1.017/1.010 Class 6
Conditional Probability and Bayes Theorem

Conditional Probability

If two events A and B are not independent we can gain information about $P(A)$ if we know that an event in B has occurred. This is reflected in **conditional probability** of A given B, written as $P(A|B)$:

$$P(A \mid B) = \frac{P(AB)}{P(B)}$$

The **unconditional** probability $P(A)$ is often called the ***a priori*** probability while the **conditional** probability $P(A|B)$ is often called the ***a posteriori*** probability. Note that conditioning may take place in either direction:

$$P(AB) = P(A|B)P(B) = P(B|A)P(A)$$

Conditional probabilities are valid probability measures that satisfy all the fundamental axioms.

If A and B are independent:

$$P(A|B) = P(A)$$

Example:

A = {Algae bloom occurs }
B = {Daily average water temperature above 25 deg. C)

Obtain probabilities from long record of daily algae and temperature observations:

Suppose $P(A) = 0.01, \quad P(B) = 0.15, \quad P(A, B) = 0.005$
Then:

$$P(A \mid B) = \frac{P(AB)}{P(B)} = \frac{0.005}{0.15} = 0.033$$

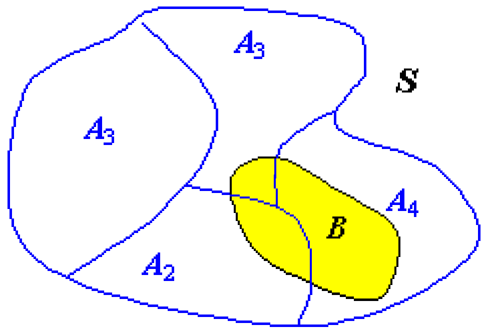
Probability of a bloom increases significantly if we know that temperature is above 25 deg. C.

Bayes Theorem

Suppose that the sample space S is divided into a collection of n mutually exclusive events (sets) called a **partition** of S:

$$S = \{A_1, A_2, A_3, ..., A_n\}$$
$$A_i A_j = 0 \quad i \neq j$$

Consider an arbitrary event B in S, as indicated in the diagram below:



The event B can be written as the union of the n disjoint (mutually exclusive) events $BA_1, BA_2, ..., BA_n$:

$$B = BA_1 + BA_2 + ... + BA_n$$

This implies **total probability theorem**:

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n)$$

The total probability theorem and the definition of the conditional probability may be used to derive **Bayes theorem**:

$$P(A_i \mid B) = \frac{P(B \mid A_i)P(A_i)}{P(B)} = \frac{P(B \mid A_i)P(A_i)}{P(B \mid A_1)P(A_1)(B) + ... + P(B \mid A_n)P(A_n)}$$

Bayes rule updates $P(A_i)$, given information on the probabilities of obtaining B when outcomes are $A_1, A_2, ..., A_n$.

Example:

Consider a group of 10 water samples. Exactly 3 are contaminated. Define following events:

Event	Definition
C	Sample contaminated
C'	Sample not contaminated

D	Contamination detected
D'	Contamination not detected

$P(C)$ = 0.3 (based on 3 out of 10 samples contaminated)

Suppose sample analysis technique is imperfect. Based on calibration tests:

$P(D|C) = 0.9$ Successful detection
$P(D|C') = 0.4$ False alarm

Bayes theorem (replace A_1 with C, A_2 with C', B with D):

$$P(C \mid D) = \frac{P(D \mid C)P(C)}{P(D \mid C)P(C) + P(D \mid C')P(C')} = \frac{(0.9)(0.3)}{(0.9)(0.3) + (0.4)(0.7)} = 0.5$$