kth coordinate projection function. Thus, $\pi_1(G_q(x))$ is the "continuous part" and $\pi_2(G_q(x))$ is the "discrete part" of the transition function.

The dynamics are as follows. The state starts at point x_0 in U_i. It evolves according to $\dot{x} = f_i(x)$. If $x(\cdot)$ hits ∂U_i at time t_1, then the state instantaneously jumps to state ξ in $\overline{U}_j$, where $G(x(t_1)) = (\xi, j)$. From there, the process continues. We refer to this as the BGM model. See Figure 4, which is reproduced from [2].

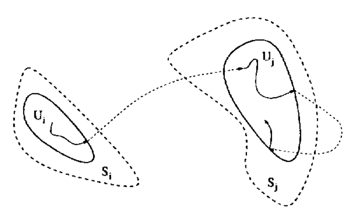

Figure 4: Example dynamics of the BGM model.

3.3 Nerode-Kohn Model

In [17], Nerode and Kohn take an automata-theoretic approach to systems composed of interacting ODEs and FA. Many cases of their approach are given, but here we only discuss their "event-driven, autonomous sequential deterministic model" (herein, NKSD) [17, p. 331].

The model consists of three basic parts: plant, digital control automaton, and interface. In turn, the interface is comprised of an analog-to-digital (AD) converter and digital-to-analog (DA) converter. See Figure 5.

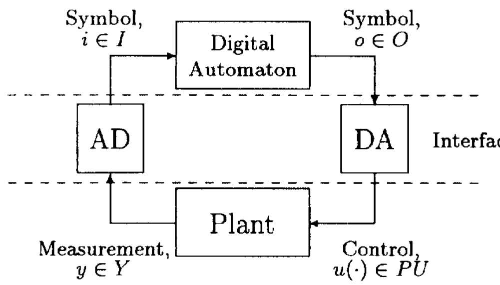

Figure 5: Hybrid system as in Nerode-Kohn model.

The plant is modeled as

$$\dot{x}(t) = f(x(t), u(t)), \qquad y(t) = h(x(t)), \tag{2}$$

where $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $y \in Y \subset \mathbb{R}^p$, $f : X \times U \to \mathbb{R}^n$, and $h : X \to Y$.[1]

The digital control automaton is a quintuple (Q, I, O, ν, η), consisting of the state space, input alphabet, output alphabet, transition function, and output function, respectively. The functions involved are $\nu : Q \times I \to Q$ and $\eta : Q \times I \to O$. In general, Q, I, and O are each isomorphic to subsets of $\mathbb{N}$. However, the interesting case is where these sets are finite, which is discussed below. The "dynamics" of the automaton are given by

$$q_{k+1} = \nu(q_k, i_k), \qquad o_k = \eta(q_k, i_k).$$

This automaton may be thought of as operating in "continuous time" by the convention that the state, input, and output symbols are piecewise right-continuous functions. Then, the convention is that the state $q(t)$ changes only when the input symbol $i(t)$ changes.

It remains to couple the plant and control automaton. This is done through the interface by introducing maps $AD : Y \times Q \to I$ and $DA : O \to PU$. Here, PU denotes the set of piecewise right-continuous functions in $U^{[0,\infty)}$.

The AD symbols are determined by (FA-state-dependent) partitions of the output space Y. These partitions are not allowed to be arbitrary, but are the "essential parts" of small topologies placed on Y for each $q \in Q$ [6, 17]. Analogously, to each $o \in O$ is associated an open set of PU. The DA signal corresponding to output symbol o is chosen from this open set of plant inputs.

In full, we have the system

$$\begin{aligned}
\dot{x}(t) &= f[x(t), DA(o(t))(t - [t])], \\
y(t) &= h[x(t)], \\
q(t) &= \nu[q(t^-), AD(y(t), q(t^-))], \\
o(t) &= \eta[q(t), AD(y(t), q(t^-))].
\end{aligned}$$

Above, $[t]$ denotes the time at which the input symbol last changed. Briefly, the combined dynamics is as follows. Assume the continuous state is evolving according to the first equation and that the FA is in state q. Then $AD(\cdot, q)$ assigns to output $y(t)$ a symbol from the input alphabet of the FA. When this symbol changes, the FA makes the associated state transition, causing a corresponding change in its output symbol o. Associated with this symbol is a control input, $DA(o)$, that is applied as input to the differential equation until the input symbol of the FA again changes.

[1] We have lumped the control and disturbance signals of [17] into a single signal u.

3.4 Antsaklis-Stiver-Lemmon Model

In [1], Antsaklis, Stiver, and Lemmon take a DEDS approach to hybrid systems. Conceptually, the model is related to that of Nerode and Kohn. We refer to it as the ASL model.

Like the NKSD model, the ASL model consists of three basic parts: the plant, the controller, and the interface. Again, see Figure 5. The plant is modeled as Eq. (2), with $X = \mathbb{R}^n$, $U = \mathbb{R}^m$, and $Y = \mathbb{R}^p$. The controller is a discrete event dynamic system, thought of as operating in continuous time as in Section 3.3:

$$q(t) = \nu(q(t^-), i(t)), \qquad o(t) = \eta(q(t)),$$

where $q(t) \in Q$, $i(t) \in I$, and $o(t) \in O$, the state space, plant symbols, and controller symbols, respectively. The sets Q, I, and O are unspecified in [1], but we take from context that they are each isomorphic to subsets of $\mathbb{N}$. The maps are $\nu : Q \times I \to Q$ and $\eta : Q \to O$.

The plant and controller communicate through an interface consisting of two memoryless maps, DA and AD. The first map, called the actuating function, $DA : O \to \mathbb{R}^m$, converts a controller symbol to a piecewise constant plant input: $u(t) = DA(o(t))$. The second map, called the plant symbol generating function, $AD : \mathbb{R}^n \to I$, is a function that maps the plant state space to the set of plant symbols: $i(t) = AD(x(t))$. The function AD is based upon a partition of the state space, where each element of the partition is associated with one plant symbol. The combined dynamics is similar to that of the NKSD model.

3.5 Brockett's Models

Several models of hybrid systems are described in [10]. We only give his "type D hybrid system" here:

$$\begin{aligned}
\dot{x}(t) &= f(x(t), u(t), z\lfloor p \rfloor), \\
\dot{p}(t) &= r(x(t), u(t), z\lfloor p \rfloor), \\
z\lceil p \rceil &= \nu(x[t], z\lfloor p \rfloor, v\lfloor p \rfloor),
\end{aligned}$$

where $x(t) \in X \subset \mathbb{R}^n$, $u(t) \in U \subset \mathbb{R}^m$, $p(t) \in \mathbb{R}$, $v\lfloor p \rfloor \in V$, $z\lfloor p \rfloor \in Z$. Also, $f : X \times U \times Z \to \mathbb{R}^n$, $r : X \times U \times Z \to \mathbb{R}$, and $\nu : X \times Z \times V \to Z$. Here, X and U are open, and V, Z are isomorphic to subsets of $\mathbb{N}$. The notation $[t]$ denotes the value of t at which p most recently became an integer. The rate equation r is required to be nonnegative for all arguments, but need have no upper bound imposed upon it. We denote such a system as BD, short for Brockett's type D model.

Brockett has mixed continuous and "symbolic" controls by the inclusion of the special "clock" or "counter" variable p. The first equation denotes the continuous dynamics and the last equation the "symbolic processing" done by the system. The control $u(t)$ is the continuous control exercised at time t; the control $v\lfloor p \rfloor$ is the pth symbolic or discrete control, that is exercised at the times when p passes through integer values.

The times when p passes through integer values can be thought of as the discrete event times of the hybrid dynamical system. Thus, we consider BD as a precise, first-order model of interactions of ODEs and DEDS. We may picture the dynamics as in Figure 6.

Brockett also has a simpler "type B system" (herein, BB) in which the third equation does not appear and v replaces z in the first two equations. He also generalizes

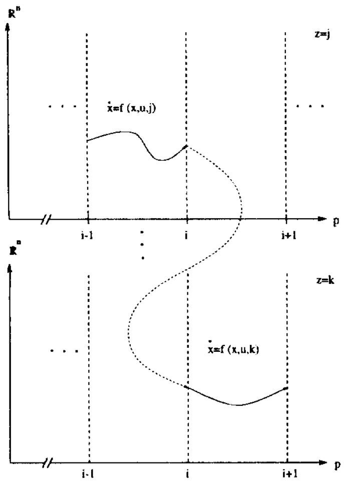

Figure 6: Dynamics of Brockett's BD model.

BD to the case of "hybrid system with vector triggering" (herein, BDV), in which one replaces the single rate and symbolic equations with a finite number of such equations.

3.6 Discussion

At the risk of oversimplification, Tavernini, NKSD, and ASL use autonomous switching; BGM uses autonomous switching and autonomous jumps; and BD uses a combination of autonomous and controlled switching.

It is not hard to see that the BGM model contains Tavernini's model. Simply choose $S_i = \mathbb{R}^n$, $U_i = \mathbb{R}^n \backslash M_i$, $i \in \underline{N}$, and $G(x) = (x, j)$ if $x \in M_{i,j}$. One may also show that the NKSD model contains the Tavernini model (see [7]).

4 Abstract Model

We first present our over-riding framework in generality. We refine it later when we set up our control problem. Our state space for $x(\cdot)$ is $S = \bigcup_{i=0}^{\infty} S_i$ where each S_i is a subset of some Euclidean space $\mathbb{R}^{d_i}$, $d_i \in \mathbb{N}$.[2] Notice that we allow the S_i to overlap and the inclusion of multiple copies of the same space. We also specify *a priori* regions $A_i, C_i, D_i \subset S_i$, $i \in \mathbb{N}$. These are the *autonomous jump sets*, *controlled jump sets*, and *jump destination sets*, respectively. Let A, C, D denote the unions $\bigcup_i A_i$, $\bigcup_i C_i$, $\bigcup_i D_i$, $i \in \mathbb{N}$, respectively. Let U, V be the sets of continuous and discrete controls, respectively. The following maps are assumed to be known:

- *vector fields* $f_i : S_i \times S_i \times U \to \mathbb{R}^{d_i}$, $i \in \mathbb{N}$.
- *transition map* $G : A \times V \to D$.
- *transition delay* $\Delta_1 : A \times V \to \mathbb{R}^+$.
- *impulse delay* $\Delta_2 : \bigcup_i (C_i \times D_i) \to \mathbb{R}^+$.

The dynamics of the control system can now be described as follows. There is a sequence of *pre-jump times* $\{\tau_i\}$ and another sequence of *post-jump times* $\{\Gamma_i\}$ satisfying $0 = \Gamma_0 \le \tau_1 < \Gamma_1 < \tau_2 < \Gamma_2 < \cdots \le \infty$, such that on each interval $[\Gamma_{j-1}, \tau_j)$ with non-empty interior, $x(\cdot)$ evolves according to Eq. (1) in some S_i, $i \in \mathbb{N}$. At the next pre-jump time (say, τ_j) it jumps to some $D_k \in S_k$ according to one of the following two possibilities:

- $x(\tau_j) \in A_i$, in which case it must jump to $x(\Gamma_j) = G(x(\tau_j), v_j) \in D$ at time $\Gamma_j = \tau_j + \Delta_1(x(\tau_j), v_j)$, $v_j \in V$ being a control input. We call this phenomenon an autonomous jump.
- $x(\tau_j) \in C_i$ and the controller chooses to[3] move the trajectory discontinuously to $x(\Gamma_j) \in D$ at time $\Gamma_j = \tau_j + \Delta_2(x(\tau_j), x(\Gamma_j))$. We call this an impulsive jump.

As for the periods $[\tau_j, \Gamma_j)$, we shall follow the convention that the system remains frozen during these intervals. See Figure 7.

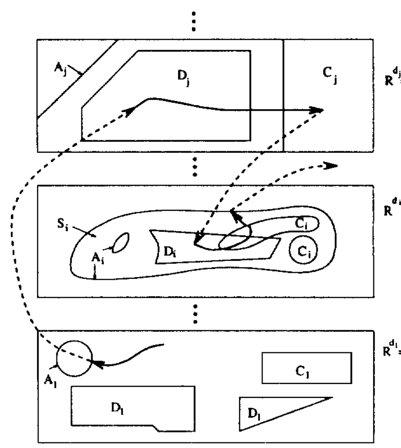

Figure 7: Example dynamics of our model.

For $t \in [0, \infty)$, let $[t] = \max_j \{\Gamma_j \mid \Gamma_j \le t\}$. The vector field $\xi(t)$ of Eq. (1) is given by

$$\xi(t) = f_i(x(t), x[t], u(t)), \tag{3}$$

where i is such that $x(t), x[t] \in S_i$ and $u(\cdot)$ is a U-valued control process.

To avoid confusion, the shorthand $G(x, v; i) = (x'; j)$ is sometimes used to explicitly denote the transition from $x \in A_i \subset S_i$ (with discrete control v) to $x' \in D_j \subset S_j$.

We now show how this framework encompasses the discrete phenomenon of Section 2 and how it compares to the models of Section 3.

Autonomous and Controlled Jumps. These are clearly taken care of with the sets A_i and C_i.

Autonomous Switching. We show that autonomous switching can be viewed as a special case of autonomous jumps. Consider the differential equation with parameters $\dot{x} = f(x, p)$, where $x \in \mathbb{R}^n$, $p \in P \subset \mathbb{R}^m$ closed, $f : \mathbb{R}^n \times P \to \mathbb{R}^n$ continuous. Let, $\nu : \mathbb{R}^n \times P \to P$ be the function governing autonomous switching. For example, in the Tavernini model, ν is the "discrete dynamics."

Then, since $\mathbb{R}^n$ has the universal extension property [16], we can extend f to a continuous function $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$. Now, consider the ODE on $\mathbb{R}^{n+m}$:

$$\dot{x} = F(x, \xi), \qquad \dot{\xi} = 0,$$

where $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ continuous. Let, the transition function be $G : \mathbb{R}^n \times P \to \mathbb{R}^n \times P$ with $G(x, p) = (x, \nu(x, p))$. (Here, V is suppressed.)

[2] The state dimension may change to take into account changes in dynamical description based on discrete events—controlled or autonomous—that change it, e.g., component failures or the collision of two inelastic particles.

[3] It does not have to

Controlled Switching. A system with controlled switching is described by

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = x_0 \in \mathbb{R}^d,$$

where $u(\cdot)$ is a piecewise constant function taking values in $U \subset \mathbb{R}^m$ and $f : \mathbb{R}^d \times U \to \mathbb{R}^d$ is a map with sufficient regularity. There is a strictly positive cost associated with the switchings of $u(\cdot)$. In our framework, let $x'(\cdot) = [x(\cdot),\ u(\cdot)]^T$ be the new state process with dynamics

$$\dot{x}'(t) = f'(x'(t)), \qquad f'(\cdot) = [f(\cdot),\ 0]^T,$$

taking values in $S = \bigcup_{i=0}^{\infty} S_i$ where each S_i is a copy of $\mathbb{R}^d \times U$. Set $C_i = D_i = S_i$, $A_i = \emptyset$ for $i \in \mathbb{N}$. Switchings of $u(\cdot)$ now correspond to impulsive jumps with the associated costs.

Tavernini and BGM Models. Let primed symbols denote those in our model with the same notation as those for BGM. It is obvious that our model includes the BGM model by choosing $S_i' = \overline{S_i}$, $A_i = \partial S_i \cup \partial U_i$, $D_i = \overline{U_i}$, $C_i = \emptyset$, and

$$G'(x;j) = (\pi_1(G(x));\ \pi_2(G(x)))$$

for $x \in U_j$. G' need not be defined on $\partial S_j \backslash \partial U_j$, but for completeness we may define $G'(x;j) = (x;j)$ for $x \in \partial S_j \backslash \partial U_j$. Since BGM contains Tavernini's model, our model does as well.

NKSD and ASL Models. Our model includes the ASL model. First, choose $S_i = \mathbb{R}^n \times \mathbb{R}^3$, $i \in I$. Then, note that the sets $AD^{-1}(i)$, $i \in I$, form a partition of Y. Define the sets $M_j = h^{-1}(AD^{-1}(j))$ and define

$$A_i \equiv \bigcup_{j \neq i, j \in I} M_j.$$

Then define $\tilde{f}_i = [f,\ 0,\ 0,\ 0]$, with dimensions representing x, q, i, and o. The model is complete by specifying

$$G(x;i) = (x, \nu(q,j), j, \eta(q); j)$$

if $x \in M_j \subset A_i$. Inclusion of NKSD is similar.

Brockett's Models. Our model includes Brockett's BD model by choosing $S = \mathbb{R}^n \times \mathbb{R}^4$ and defining

$$\tilde{f} = [f,\ r,\ 0,\ 0,\ 0],$$

with dimensions representing x, $q = p - \lfloor p \rfloor$, $i = \lfloor p \rfloor$, v, and z. Also, set $A = \mathbb{R}^n \times \{1\} \times \mathbb{R}^3$, $D = \mathbb{R}^n \times \{0\} \times \mathbb{N}^3$, and $G((x, 1, i, v, z), v') = (x, 0, i+1, v', \nu(x, z, v))$. It is clear that this can be modified to include BB and BDV.

Automata. A variety of automata are automatically subsumed by inclusion of the Tavernini, BGM, NKSD, ASL, and Brockett models.

5 The Control Problem

In this section, we define a control problem and elucidate all assumptions used in deriving the results in the sequel.

5.1 Problem

Let $a > 0$ be a *discount factor*. We add to our previous model the following known maps:

- *running cost* $k : S_i \times S_i \times U \to \mathbb{R}^+$.
- *transition cost* $c_1 : A \times V \to \mathbb{R}^+$.
- *impulse cost* $c_2 : \bigcup_i (C_i \times D) \to \mathbb{R}^+$, satisfying for all $i, j \in \mathbb{N}$ the conditions

$$c_2(x,y) \ \geq \ c_0 > 0, \tag{4}$$

$$c_2(x,y) \ < \ c_2(x,z) + e^{-a\Delta_2(x,z)} c_2(z,y), \tag{5}$$

for all $x \in C_i$, $y \in D$, and $z \in D \cap C_j$.

Thus, autonomous jumps are done at a cost of $c_1(x(\tau_j), v_j)$ paid at time τ_j; impulsive jumps at a cost of $c_2(x(\tau_j), x(\Gamma_j))$ paid at time τ_j. Note that Eq. (4) rules out from consideration infinitely many impulsive jumps in a finite interval and Eq. (5) rules out the merging of post-jump time of an impulsive jump with the pre-jump time of the next impulsive jump.

In addition to the costs associated with the jumps as above, the controller also incurs a *running cost* of $k(x(t), x[t], u(t))$ per unit time during the intervals $[\Gamma_{j-1}, \tau_j)$, $j \in \mathbb{N}$. The total discounted cost is defined as

$$\int_{\mathbb{T}} e^{-at} k(x(t), x[t], u(t))\, dt + \sum_i e^{-a\sigma_i} c_1(x(\sigma_i), v_i) + \sum_i e^{-a\zeta_i} c_2(x(\zeta_i), x(\zeta_i')), \tag{6}$$

where $\mathbb{T} = \mathbb{R}^+ \backslash (\bigcup_i [\tau_i, \Gamma_i))$, $\{\sigma_i\}$ (resp. $\{\zeta_i\}$) are the successive pre-jump times for autonomous (resp. impulsive) jumps and ζ_j' is the post-jump time for the jth impulsive jump. The *decision* or *control* variables over which Eq. (6) is to be minimized are the continuous control $u(\cdot)$, the discrete control $\{v_i\}$ exercised at the pre-jump times of autonomous jumps, the pre-jump times $\{\zeta_i\}$ of impulsive jumps, and the associated *destinations* $\{x(\zeta_i')\}$.

Our framework clearly includes conventional impulse control [4].

5.2 Assumptions

Throughout the sequel, we make use of the following further assumptions on our abstract model, which are collected here for clarity and convenience.

For each $i \in \mathbb{N}$, the following hold: S_i is the closure of a connected open subset of Euclidean space $\mathbb{R}^{d_i}$, $d_i \in \mathbb{N}$, with Lipschitz boundary ∂S_i. $A_i, C_i, D_i \subset S_i$ are closed. In addition, ∂A_i is Lipschitz and contains ∂S_i.

The maps G, Δ_1, Δ_2, c_1, c_2, and k are bounded uniformly continuous. The vector fields f_i, $i \in \mathbb{N}$, are bounded (uniformly in i), uniformly Lipschitz continuous in the first argument, uniformly equicontinuous with respect to the rest. U, V are compact metric spaces. Below, $u(\cdot)$ is a U-valued control process, assumed to be measurable.

All the above are fairly mild assumptions. The following are more technical assumptions. They may be traded for others as discussed in Section 9. However, in the sequel we construct examples pointing out the necessity of such assumptions or ones like them.

Assumption 1 *$d(A_i, C_i) > 0$ and $\inf_{i \in \mathbb{N}} d(A_i, D_i) > 0$, d being the appropriate Euclidean distance.*

Assumption 2 *Each D_i is bounded and for each i, there exists an integer $N(i) < \infty$ such that for $x \in C_i$, $y \in D_j$, $j > N(i)$, $c_2(x,y) > \sup_z J(z)$. (J is the cost-to-go function, defined below.)*

Assumption 3 *For each i, ∂A_i is an oriented C^1-manifold without boundary and at each point x on ∂A_i, $f_i(x, z, u)$ is "transversal" to ∂A_i for all choices of z, u. By this we require that* (i) *the flow lines be transversal in the usual sense*[4] *and* (ii) *the vector field does not vanish on ∂A_i.*

Assumption 4 *Same as Assumption 3 but with C_i replacing A_i.*

[4]Transversality implies that ∂A_i is $(d_i - 1)$-dimensional.

6 Existence of Optimal Controls

Let $J(x)$ denote the infimum of Eq. (6) over all choices of $u(\cdot), \{v_i\}, \{\zeta_i\}, \{x(\zeta_i')\}$ when $x(0) = x$. We have

Theorem 6.1 *A finite optimal cost exists for any initial condition.*

Corollary 6.2 *There are only finitely many autonomous jumps in finite time.*

To see why an assumption like Assumption 1 is necessary for the above results, one need only consider the following one-dimensional example:

Example 6.3 *Let $S_i = [0,2]$, $A_i = \{0,2\}$, and $f_i(\cdot,\cdot,\cdot) \equiv -1$ for each $i \in \mathbb{N}$. Also for each i, define $C_i = \emptyset$, $D_i = 1/i^2$ and $G(A_i, \cdot) \equiv 1/(i+1)^2$. Finally, let $\Delta_1(\cdot,\cdot) \equiv 0$ and $c_1(\cdot,\cdot) \equiv 1$. Starting in S_1 at $x(0) = 1$, we see that $x(\sum_{i=1}^N 1/i^2) = 1/(N+1)^2$. Since the sum of inverse squares converges, we will accumulate an infinite number of jumps and infinite cost by time $t = \pi^2/6$.*

Next, we show that $J(x)$ is attained for all x if we extend the class of admissible $u(\cdot)$ to "relaxed" controls. The "relaxed" control framework [20] is as follows: We suppose that $U = \mathcal{P}(U')$, defined as the space of probability measures on a compact space U' with the topology of weak convergence [5]. Also

$$f_i(x,z,u) = \int f_i'(x,z,u)u(dy), \quad i \in \mathbb{N},$$

$$k(x,z,u) = \int k'(x,z,u)u(dy),$$

for suitable $\{f_i'\}$ and k' satisfying the appropriate continuity/Lipschitz continuity requirements. The relaxed control framework and its implications in control theory are well known and the reader is referred to [20] for details.

Theorem 6.4 *An optimal trajectory exists for any initial condition.*

It is easy to see why Theorem 6.4 may fail in absence of Assumption 2:

Example 6.5 *Suppose, for example, $k(x,z,u) \equiv \alpha_i$ and $c_1(x,v) \equiv \beta_i$ when $x \in S_i$, $c_2(x,y) \equiv \gamma_{i,j}$ when $x \in S_i$, $y \in S_j$, with α_i, β_i, $\gamma_{i,j}$ strictly decreasing with i,j. It is easy to conceive of a situation where the optimal choice would be to "jump to infinity" as fast as you can.*

The theorem may also fail in the absence of Assumption 3 as the following two-dimensional system shows:

Example 6.6 *Consider*

$$\dot{x}_1(t) = 1, \quad \dot{x}_2(t) = u, \quad x_1(0) = x_2(0) = 0,$$

with $u \in [0,1]$ and cost

$$\int_0^\infty e^{-t} \min\{|x_1(t) + x_2(t)|, 10^{20}\}\, dt,$$

with the provision that the trajectory jumps to $[10^{10}, 10^{10}]$ on hitting a certain curve A. For A, consider

- *the line segment $\{x_1 = 1, -1 \le x_2 \le 0\}$, a C^1-manifold with boundary;*
- *the circle $\{(x_1,x_2) \mid (x_1-1)^2 + (x_2+1)^2 = 1\}$, a C^1-manifold without boundary, but the vector field $(1,u)$ with $u = 0$ is not transversal to it at $(1,0)$.*

It is easy to see that the optimal cost is not attained in either case.

Also, it is not enough that the flow lines for each control be transversal in the usual sense as the following one-dimensional example shows:

Example 6.7 *Let $S_1 = S_2 = \mathbb{R}^+$. Consider*

$$f_1(x,y,u) = -x + u, \qquad f_2(x,y,u) = 0, \qquad u \in [-1,0],$$

with running cost $\min\{K, |x|\}$ and $G(0,\cdot;1) \equiv (K;2)$. Choosing, for example, $K > 1$ one sees that the optimal cost cannot be attained for any $1 \ge x(0) > 0$.

Coming back to the relaxed control framework, say that $u(\cdot)$ is a *precise* control if $u(\cdot) = \delta_{q(\cdot)}(dy)$ for a measurable $q : [0,\infty) \to U'$ where δ_z denotes the Dirac measure at $z \in U'$. Let M denote the set of measures on $[0,T] \times U'$ of the form $dt\, u(t,dy)$ where $u(\cdot)$ is a relaxed control, and M_0 its subset corresponding to precise controls. It is known that M_0 is dense in M with respect to the topology of weak convergence [20]. In conjunction with Assumption 4, this allows us to deduce

Theorem 6.8 *Under Assumptions 2–4, for every $\epsilon > 0$ an ϵ-optimal control policy exists wherein $u(\cdot)$ is precise.*

7 The Value Function

In the foregoing, we had set $[0] = 0$ and thus $x[0] = x(0) = x_0$. More generally, for $x(0) = x_0 \in S_{i_0}$, we may consider $x[0] = y$ for some $y \in S_{i_0}$, making negligible difference in the foregoing analysis. Let $V(x,y)$ denote the optimal cost corresponding to this initial data. Then in dynamic programming parlance, $(x,y) \mapsto V(x,y)$ defines the "value function" for our control problem. We next explore some properties of this value function.

In view of Assumption 3, we can speak of the *right side* of ∂A_i as the side on which $f_i(\cdot,\cdot,\cdot)$ is directed towards ∂A_i, $i \in \mathbb{N}$. A similar definition is possible for the right side of ∂C_i (in light of Assumption 4).

Definition 7.1 (From the right) *Say that $(x_n, y_n) \to (x_\infty, y_\infty)$ from the right in $\bigcup_i (S_i \times S_i)$ if $y_n \to y_\infty$ and either $x_n \to x_\infty \notin \bigcup_i (\partial A_i \cup \partial C_i)$ or $x_n \to x_\infty \in \bigcup_i (\partial A_i \cup \partial C_i)$ from the right side.*

V is said to be *continuous from the right* if $(x_n, y_n) \to (x_\infty, y_\infty)$ from the right implies $V(x_n, y_n) \to V(x_\infty, y_\infty)$.

Theorem 7.2 *V is continuous from the right.*

For brevity, let $V(z)$ denote $V(z,z)$, $\mathcal{C} = \bigcup_i (C_i \times S_i)$, and

$$F(x,y,u) \equiv \langle \nabla_x V(x,y), f_i(x,y,u) \rangle - aV(x,y) + k(x,y,u).$$

We now propose the following system of *generalized quasi-variational inequalities* (*GQVIs*) $V(\cdot,\cdot)$ is expected to satisfy, which are formally derived in [8]. For $(x,y) \in S_i \times S_i$,

$$\min_u F(x,y,u) \le 0. \tag{7}$$

On $\bigcup_i (A_i \times S_i)$,

$$V(x,y) \le \min_v \left\{ c_1(x,v) + e^{-a\Delta_1(x,v)} V(G(x,v)) \right\}. \tag{8}$$

On $\mathcal{C}$,

$$V(x,y) \le \min_{z \in D} \left\{ c_2(x,z) + e^{-a\Delta_2(x,z)} V(z) \right\}, \tag{9}$$

$$\left(V(x,y) - \min_{z\in D} \left\{ c_2(x,z) + e^{-a\Delta_2(x,z)} V(z) \right\} \right) \cdot (\min_u F(x,y,u)) = 0. \tag{10}$$

Eq. (10) states that at least one of Eqs. (7), (9) must be an equality on $\mathcal{C}$. Eqs. (7)–(10) generalize the traditional quasi-variational inequalities encountered in impulse control [4]. We do not address the issue of well-posedness of Eqs. (7)–(10). The following "verification theorem," however, can be proved by routine arguments.

Theorem 7.3 *Suppose Eqs.* (7)–(10) *have a "classical" solution V that is continuously differentiable "from the right" in the first argument and continuous in the second. Suppose $x(\cdot)$ is an admissible trajectory with initial data (x_0, y_0) and $u(\cdot)$, $\{v_i\}$, $\{\sigma_i\}$, $\{\zeta_i\}$, $\{\tau_i\}$, $\{\Gamma_i\}$ the associated controls and jump times, such that the following hold:*
(i) *For a.e. $t \in \mathbb{T}$, i such that $x(t) \in S_i$,*

$$F(x(t), x[t], u(t)) = \min_u F(x(t), x[t], u).$$

(ii) *For all i, $V(x(\sigma_i), x[\sigma_i]) =$*

$$c_1(x(\sigma_i), v_i) + \exp\{-a\Delta_1(x(\sigma_i), v_i)\}V(G(x(\sigma_i), v_i)).$$

(iii) *For all i, $V(x(\zeta_i), x[\zeta_i]) =$*

$$c_2(x(\zeta_i), x(\zeta_i')) + \exp\{-a\Delta_2(x(\zeta_i), x(\zeta_i'))\}V(x(\zeta_i')).$$

Then $x(\cdot)$ is an optimal trajectory.

8 Example Problems

Here, we give solve some example problems in our framework. First, going back to Example 6.7 we have

Example 8.1 *Consider Example 6.7 except with the controls restricted in $[-1, -\epsilon]$, $0 < \epsilon < 1$. Then the flows are transversal and do not vanish on $A_1 = \{0\}$ for any u. In this case, the optimal control exists. For example, if $K > 1/\epsilon$, one can show that $u(\cdot) \equiv -\epsilon$ is optimal.*

More interestingly, consider the system of Example 2.1. As a control problem, consider minimizing

$$J = \int_0^\infty \tfrac{1}{2}(qx^2 + u^2)e^{-at}dt \equiv \int_0^\infty k(x,u)e^{-at}dt$$

We first solve for $V(x, s = H(x))$ and then u. By symmetry, we expect $V(-\Delta, 1) = V(\Delta, -1)$. From the GQVIs, we expect V to satisfy

$$\min_u \{-aV(x,s) + V_x(x,s)\cdot f(x,u) + k(x,u)\} = 0,$$
$$V(\Delta, 1) = c + V(\Delta, -1), \quad V(-\Delta, -1) = c + V(-\Delta, 1),$$

where s can take on the values ± 1 and c represents the cost associated with the autonomous switchings.

We have solved these equations numerically for the case $c = 0$, $a = 1$, $\Delta = 0.1$. The resulting control u, plotted against x (for $s = 1$) and q, is shown in Figure 8. As the state is increasingly penalized, the control action increases in such a way to "invert" the hysteresis function H.

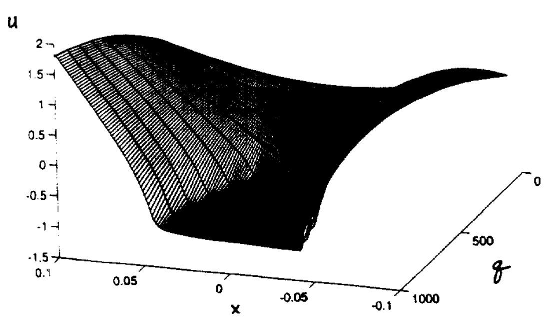

Figure 8: Optimal control u versus x and q, for the case $a = 1$, $c = 0$, $\Delta = 0.1$.

9 Conclusions

We examined the phenomena that arise in hybrid systems and reviewed several hybrid systems models from the literature. We then proposed a very general framework for hybrid control problems that encompasses these hybrid phenomena and the reviewed models. A specific control problem was then studied in this framework, leading to an existence result for optimal controls. The "value function" associated with this problem is expected to satisfy a set of "generalized quasi-variational inequalities."

The foregoing presents some initial steps towards developing a unified "state space" paradigm for hybrid control. Several open issues suggest themselves. We conclude with a brief list of some of the more striking ones.

- A daunting problem is to characterize the value function as the unique viscosity solution of the GQVIs Eqs. (7)–(10).
- Many of our assumptions can possibly be relaxed at the expense of additional technicalities or traded off for alternative sets of assumptions that have the same effect. For example, the condition $d(C_i, A_i) > 0$ could be dropped by having c_2 penalize highly the impulsive jumps that take place too close to A_i.
- Example 6.6 shows that Assumption 3 cannot be dropped. It remains open how to relax Assumptions 3 and 4. This might be accomplished through additional continuity assumptions on G, Δ_1, and c_1.
- An important issue is to develop good computational schemes to compute near-optimal controls, which is currently a topic of further research. (See [11] for some related work.) This is daunting in general as [7] shows that the hybrid systems models in Section 3 can simulate arbitrary Turing machines (TMs), with state dimension as small as three. It is not hard to conceive of control problems where the cost is less than 1 if the corresponding TM does not halt, but greater than 3 if it does. Allowing an initial impulsive jump that would result in a cost of 2, finding the optimal control is equivalent to solving the halting problem.
- Another possible extension is replacing the S_i by smooth manifolds with boundary embedded in a Euclidean space. See [9] for some related work.

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