STATE IDENTIFICATION OF A CLASS OF LINEAR DISTRIBUTED SYSTEMS

by

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This paper is concerned with the state identification problem for a class of linear distributed parameter systems. Since the system is described by a partial differential equation, its solution requires knowledge of initial conditions and environment forcing terms which include the boundary conditions. The problem studied here is the following: Given i) inexact measurements of the initial conditions and environmental interactions ii) inexact and possibly incomplete measurements of the state of the system, determine on the basis of the above data the true initial and boundary conditions associated with a given partial differential equation which is in some sense optimal with respect to the given data.

The basis for selecting the estimates of the boundary and initial conditions associated with a given partial differential equation, that is, the criterion of optimality, is that of "least squares". Theoretical results as well as a computational scheme with numerical results are presented.

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A. Introduction

This paper is concerned with the state identification problem for a class of linear distributed parameter systems. Since the system is described by a partial differential equation, its solution requires knowledge of initial conditions and environmental forcing terms which include the boundary conditions. The problem studied here is the following:

Given i) inexact measurements of the initial conditions and environmental interactions ii) inexact and possibly incomplete measurements of the state of the system, determine on the basis of the above data the true initial and boundary conditions associated with a given partial differential equation which is in some sense optimal with respect to the given data.

The basis for selecting the estimates of the boundary and initial conditions associated with a given partial differential equation, that is, the criterion of optimality, is that of "least squares". To be more precise, we mean the following:

Given:

(1) The measurement data, which we denote here by Z, and

(2) An (arbitrary) solution of the partial differential equation, denoted here by $Y(\underline{v})$, where $\underline{v}$ is an arbitrary <u>estimate</u> of the true initial state and boundary conditions, then

Obtain:

(1) $\underline{v}$ which extremizes the <u>error functional</u>

$$J(\underline{v}) = ||Z - Y(\underline{v})||^2 \quad ,$$

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where $||\cdot||^2$ is some appropriate squared metric.

The identification problem, treated here is thus a variational problem--that of characterizing extremals to a given functional, constrained by a partial differential equation. We obtain a characterization of these extremals which is both necessary and sufficient, using the theory of variational inequalities.[1] Two methods for the numerical recovery of the extremals from this characterization are presented. One of these is a Ricatti-like decoupling and the other is a "direct method" involving conjugate directions of search on a quadratic error surface. Collateral work may be found in the recent paper of Balakrishnan and Lions.[2]

The identification problem, as introduced, is customarily given a stochastic treatment. In that context, the error associated with the measurement data Z is considered to be a random variable, whose values are "distributed" in a known way. The state identification or filtering problem, as it is called in this context, is to determine the a posteriori probability density of the state, given the measurements Z .

Under special statistical hypothesis on the error processes, namely that they be purely random with Gaussian probability density and in addition, are additive--that is

$$Z = Y(\underline{u}) + E$$

where $\underline{u}$ is the true "state of nature" and E is the error process, then if the system state evolution process is also linear, the a posteriori density of the states is also Gaussian. It can be shown[3] that the filtered estimate (given in terms of the sufficient statistics of the Gaussian distribution of the states, the mean and variance) coincides with the "least-squares" estimate. Thus, under these special hypotheses the variational and stochastic approaches yield identical results.

We remark that the variational problems arising in distributed optimal control are amenable to the solution techniques suggested in the sequel. In particular, optimal boundary controllers are recovered efficiently by the "direct method" already mentioned.

B. Definitions and Mathematical Preliminaries

Let Ω be a simply connected, bounded open set in R^r . Points of Ω are denoted by $x = (x_1\ x_2\ \ldots\ x_r)$. Γ is the boundary of Ω . Let t denote time, $t \in (0,T]$. Define the sets:

$$\Sigma = \Gamma\times(0,T] \;;\; Q = \Omega\times(0,T]$$

We adopt a notational convention regarding the functions f :

$f(x,t)$ is a point in R^1 , $(x,t)\in Q$

$f(\cdot,t)$ is an element of a Hilbert space $K(\Omega)$

$f(\cdot,\cdot)$ is an element of the Hilbert space $L^2(0,T;K)$,

where $L^2(0,T;K)$ is the space of functions (equivalence classes) which are square integrable with values in K . When appropriate, we shall consider derivatives of f to be taken in the distribution sense, that is, given a function $\phi(\cdot)\in C^1(\Omega)$ with compact support in Ω, then for $f(\cdot)\in K(\Omega)$, the mapping

$$\frac{\partial f(x)}{\partial x_i} : \phi(x) \to -\int_\Omega f \frac{\partial\phi}{\partial x}\, dx \;;\; i = 1, 2, \ldots r$$

is called the distribution derivative of the function f . Higher order derivatives are taken in an analogous way.

Define the second order elliptic operator $A[\cdot]$:

$$A[\psi] = -\sum_{i,j=1}^{r} \frac{\partial}{\partial x_i}\left[a_{ij}(x,t)\frac{\partial\psi(x,t)}{\partial x_j}\right] + a_o(x,t)\psi$$

where $a_{ij}(x,t)$, $(i,j = 1, 2 .. r)$ are bounded, measurable and exhibit the coercive property:

$$\sum_{i,j=1}^{r} a_{ij}(x,t)\,\xi_i\,\xi_j \geq \alpha(\xi_1^2 + \ldots + \xi_r^2) \quad \text{for all} \quad \xi_i \in R^1 \;, (i = 1, 2 .. r)$$

$$a_o(x,t) \geq \alpha$$

$$\alpha > 0 \;,\; (x,t)\in Q \;.$$

By $A_s[\cdot]$ we mean the operator $A[\cdot]$ with an additional symmetry condition:

$$a_{ij}(x,t) = a_{ji}(x,t) \quad (i,j = 1, 2 .. r) \;,\; \text{for all} \;\; (x,t)\in Q \;.$$

The case where the coeficients $a_{ij}(x,t) = a_{ij}(x)$, $a_o(x,t) = a_o(x)$ leads to the classical Sturm Liouville operator, denoted by A_{st} .

We shall be concerned with the properties of solutions to the Sturm Liouville problem

$$A_{st}[w] - \lambda\rho w = 0 \qquad (1)$$

with any one of the boundary conditions

(I) $w(s) = 0 \qquad\qquad s \in \Gamma$

(II) $\dfrac{\partial w(s)}{\partial \nu} = 0 \qquad\qquad s \in \Gamma$

(III) $\dfrac{\partial w(s)}{\partial \nu} + \beta(s)\, w(s) = 0 \qquad s \in \Gamma \;;\;\; \beta(s) > 0$ for all $s \in \Gamma$.

Solutions to (1) with any one of (I) (II) (III) are complete in $L^2(\Omega)$. Of special interest are the solutions to

$$\left.\begin{array}{ll} \dfrac{\partial^2 w}{\partial x^2} + \lambda w = 0 & x \in (0,1) \\ w(0) = 0 & \\ w(1) = 0 & \end{array}\right\} \qquad (1)^*$$

namely,

$$\{\sqrt{2}\,\sin\sqrt{\lambda_i}\,x\}_{i=1,2\ldots} \;\;;\;\; \lambda_i = (i\pi)^2$$

C. <u>The Distributed Systems</u>

We consider in detail identification problems associated with the distributed system whose evolution equation is linear, parabolic, with inhomogeneous boundary conditions of the Dirichlet type:

$$\left.\begin{array}{lll} \dfrac{\partial y(x,t)}{\partial t} + A[y(x,t)] = f(x,t) & (x,t) \in Q \\ y(s,t) \qquad\qquad\quad = u_1(s,t) & (s,t) \in \Sigma \\ y(x,o) \qquad\qquad\quad = u_2(x) & x \in \Omega \end{array}\right\} \qquad (2)$$

Hypothesis on $f(x,t)$ and $u_1(s,t)$, $u_2(x)$ are:

$$\left.\begin{array}{l} f(\cdot,\cdot) \in L^2(Q) \\ u_1(\cdot,\cdot) \in L^2(\Sigma) \\ u_2(\cdot) \in L^2(\Omega) \end{array}\right\} \qquad (3)$$

For the system (2) with hypothesis (3), we have the following Lemma:

<u>Lemma 1 (Lions-Magenes)</u>

There exists one and only one solution to (2) with (3) such that $y(\cdot,\cdot) \in L^2(Q)$. In addition,

$$\frac{\partial y}{\partial x}(\cdot,\cdot) \notin L^2(Q) \;.$$

<u>Remark</u> All our results hold in the case where the boundary conditions on (2) are Neumann or "Mixed". Moreover, systems whose evolution equation

is of second order hyperbolic type, namely:

$$\frac{\partial^2 y(x,t)}{\partial t^2} + A_s[y(x,t)] = f(x,t)$$

with any of the three boundary conditions fall within the jurisdiction of our results.[3]

D. Mathematical Statement of the Identification Problem

Given: (i) System evolution process-equation (2).

(ii) Input measurements:

$$\underline{z} = \begin{bmatrix} z_1(s,t) \\ z_2(x) \end{bmatrix} = \begin{bmatrix} u_1^*(s,t) + K_1 N_1(t) \\ u_2^*(x) + K_2 N_2(t) \end{bmatrix} \tag{5}$$

where $\underline{u}^* = [u_1^*(s,t) \,\vdots\, u_2^*(x)]^T$ is the true "state of nature", and K_1 and K_2 are constants.

(iii) Output measurements:

(a) $$z(x,t) = y(x,t;\underline{u}^*) + K_o N_o(t) \tag{6}$$

(b) $$z(x^i,t) = y(x^i,t;\underline{u}^*) + K_o^i N_o^i(t) \; ; \; i = 1,2..\nu \tag{7}$$

where $x^i \in \Omega$, $N_o(t)$, $N_o^i(t)$, $N_1(t)$ and $N_2(t)$ are random error processes and K_o is a constant.

Identification Problem: Obtain $\underline{u}$, a "refined estimate" of $\underline{u}^*$, based on the data contained in the input and output measurements. The "refined estimate" is defined as that $\underline{u}$ in an admissible set of functions V which extremizes a certain quadratic error functional $J(\underline{v})$. That is, choose $\underline{u}$ such that

$$J(\underline{u}) = \inf_{\underline{v} \in V} J(\underline{v}) \; ; \quad V = L^2(\Sigma) x L^2(\Omega) \quad .$$

It is possible to consider a large variety of error functionals $J(\underline{v})$. This variety is induced by the type of measurement data available ((ii) and (iii)). A careless choice of functional $J(\underline{v})$ can lead to erroneous results. We postpone a discussion of "well set" functionals to Section E. Two specific error functionals considered in this study are induced by the two output measurements (iiia) and (iiib). They are:

$$\text{(a)}\quad J(\underline{v}) = \int_Q [y(x,t;\underline{v}) - z(x,t)]^2 dx\, dt + \int_\Sigma [v_1(s,t) - z_1(s,t)]^2 ds\, dt$$

$$+ \int_\Omega [v_2(x) - z_2(x)]^2 dx \quad . \tag{8}$$

$$\text{(b)}\quad J(\underline{v}) = \int_0^T \sum_{i=1}^{\nu} [y(x^i,t;\underline{v}) - z(x^i,t)]^2 dt + \int_\Sigma [v_1(s,t) - z_1(s,t)]^2 ds\, dt$$

$$+ \int_\Omega [v_2(x) - z_2(x)]^2 dx \quad . \tag{9}$$

Remarks The output measurement process (iiia) is physically unrealistic as it is not possible to measure the entire spatial profile. For the same reason, so is the input measurement process $z_2(x)$. The latter case can be rationalized however, by asserting that $z_2(x)$ is obtained by computing an initial steady state profile which is in error. Although not considered in this paper, it is possible to treat other measurement processes (provided they are appropriately formulated) by using the methods of this paper.

For notational convenience, we shall consider (in detail) the identification problem associated with (8) and report formally the results for (9).

E. Characterization of Extremals

The characterization of extremals to $J(\underline{v})$ is afforded by the results of Lions and Stampacchia.[1] We first introduce the appropriate framework.

Let $a(\underline{v},\underline{w})$ be a coercive continuous bilinear form, $\underline{v},\underline{w} \in V = L^2(\Sigma) \times L^2(\Omega)$

$\ell(\underline{v})$ be a continuous linear form.

Then, if

$$J(\underline{v}) = a(\underline{v},\underline{v}) - 2\ell(\underline{v}) + c \tag{10}$$

we have the following theorem:

Theorem 1 (Lions-Stampacchia):[1] There exists one and only one $\underline{u} \in V$ such that

$$J(\underline{u}) \leq J(\underline{v}) \quad \text{for all } \underline{v} \in V$$

and it is characterized by

$$a(\underline{u},\underline{v}) - \ell(\underline{v}) = 0 \quad \text{for all } \underline{v} \in V \tag{11}$$

Theorem 1 is an appropriate "maximum principle" for the purposes of solving the given identification problem. It is necessary to check whether $J(\underline{v})$ given by (8) (or (9)) has the representation (10). Using (8), we can define

$$a(\underline{v},\underline{v}) = \int_Q [y(x,t;\underline{v}) - y(x,t;\underline{o})]^2 dx\, dt + \int_\Sigma v_1(s,t)^2 ds\, dt + \int_\Omega v_2(x)^2 dx \tag{12}$$

$$\ell(\underline{v}) = -\{\int_Q [y(x,t;\underline{v}) - y(x,t;\underline{o})][y(x,t;\underline{o}) - z(x,t)]dx\, dt - \int_\Sigma v_1(s,t)\, z_1(s,t) ds\, dt - \int_\Omega v_2(x)\, z_2(x) dx\} \tag{13}$$

$$c = \int_Q [y(x,t;\underline{o}) - z(x,t)]^2 dx\, dt + \int_\Sigma z_1(s,t)^2 ds\, dt + \int_\Omega z_2(x)^2 dx \tag{14}$$

Then it is clear that $J(\underline{v})$, given by (8), can be written:

$$J(\underline{v}) = a(\underline{v},\underline{v}) - 2\ell(\underline{v}) + c$$

with $a(\underline{v},\underline{v})$, $\ell(\underline{v})$ and c given by (12), (13) and (14), respectively. Moreover, the hypothesis on $a(\underline{v},\underline{v})$, $\ell(\underline{v})$ and $\underline{c}$ are satisfied. Hence, by Theorem 1, the refined estimate $\underline{u}$, which minimizes $J(\underline{v})$, is uniquely characterized by:

$$\int_Q [y(x,t;\underline{u}) - z(x,t)][y(x,t;\underline{v}) - y(x,t;\underline{o})]dx\, dt + \int_\Sigma [u_1(s,t) - z_1(s,t)][v_1(s,t)]ds\, dt + \int_\Omega [u_2(x) - z_2(x)]\, v_2(x) dx = 0 . \tag{15}$$

Equation (15) is not of immediate utility. However, by defining a system adjoint to (2), (15) can be manipulated to yield a more workable result. Thus, define $p(x,t)$, the adjoint variable to $y(x,t)$, which evolves according to:

$$\left.\begin{aligned} -\frac{\partial p(x,t)}{\partial t} + A[p(x,t)] &= y(x,t;\underline{u}) - z(x,t) && x,t \in Q \\ p(s,t) &= 0 && s,t \in \Sigma \\ p(x,T) &= 0 && x \in \Omega \end{aligned}\right\} \qquad (16)$$

It can be shown[3] that (15) is equivalent to:

$$\left.\begin{aligned} -\frac{\partial p(s,t)}{\partial \nu^*} + u_1(s,t) - z_1(s,t) &= 0 && s,t \in \Sigma \\ p(x,o) + u_2(x) - z_2(x) &= 0 && x \in \Omega \end{aligned}\right\} \qquad (17)$$

Thus the simultaneous solution of (2), (16) and (17) defines the defined estimate $\underline{u}$ and yelds the refined estimate of the state, $y(x,t;\underline{u})$.

Remark The extremal to the functional $J(\underline{v})$ given by (9) is given by solving (2) and (17) simultaneously with an equation for $p(x,t)$ given by

$$\left.\begin{aligned} -\frac{\partial p(x,t)}{\partial t} + A[p(x,t)] &= \sum_{i=1}^{\nu} y(x,t;\underline{u}) - z(x,t)]\,\delta(x-x^i) \quad (i=1,2..\nu) \\ p(s,t) &= 0 \\ p(x,T) &= 0 \end{aligned}\right\} \qquad (18)$$

It can be shown,[4] that (16) and (18) have solutions such that $\frac{\partial p}{\partial \nu^*}(\cdot,\cdot) \in L^2(\Sigma)$, so that (17) makes sense.

We remarked in Section D that it was possible to construct functionals $J(\underline{v})$ which were not "well set". By well set, we mean that a representation for $J(\underline{v})$ given by (10) is possible. As an example of a non well set problem, consider

$$J(\underline{v}) = \int_\Omega [y(x,T;\underline{v}) - z(x,T)]^2 dx + \int_\Sigma [v_1(s,t) - z_1(s,t)]^2 ds\, dt + \int_\Omega [v_2(x) - z_2(x)]^2 dx \qquad (19)$$

As before, we can define

$$a(\underline{v},\underline{v}) = \int_\Omega [y(x,T;\underline{v}) - y(x,T;\underline{o})]^2 dx + \int_\Sigma v_1(s,t)^2 ds\, dt$$

$$+ \int_\Omega v_2(x)^2 dx \tag{20}$$

Now, $a(\underline{v},\underline{v})$ is not continuous[3] and the representation fails. It is possible to construct several such ill-posed problems.[2,3] Appropriate reconstruction can, however, relieve these difficulties.[2,3]

F. Recovery of the Extremals

As we announced in Section A, two methods for the recovery of extremals from the characterization given by (2), (16) or (18) and (17) can be proposed. Consider first a Ricatti-like Decoupling.

F.1 Ricatti-like Decoupling

We note that (2), (16) or (18) and (17) constitutes a two point (time) boundary-value problem. That is, the "initial" conditions on $y(x,t)$ and $p(x,t)$ are split. It is possible to determine an equation for $y(x,T)$, with which the system of equations (2), (16) or (18) and (17) can be solved (in principle) as an initial value problem. However, $y(x,T)$ --that is, $y(x,T;\underline{u})$ is the refined state estimate at the terminal time T, which is fixed, but arbitrary. Thus we shall consider the identification problem to be solved once having obtained an equation for $y(x,T;\underline{u})$. We give the result as a theorem:

Theorem 2 Given the system of Equations (2), (16) and (17), then if $P(x,\xi,t)$ satisfies

(a) $$\frac{\partial P(x,\xi,t)}{\partial t} - A_\xi[P(x,\xi,t)] - A_x[P(x,\xi,t)] - \delta(\xi - x)$$

$$+ \int_{\Gamma_s} \frac{\partial P(x,s,t)}{\partial \nu^*} \frac{\partial P(s,\xi,t)}{\partial \nu^*} ds = 0 , \quad (x,\xi,t) \in \Omega\times\Omega\times(0,T] \tag{21}$$

$$P(x,s,t) = P(s,\xi,t) = 0 \tag{22}$$

$$P(x,\xi,o) = \delta(x - \xi) \tag{23}$$

(b) $P(\cdot,\cdot,t) \in H^2(\Omega\times\Omega)$, $\frac{\partial P}{\partial t}(\cdot,\cdot,t) \in L^2(\Omega\times\Omega)$, $t \in (0,T]$,

$(H^2(\Omega\times\Omega)$ is the second Sobolev space) then:

(i) There exists one and only one $\hat{y}(\cdot,\cdot) \in L^2(Q)$ such that

$$y(x,T;\underline{u}) = \hat{y}(x,T) \tag{24}$$

where $\hat{y}(x,t)$ is the unique solution of the following linear integral equation of the second kind

$$\int_\Omega P(x,\xi,t)\left(\frac{\partial \hat{y}(\xi,t)}{\partial t} + A[\hat{y}(\xi,t)] - f(\xi,t)\right)d\xi = z(x,t) - \hat{y}(x,t) \tag{25}$$

The conditions satisfied by $\hat{y}(x,t)$ on the closure of Q are:

$$\left.\begin{aligned} \hat{y}(s,t) &= z_1(s,t) \qquad (s,t) \in \Sigma \\ \hat{y}(x,o) &= z_2(x) \qquad x \in \Omega \end{aligned}\right\} \tag{26}$$

For a proof of this theorem, see Phillipson.[3] The numerical solution of equations (21) through (26) is not trivial. However an approximate solution is possible, using an eigenvalue expansion.[3] With the definitions

$$P(x,\xi,t) \overset{\text{l.i.m}}{=} P_m(x,\xi,t) \; ; \; P_m(x,\xi,t) = \sum_{i,j=1}^{m} P_{ij}(t)\, w_i(x)\, w_j(\xi)$$

$$\hat{y}(x,t) \overset{\text{l.i.m}}{=} \hat{y}_m(x,t) \; ; \; \hat{y}_m(x,t) = \sum_{i=1}^{m} \hat{y}_i(t)\, w_i(x)$$

where $w_i(x)$ and $w_j(x)$ satisfy (1), it can be shown[3] that (21) through (26) yield the familiar "lumped" results:

$$\frac{d}{dt}(P^{-1}(t)) + P^{-1}(t)A + AP^{-1}(t) + P^{-1}(t)\, IP^{-1}(t) - WW^T = 0 \tag{27}$$

$$P^{-1}(0) = I \tag{28}$$

$$\frac{d\hat{y}(t)}{dt} + A\hat{y}(t) - f(t) + z_1(t) = P^{-1}(t)[z(t) - \hat{y}(t)] \tag{29}$$

$$\hat{y}(0) = z_2 \tag{30}$$

where

$P(t) = \{P_{ij}(t)\}_{i,j=1,2..m}$

$\hat{y}(t) = \{\hat{y}_i(t)\}_{i=1,2..m}$

$A = \text{diag}\{\lambda_i\}_{i=1,2..m}$

$$WW^T = \int_\Gamma \frac{\partial w}{\partial \nu^*} \frac{\partial w^T}{\partial \nu^*} ds \; ; \; \frac{\partial w}{\partial \nu^*} = \left\{\frac{\partial w_i}{\partial \nu^*}\right\}_{i=1,2..m}$$

$$z_1(t) = (z_{1i}(t))_{i=1,2..m} \; ; \; z_{1i} = \int_\Gamma z_1(s,t) \frac{\partial w_i}{\partial \nu^*} ds$$

$$z_2 = (z_{2i})_{i=1,2..m} \; ; \; z_{2i} = \int_\Omega z_2(x)\, w_i(x)\, dx$$

$$f(t) = (f_i(t))_{i=1,2..m} \; ; \; f_i(t) = \int_\Omega f(x,t)\, w_i(x)\, dx \; .$$

In Section G, we report results using the suggested decoupling and subsequent approximation, for a simulated example. There, we also give results pertaining to the "discrete measurement" case induced by the functional (9).

F.2 A Direct Variational Method

To recapitulate, the problem is to select $\underline{u} \in V$ such that

$$J(\underline{u}) = \inf_{\underline{v} \in V} J(\underline{v}) \; . \tag{31}$$

As we have seen, there is a unique $\underline{u} \in V$ with the property (31) and it is characterized by

$$a(\underline{u},\underline{v}) - \ell(\underline{v}) = 0 \tag{32}$$

Equation (32) is the derivative of the functional $J(\underline{v})$ evaluated at $\underline{u}$. In terms of the gradient $\underline{G}(\underline{u})$, (32) is equivalent to:

$$(\underline{G}(\underline{u}),\underline{v})_V = 0$$

with $G(\underline{u})$ given by (17). The direct method for determining $\underline{u}$ involves searching on the quadratic surface $J(\underline{v})$ along directions $\underline{s}^k(\underline{G})$ which lead eventually to $\underline{u}$. That is,

$$\underline{u}^{k+1} = \underline{u}^k + \alpha^k \underline{s}^k(\underline{G}) \tag{33}$$

Because of the demonstrated efficiency of conjugate directions of search, we shall employ them here. The algorithm is as follows:

(i) Select $\underline{u}^o \in V$ (Initial guess)

(ii) Evaluate $\underline{G}(\underline{u}^o)$ via (17). If $G(\underline{u}^o) = 0$, then by (32), $\underline{u}^o$ is the solution. If $\underline{G}(\underline{u}^o) \neq 0$, then for the (i+1)st iteration, (i=0,1... ,) proceed as follows:

(iii) $\underline{u}^{i+1} = \underline{u}^i + \alpha^i \underline{s}^i \; ; \; \underline{s}^o = -\underline{G}(\underline{u}^o)$.

$$\underline{s}^{i+1} = -\underline{G}(\underline{u}^{i+1}) + \beta^i \underline{s}^i$$

$$\beta^i = \frac{(\underline{G}(\underline{u}^{i+1}) \, , \, \underline{G}(\underline{u}^{i+1}))_V}{(\underline{G}(\underline{u}^i) \, , \, \underline{G}(\underline{u}^i))_V}$$

in addition, α^i is chosen so that

$$J(\underline{u}^{i+1}) = \operatorname*{Inf}_{\gamma^i \in R^1} J(\underline{u}^i + \gamma^i \underline{s}^i)$$

It is possible to obtain an explicit expression for α^i :

$$\alpha^i = -\frac{a(\underline{s}^i,\underline{u}^i) - \ell(\underline{s}^i)}{a(\underline{s}^i,\underline{s}^i)} = \frac{(\underline{G}(\underline{u}^i),\underline{G}(\underline{u}^i))_V}{a(\underline{s}^i,\underline{s}^i)} \; .$$

We review some properties of the algorithm in the following theorems:

Theorem 3 If $\underline{G}(\underline{u}^i) \neq 0$, $J(\underline{u}^{i+1}) < J(\underline{u}^i)$

Corollary: The sequence of real numbers $J(\underline{u}^i)$ is monotone decreasing and has a limit in the extended reals:

$$\lim_{i \to \infty} J(\underline{u}^i) = J_\infty = \operatorname*{Inf}_{\underline{v} \in V} J(\underline{v}) \; .$$

Theorem 4 The sequence $(\underline{u}^i)$ converges weakly to a unique $\underline{u} \in V$ and the limit $\underline{u}$ has the property that

$$J(\underline{u}) = \operatorname*{Inf}_{\underline{v} \in V} J(\underline{v})$$

that is,

$$\lim_{i \to \infty} \underline{u}^i \xrightarrow{\text{weakly}} \underline{u} \in V \quad \text{(unique)}$$

$$\text{and} \quad J(\underline{u}) = J_\infty \; .$$

We remark that at each iteration, the evaluation of $\underline{G}(\underline{u}^i)$ involves the numerical solution of (2) forwards in time, then (16) backwards in time, which are the (numerically) stable directions of solution.

As before, the numerical solution requires an approximation of the solutions to a set of partial differential equations. Again, we use the eigenfunctions of (1) to achieve this approximation. We discuss the results in the next section.

G. Numerical Results

The system chosen was the following:

$$\frac{\partial y(x,t)}{\partial t} - \frac{\partial^2 y(x,t)}{\partial x^2} = 212 \qquad (x,t) \in (0,1)x(0,T] \qquad (34)$$

$$y(o,t) = u_1(o,t) \qquad t \in (0,T]$$

$$y(1,t) = u_1(1,t) \qquad t \in (0,T]$$

$$y(x,o) = u_2(x) \qquad x \in (0,1)$$

Input Measurements

$$z_1(o,t) = u_1^*(o,t) + k_1\, N_1(t)\ ; \quad u_1^*(o,t) = 70 + 10 \sin 2\ \pi t$$

$$z_1(1,t) = u_1^*(1,t) + k_2\, N_2(t)\ ; \quad u_2^*(1,t) = 54.5$$

$$z_2(x) \ = u_2^*(x) \ + k_3 \qquad ; \quad u_2^* \ = 70\ e^{-0.25x}$$

$$k_1 = 4.2,\ k_2 = 0,\ k_3 = 2.8$$

$N_1(t)$ purely random function with amplitude ± 1.0.

Output Measurements

(a) $z(x,t) = y(x,t;\underline{u}^*) + k_o\, N_o(t)$

(b) $z(x^i,t) = y(x^i,t;\underline{u}^*) + k_o^i\, N_o^i(t)$; $(x^i = 0.2,\ 0.4,\ 0.6,\ 0.8)$.

$k_o^i = 8.0$, N_o, N_o^i are random telegraph signals with amplitude ± 1.0.

The eigenfunctions appropriate to the suggested approximations are those given by $(1)^*$.

The results obtained for our example using the method of Section F.1 are shown in Figures 1, 2 and 3. An eight term expansion was adopted, and several selected variables are shown. It should be stated that the integration step size necessary to obtain a numerically stable solution for the $\{P_{ij}(t)\}$ was small (0.001) and this resulted in a large computational effort. The total time for solution was of the order of three minutes. On the other hand, using the direct method of Section F.2, three iterations, (sufficient to recover $\underline{u}$ such that $G(\underline{u}) \doteq 0$) were accomplished in only 75 seconds. Some selected results are shown in

Figures 4, 5, 6, 7, and 8. Note that $J(\underline{v})$ is minimized rapidly (Figure 9) and observe that $J(\underline{v})_{\text{measurement (a)}} < J(\underline{v})_{\text{measurement (b)}}$.

We remark that $G(\underline{u}^i)$ given by (17) was approximated by $G_m(\underline{u}^i)$:

$$G_m(\underline{u}^i) = - \sum_{i=1}^{4} p_i(t) \frac{\partial w_i(s)}{\partial \nu^*} + u_1^i(s,t) - z_1(s,t)$$

$$\sum_{i=1}^{4} p_i(o)\, w_i(x) + u_2^i(x) - z_2(x) \quad .$$

We observed that for the example chosen, the last three terms in the summation were identically zero, that is,

$$G_m(\underline{u}) = G(\underline{u})$$

However in general, it is not clear in what sense $G_m(\underline{u}) \to G(\underline{u})$, and we are attempting to establish an appropriate result.

H. Summary and Conclusions

A special variational phrasing of a distributed identification problem resulted in a framework in which solutions were characterized using the theory of variational inequalities. Numerical techniques were suggested for recovering extremals to the variational problem, one of which, the direct method, yielded promising results. This direct method is also applicable to the problem of determining optimal boundary controls for certain distributed optimal control problems.

References

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2. Balakrishnan, A.V., Lions, J.L., "State Estimation for Infinite-Dimensional Systems", J. Comp. and System Sciences, Vol. 1, (1967) pp. 391-403.
3. Phillipson, G.A., "The State Identification of a Class of Distributed Systems", Ph.D. Thesis, Case Western Reserve University, June 1967.
4. Lions, J.L., Magenes, E., "Remarques sur les Problemes aux Limites pour Operateurs Paraboliques", C.R. Acad. Sc., Paris, Vol. 25, (1960) pp. 2118-2120.

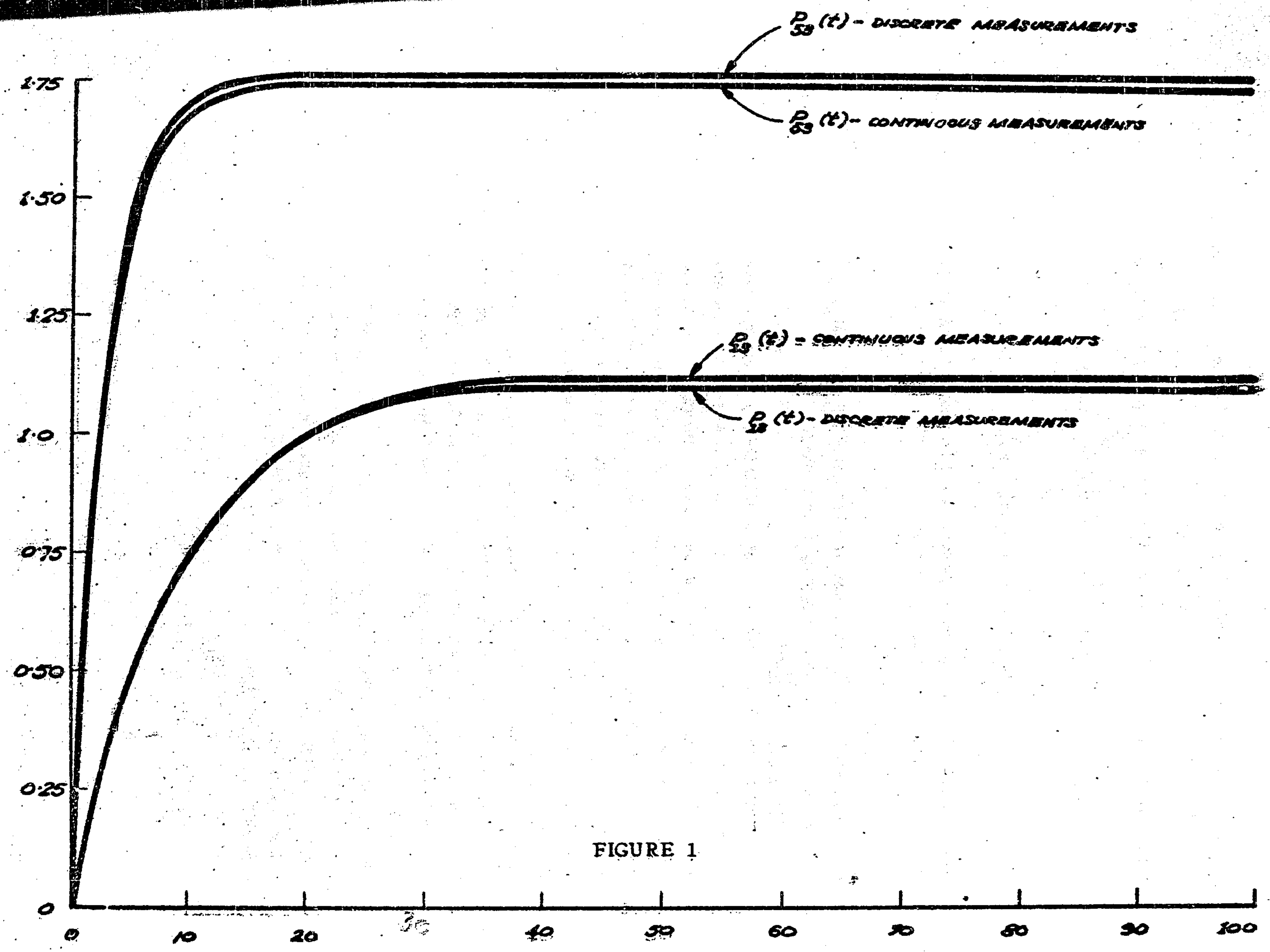
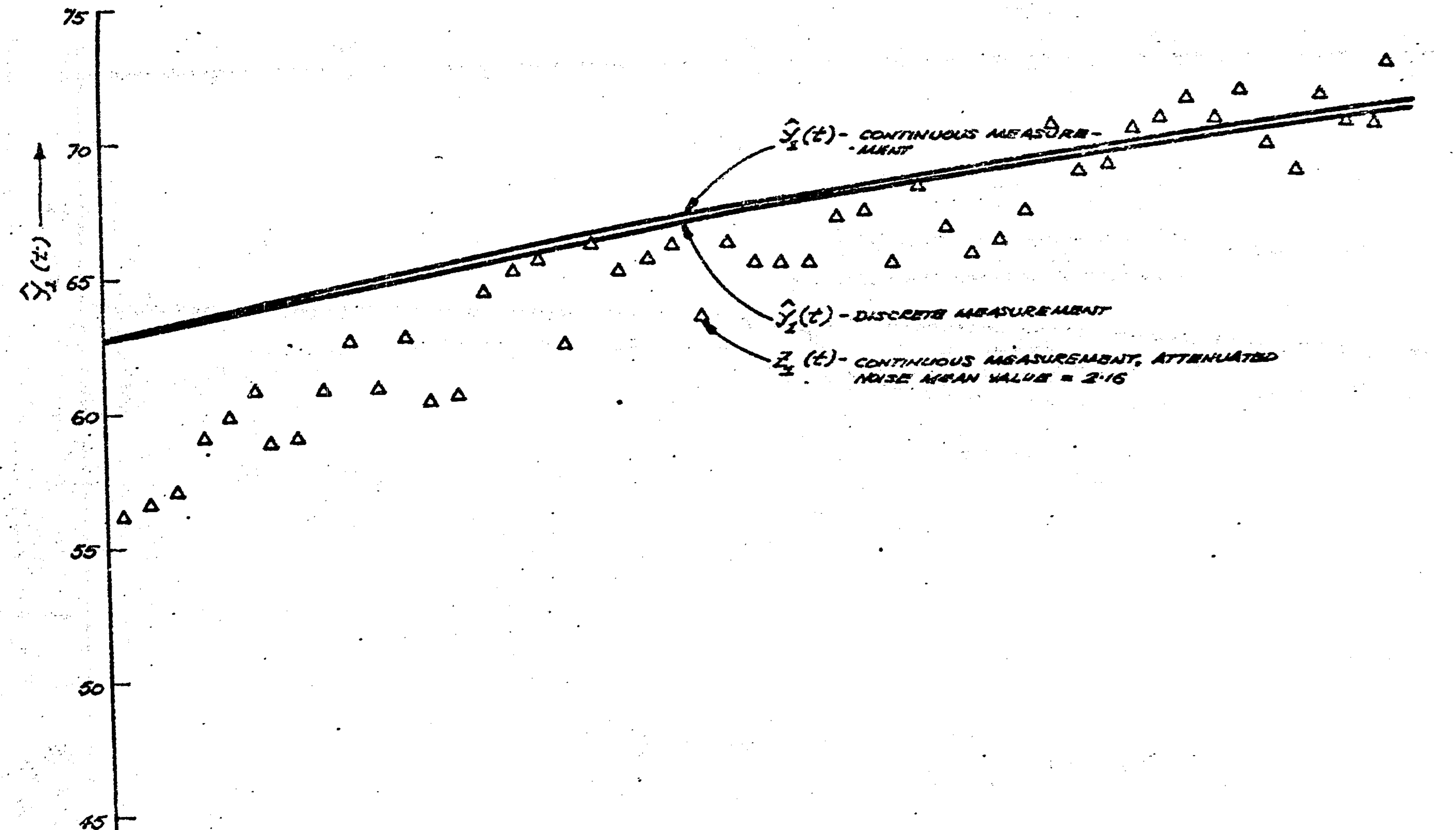

P(t) - DISCRETE MEASUREMENTS
P(t) - CONTINUOUS MEASUREMENTS
P(t) = CONTINUOUS MEASUREMENTS
P(t) - DISCRETE MEASUREMENTS
1·75
1·50
1·25
1·0
0·75
0·50
0·25
0
0
10
20
40
50
60
70
80
90
100

FIGURE 1



$\hat{y}_1(t)$ - CONTINUOUS MEASURE-MENT
$\hat{y}_1(t)$ - DISCRETE MEASUREMENT
$z_1(t)$ - CONTINUOUS MEASUREMENT, ATTENUATED NOISE MEAN VALUE = 2·16
$\hat{y}_1(t)$
t
0.01

FIGURE 2

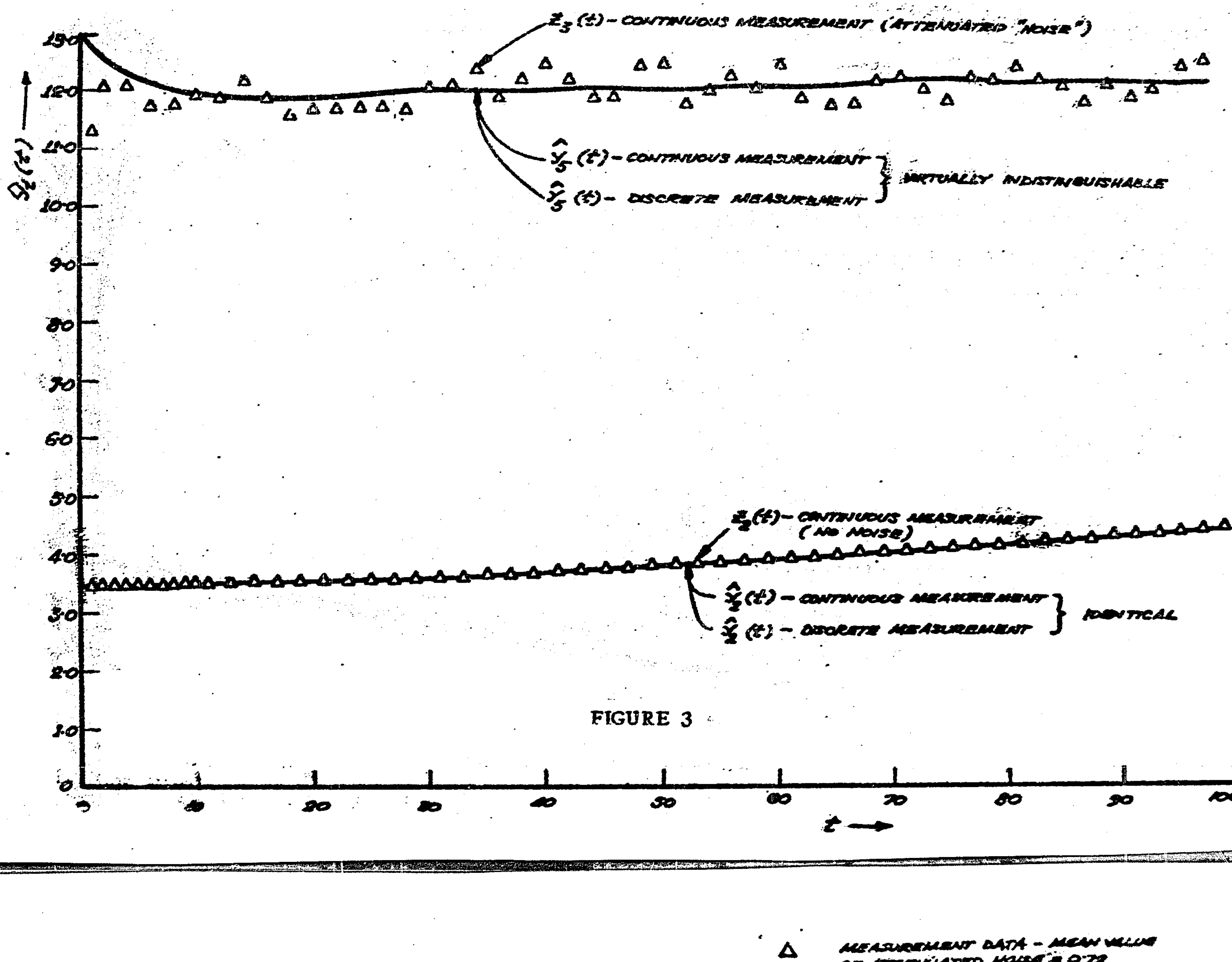

$z_3(t)$ - CONTINUOUS MEASUREMENT (ATTENUATED "NOISE")
$\hat{y}_5(t)$ - CONTINUOUS MEASUREMENT
$\hat{y}_5(t)$ - DISCRETE MEASUREMENT
VIRTUALLY INDISTINGUISHABLE
$z_2(t)$ - CONTINUOUS MEASUREMENT (NO NOISE)
$\hat{x}_2(t)$ - CONTINUOUS MEASUREMENT
$\hat{x}_2(t)$ - DISCRETE MEASUREMENT
IDENTICAL
t
0
10
20
30
40
50
60
70
80
90
100
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10.0
11.0
12.0
13.0
Δ MEASUREMENT DATA - MEAN VALUE OF ATTENUATED NOISE = 0.72

FIGURE 3

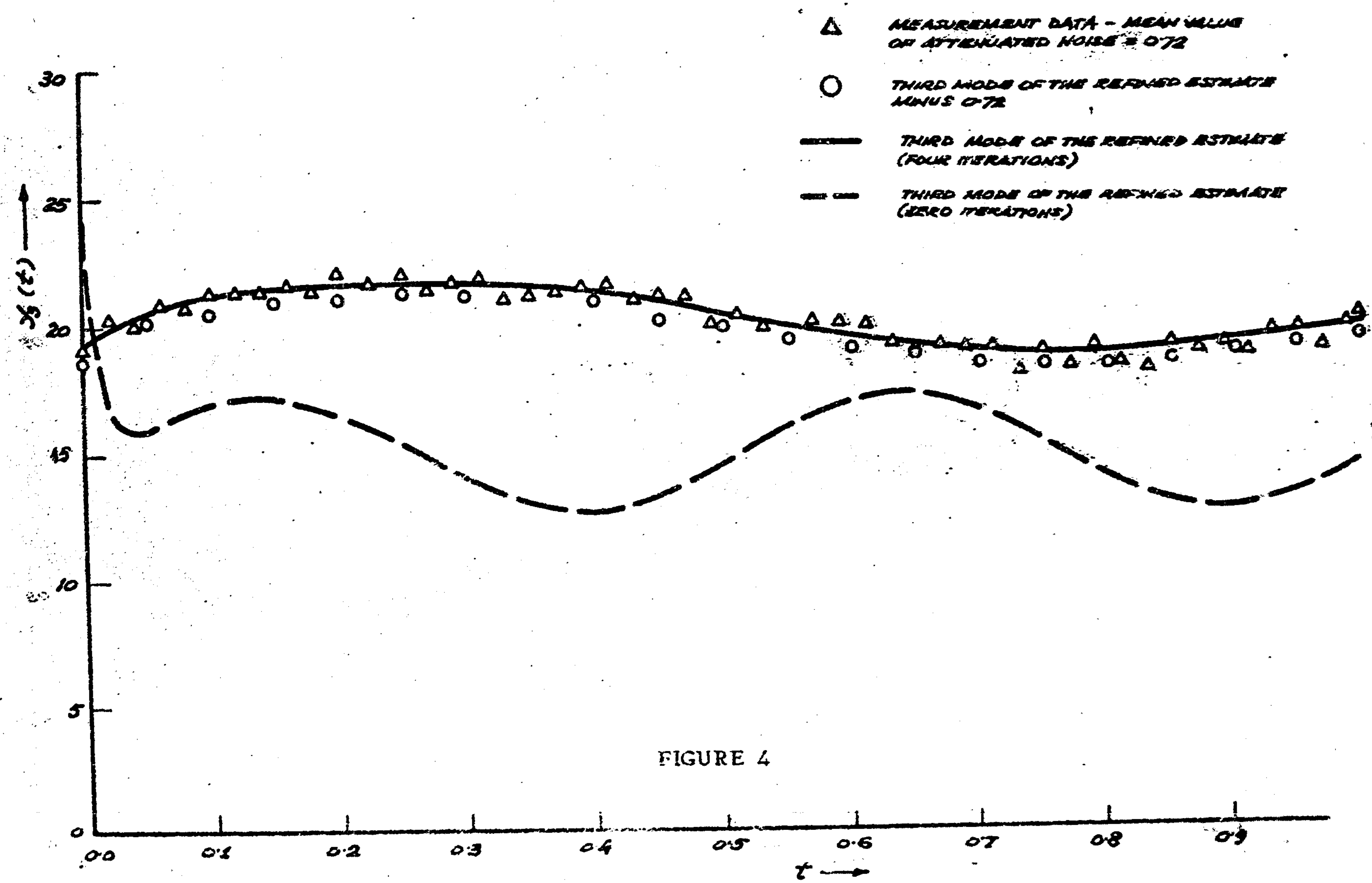
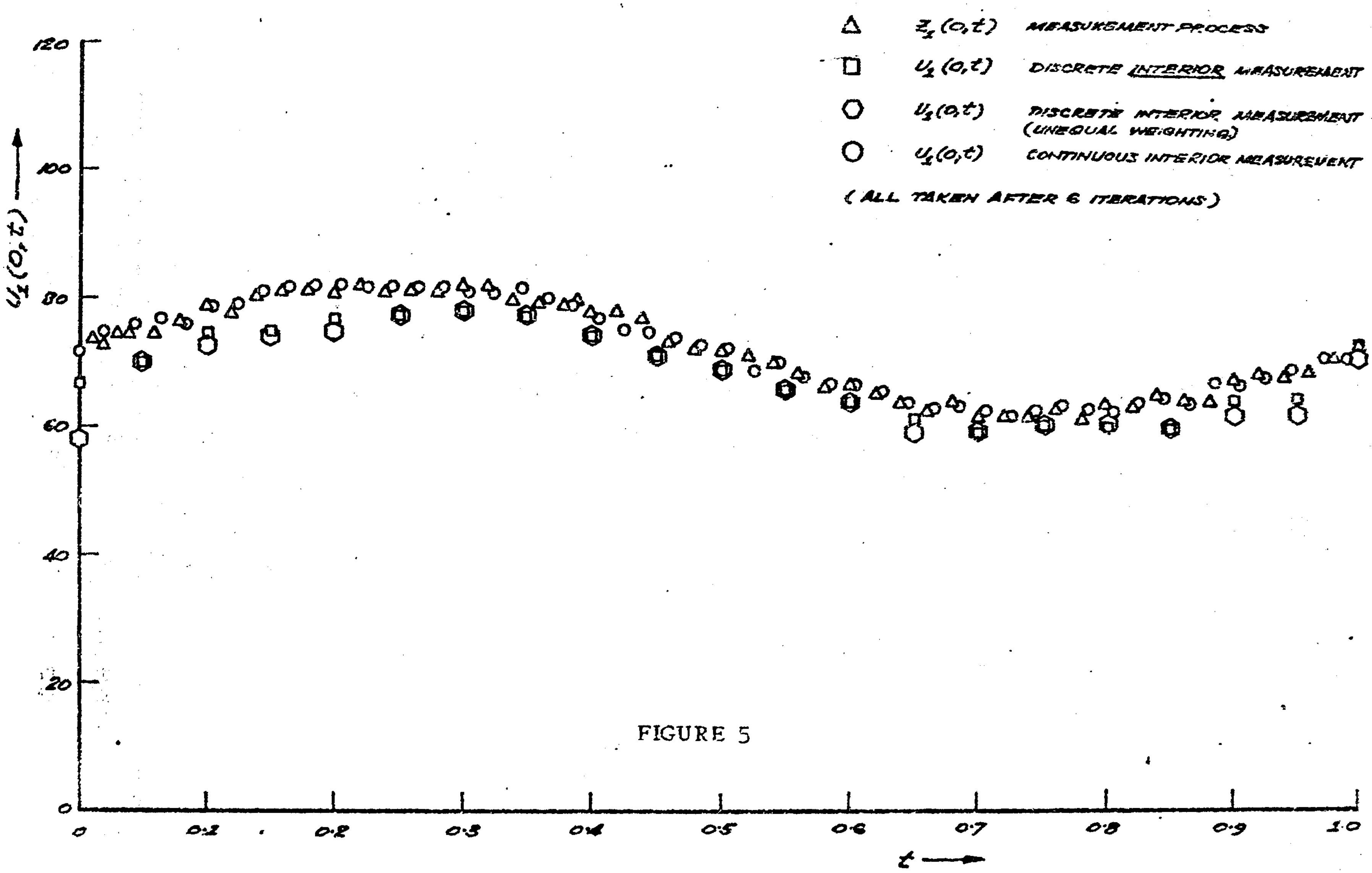
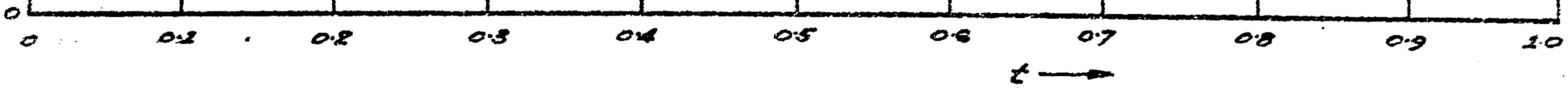

MEASUREMENT DATA - MEAN VALUE OF ATTENUATED NOISE = 0·72
THIRD MODE OF THE REFINED ESTIMATE MINUS 0·72
THIRD MODE OF THE REFINED ESTIMATE (FOUR ITERATIONS)
THIRD MODE OF THE REFINED ESTIMATE (ZERO ITERATIONS)
0
5
10
15
20
25
30
0·0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
t →

FIGURE 4



$U_1(0,t)$
120
100
80
60
40
20
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
t
$Z_1(0,t)$ MEASUREMENT PROCESS
$U_1(0,t)$ DISCRETE INTERIOR MEASUREMENT
$U_1(0,t)$ DISCRETE INTERIOR MEASUREMENT (UNEQUAL WEIGHTING)
$U_1(0,t)$ CONTINUOUS INTERIOR MEASUREMENT
(ALL TAKEN AFTER 6 ITERATIONS)

FIGURE 5


0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
t

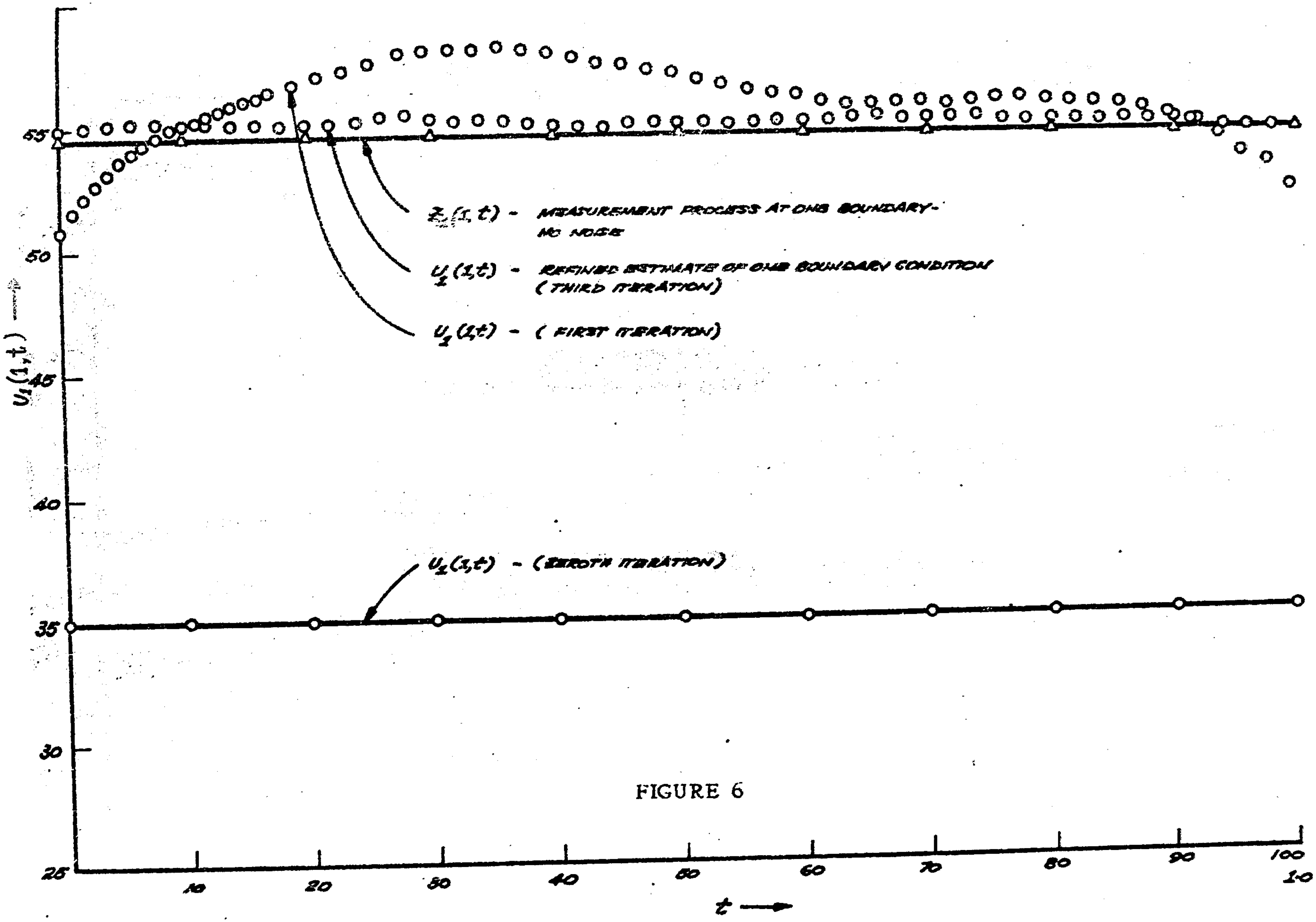

$U_1(1,t)$
55
50
45
40
35
30
25
$Z_1(1,t)$ - MEASUREMENT PROCESS AT ONE BOUNDARY - NO NOISE
$U_1(1,t)$ - REFINED ESTIMATE OF ONE BOUNDARY CONDITION (THIRD ITERATION)
$U_1(1,t)$ - (FIRST ITERATION)
$U_1(1,t)$ - (ZEROTH ITERATION)
10
20
30
40
50
60
70
80
90
100
1.0
t

FIGURE 6

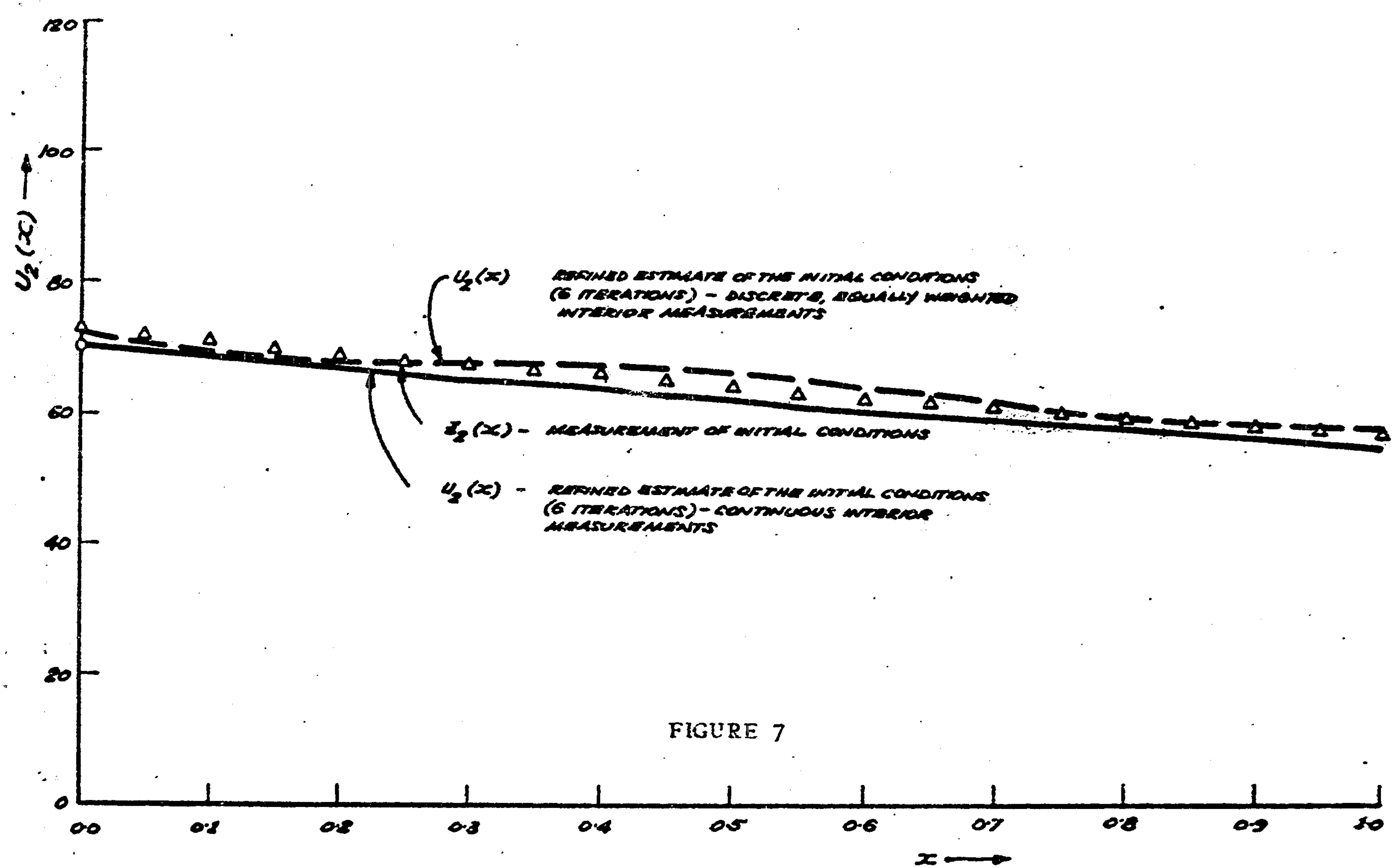


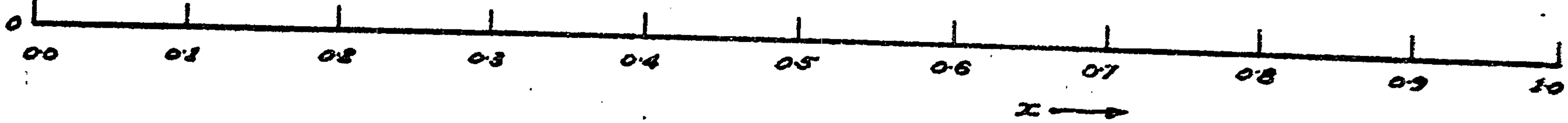
FIGURE 7

{ $G_1(0,t)$ - GRADIENT OF THE FUNCTIONAL $J(U)$ WITH RESPECT TO $U_1(0,t)$ (DISCRETE INTERIOR MEASUREMENTS) }

FIRST ITERATION

THIRD ITERATION

ZEROTH ITERATION


0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
x

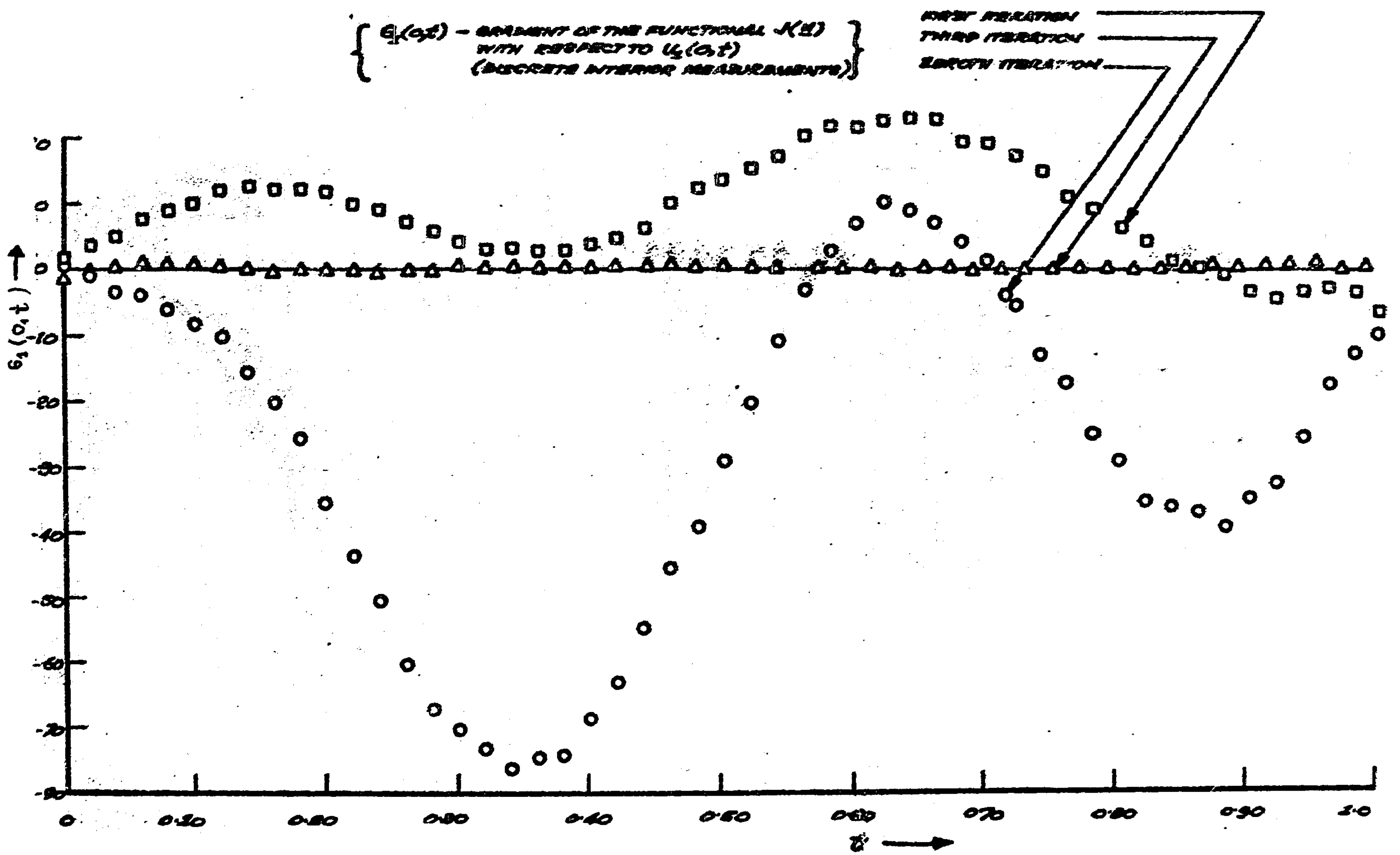
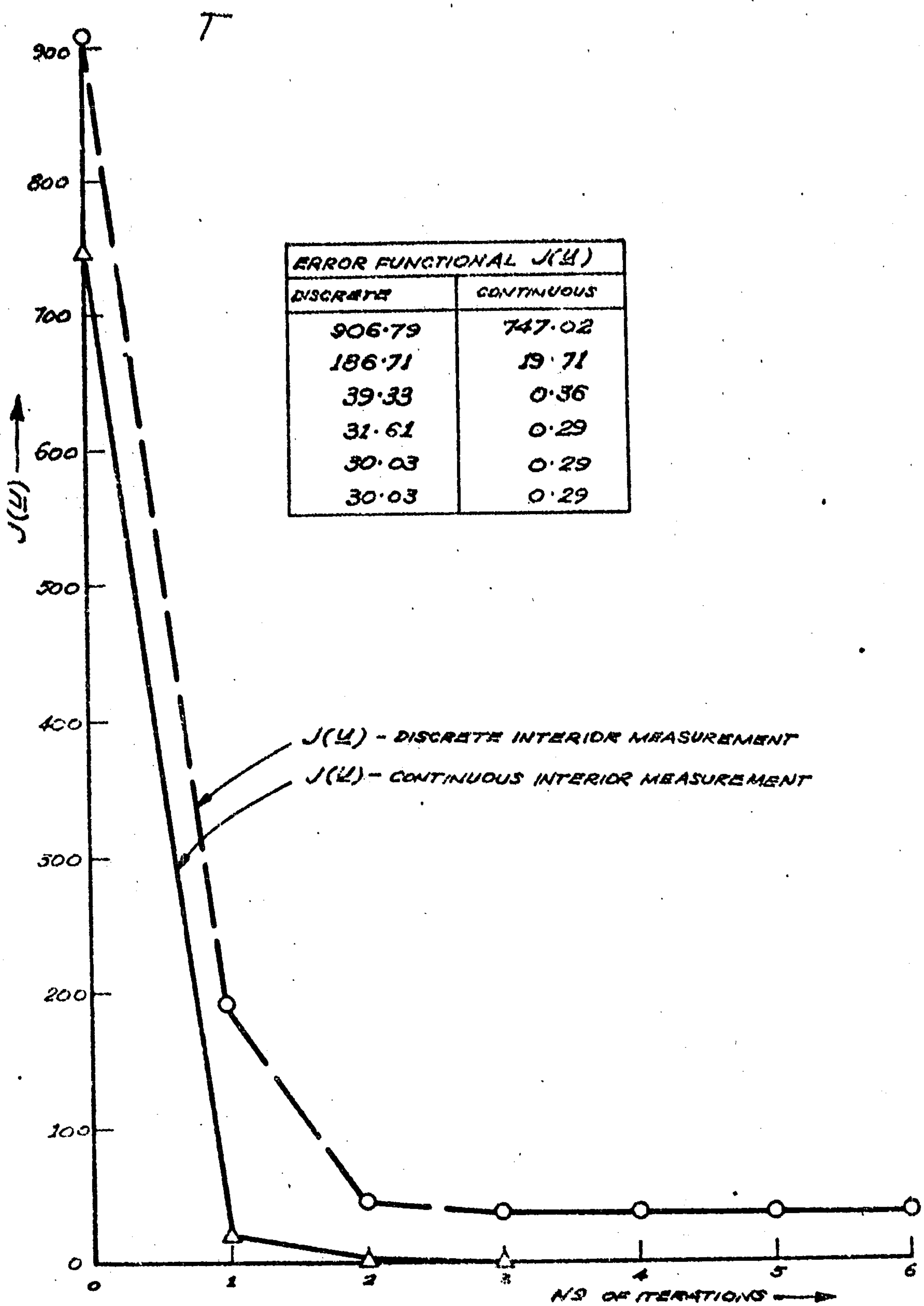

$G_1(0,t)$ – GRADIENT OF THE FUNCTIONAL $J(\Xi)$ WITH RESPECT TO $U_2(0,t)$ (DISCRETE INTERIOR MEASUREMENTS)
FIRST ITERATION
THIRD ITERATION
ZEROTH ITERATION
$G_1(0,t)$
t

FIGURE 8



ERROR FUNCTIONAL J(U)	
DISCRETE	CONTINUOUS
906·79	747·02
186·71	19·71
39·33	0·36
31·61	0·29
30·03	0·29
30·03	0·29

FIGURE 9