Exercise 1: Computing joint probabilities and correlations of a time series

Suppose that the total streamflows (in \( m^3/\text{month} \)) \( Q_t \) and \( Q_{t+1} \) over two successive months \( t \) and \( t+1 \) are related by the following equation:

\[
Q_{t+1} = 0.6Q_t + e_t \quad t = 1, \ldots, 10
\]

In deterministic applications such time series expressions are solved one step at a time, starting with a specified value of the initial value of the dependent (in this case, the initial streamflow \( Q_1 \)) and using a specified series of \( e_t \) values (for \( t = 10 \)). The result is a series of \( Q_t \) values for \( t > 1 \).

When \( Q_1 \) and the \( e_t \) values are uncertain, we can use the time series equation to derive the mean, variance, and covariance of the streamflow for \( t > 1 \). These give a probabilistic description of the temporally variable streamflow.

Suppose that the random excitation values \( e_1, \ldots, e_{10} \) are independent normally distributed random variables, each with mean 4.0 and variance 0.64 (standard deviation 0.8). In addition, suppose that the initial streamflow \( Q_1 \) is normally distributed with mean 10.0 and variance 1.0 (standard deviation 1.0). Assume that \( Q_1 \) and the \( e_t \) values are independent.

Use the definitions of variance and correlation to evaluate the correlation between the two successive streamflows \( Q_2 \) and \( Q_3 \). First use the defining time series difference equation and the properties of the mean and variance of a linear function to find the means and variances of \( Q_2 \) and \( Q_3 \). Then use the same equation to derive an expression for the covariance and correlation between \( Q_2 \) and \( Q_3 \). Are \( Q_2 \) and \( Q_3 \) independent? Why? What do you think you would get for the means and variances of \( Q_9 \) and \( Q_{10} \)? Why?

Use MATLAB to determine the joint probability that \( Q_2 \) and \( Q_3 \) are both greater than 12.0. Also, plot a typical replicate of streamflow vs. time.

Some relevant MATLAB functions: `normrnd`, `sum`

Exercise 2: The Central Limit Theorem

The Central Limit Theorem states that the distribution of a linear function of independent random variables approaches the normal distribution as the number of variables increases. Use MATLAB to demonstrate this by carrying out the following virtual experiment:
Generate a random array of 1000 replicates (rows) each consisting of 32 independent identically distributed sample values (columns), using one of the non-normal random number generators provided in the MATLAB statistics toolbox (e.g. exponential, uniform, lognormal, etc.). Select your own values for all required distributional parameters. Plot histograms and CDFs (over all replicates) for the sample mean:

$$m_x = \frac{1}{N} \sum_{i=1}^{N} x_i$$

For $N = 1, 4, 16, \text{and } 64$. Plot all 4 CDFs on the same axis using the MATLAB normplot function, which displays normally distributed data as a straight line. Your CDFs should more closely approach a straight line as $N$ increases, indicating convergence to a normal probability distribution. The approach to normality should also be apparent in your 4 histograms (try plotting all four on one page, using the MATLAB subplot function)

Some relevant MATLAB functions: exprnd, chi2rnd, lognrnd, hist, normplot, mean, subplot.

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