

Review of Discrete-Time Signal Processing

Ying-Jui Chen
Massachusetts Institute of Technology
yrchen@mit.edu

March 1, 2002

Abstract

Discrete-time signals and effects of linear systems, or filters, (probably time-varying) on them. Frequency-domain representations of signals and systems. Multirate systems.

Discrete-Time Signals

- A quantity $x[n]$ defined only for $n \in \mathbb{Z}$.
- Can be connected to *continuous-time* signals through sampling.
- Decomposition in terms of shifted unit impulses:

$$x[n] = \underbrace{\sum_{k=-\infty}^{\infty} x[k]}_{\text{function of } n} \underbrace{\delta[n-k]}_{\text{func. of } n}$$

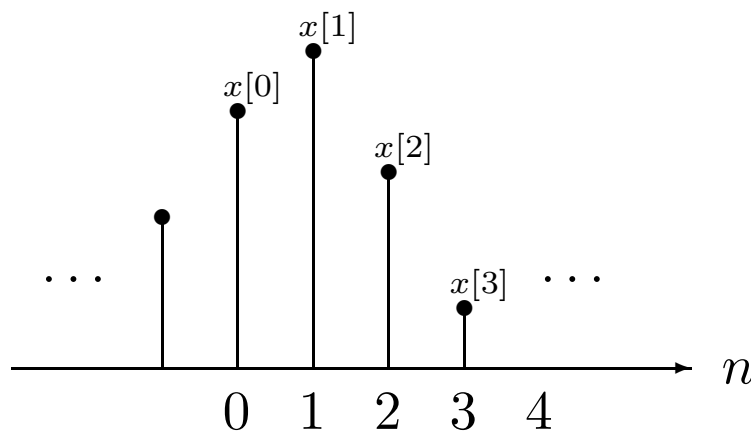


Figure 1: A discrete-time signal $x[n]$.

Linear Systems

Assume T is a linear system, or filter:

$$T\{x[\cdot]\}[n] = \sum_{k=-\infty}^{\infty} x[k] T\{\delta[\cdot - k]\}[n].$$

$T\{\delta[\cdot - k]\}[n]$ is the system response to a unit impulse located at time $n = k$. Call

$$h_k[n] \triangleq T\{\delta[\cdot - k]\}[n]$$

the k -th impulse response of T .

If T does not vary with time (time-invariant),

$$h_k[n] = h_0[n - k] \triangleq h[n - k] \quad (1)$$

\Downarrow

$$T\{x[\cdot]\}[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k] \quad (2)$$

$$\triangleq \underbrace{(x * h)[n]}_{\text{convolution}} \quad \text{LTI filter only} \quad (3)$$

LTI Systems

- Finite-impulse-response (FIR) systems
- Infinite-impulse-response (IIR) systems
- output $y[n] =$ convolution between input $x[n]$ and impulse response $h[n]$.
 - If we associate the signal $x[n]$ with a polynomial $P_x(z)$ where the coefficient of z^n is $x[n]$, i.e.,

$$P_x(z) = \sum_n x[n]z^n.$$

Similarly for $P_h(z)$ and $P_y(z)$.

- The convolution formula suggests that $y[n]$ can be read off from the product (also polynomial) of $P_h(z)$ and $P_x(z)$:

$$P_h(z)P_x(z) = \sum_n y[n]z^n.$$

- Causality, Stability, etc.

Frequency-Domain Representation of Discrete-Time Signals

- To expand $x[n]$ in terms of sinusoids $e^{j\omega n}$ of different frequencies ω . $j = \sqrt{-1}$.
- If $x[n]$ is periodic with period N ,

$$x[n] = \sum_{k=0}^{N-1} C_k \underbrace{e^{j\omega_k n}}_{\text{func of } n}, \quad \omega_k = \frac{2\pi k}{N}.$$

C_k : strength of k -th frequency component in $x[n]$, forming the line spectrum of $x[n]$.

- If $x[n]$ is aperiodic,

$$x[n] = \int_{2\pi} C(\omega) e^{j\omega n} d\omega.$$

$C(\omega)$: continuous spectrum of $x[n]$.

- Fourier Transform: the process of finding the frequency content/spectrum of a signal $x[n]$:

Periodic

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}, \quad \text{DFT}$$

Aperiodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad \text{DTFT}$$

- Inverse Fourier Transform: the process of reconstructing a signal $x[n]$ from its frequency content/spectrum:

Periodic

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi nk}{N}}$$

Aperiodic

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

- The spectrum is periodic. Notion of “highest frequency” in discrete-time case.

- Properties of DTFT:
 - 1.
 - 2.
 - 3.
 - 4.
 - 5.

z -Transform

- Extension of DTFT. Definition:

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n]z^{-n}$$

- DTFT is z -transform evaluated at

$$z = e^{j\omega}, \quad 0 \leq \omega \leq 2\pi,$$

the unit circle on the z -plane. Again, the DTFT spectrum is 2π -periodic.

- Properties of z -transform:

1. $x[n - \ell] \longleftrightarrow z^{-\ell} X(z)$
2. $x[-n] \longleftrightarrow X(z^{-1})$
3. $z_0^n x[n] \longleftrightarrow X(z/z_0)$, $(-1)^n x[n] \longleftrightarrow X(-z)$
4. $(\uparrow M)x[n] \longleftrightarrow X(z^M)$
5. $(\downarrow M)x[n] \longleftrightarrow \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} e^{j\frac{2\pi k}{M}}\right)$

- Pole-zero plot (linear- ϕ , min- ϕ , real $h[n]$, etc.)
If $h[n]$ is real, then the zeros and/or poles of

$H(z)$ appear in complex-conjugate pairs (z_0, z_0^*) .
Furthermore, if $h[n]$ is also linear-phase,

$$H(z^{-1}) = H(z) \text{ or } z^{-N+1}H(z^{-1}) = H(z)$$

they come in a pair of four: $(z_0, z_0^*, z_0^{-1}, (z_0^*)^{-1})$.

- Examples:

1. $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |a/z| < 1$$

2. $x[n] = -a^n u[-n - 1]$

$$\begin{aligned} X(z) &= - \sum_{n=-\infty}^{-1} (az^{-1})^n = - \sum_{m=1}^{\infty} (a^{-1}z)^m \\ &= \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}, \quad |z/a| < 1 \end{aligned}$$

$$3. \quad y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = 5x[n] + 2x[n-1] - \frac{1}{3}x[n-2]$$

$$\begin{aligned} H(z) &\triangleq \frac{Y(z)}{X(z)} = \frac{5 + 2z^{-1} - \frac{1}{3}z^{-2}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} \\ &= 2 - \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{4}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

Can find $h[n]$ based on possible ROC's.

Halfband Condition

- Assume centered z -transform $H(z)$.
- $H(z) + H(-z) = 2$
- $h[2n] = \delta[n]$, $h[2n + 1]$ arbitrary.
- $\{0.5, 1, 0.5\}$
- $\{-1, 0, 9, 16, 9, 0, -1\}/16$
- $\{-1, 0, 3, 16, 5, 0, 3\}/16$
- $\{\sqrt{2}, 1, 0\}$

Polyphase Representation

- Useful when combined with multirate systems (rate change by $(\downarrow M)$ and/or $(\uparrow L)$). Noble Identities.
- $(\downarrow 2)x[n]$: keeps only even-numbered samples, $x[2n]$.
- $(\downarrow M)x[n]$: keeps only $x[Mn]$.
- Therefore, it is natural to separate a signal $x[n]$ into different “phases.”
 - If $M = 2$, we can divide $x[n]$ into
 - * even phase: $\{\dots, x[-2], x[0], x[2], x[4], \dots\}$
 - * odd phase: $\{\dots, x[-1], x[1], x[3], x[5], \dots\}$
 - For general M , there are M phases.
 - * k -th phase: $\{x[Mn + k] \mid n \in \mathbb{Z}\}$
 - * $k = 0, 1, 2, \dots, M - 1$.
- In general,

$$X(z) = \sum_{k=0}^{M-1} z^{-k} X_k(z^M),$$
 where $X_k(z)$ is the k -th phase of $X(z)$.
- $(\downarrow M)x[n]$: only the 0-th phase, $\{x[Mn]\}$, survives.

- Examples:

$$- H(z) = 3 + 2z^{-1} - 2z^{-3} + 5z^{-4} + z^{-5}$$

$$\begin{aligned} H(z) &= [3 + 5z^{-4}] + z^{-1}[2 - 2z^{-2} + z^{-4}] \\ &= H_{\text{even}}(z^2) + z^{-1}H_{\text{odd}}(z^2). \end{aligned}$$

$$\therefore H_{\text{even}}(z) = 3 + 5z^{-2}$$

$$H_{\text{odd}}(z) = 2 - 2z^{-1} + z^{-2}$$

$$- H(z) = \frac{b}{1-az^{-1}}$$

$$\begin{aligned} H(z) &= \frac{b(1 + az^{-1})}{(1 - az^{-1})(1 + az^{-1})} \\ &= \frac{b + abz^{-1}}{1 - a^2z^{-2}} \end{aligned}$$

$$\therefore H_{\text{even}}(z) = \frac{b}{1 - a^2z^{-1}}$$

$$H_{\text{odd}}(z) = \frac{ab}{1 - a^2z^{-1}}$$

Polyphase matrix for filter bank

Type 1 (analysis bank)

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{0,even}(z^2) & H_{0,odd}(z^2) \\ H_{1,even}(z^2) & H_{1,odd}(z^2) \end{bmatrix}}_{H_p(z^2)} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

Type 2 (synthesis bank) **[corrected]**

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}^T = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T \underbrace{\begin{bmatrix} F_{0,odd}(z^2) & F_{1,odd}(z^2) \\ F_{0,even}(z^2) & F_{1,even}(z^2) \end{bmatrix}}_{F_p(z^2)}$$

Noble Identities & Filterbanks

Analysis bank:

$$\begin{aligned} \left((\downarrow 2) \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} \right)_{2 \times 1} &= (\downarrow 2) H_p(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \\ &= H_p(z) (\downarrow 2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} \end{aligned}$$

$$\begin{aligned} &(\downarrow 2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} X(z) \\ &= (\downarrow 2) \begin{bmatrix} X_{even}(z^2) + z^{-1} X_{odd}(z^2) \\ z^{-1} X_{even}(z^2) + z^{-2} X_{odd}(z^2) \end{bmatrix} \\ &= \begin{bmatrix} X_{even}(z) \\ z^{-1} X_{odd}(z) \end{bmatrix} \end{aligned}$$

$$\left((\downarrow 2) \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} \right) X(z) = H_p(z) \begin{bmatrix} X_{even}(z) \\ z^{-1} X_{odd}(z) \end{bmatrix}$$

Synthesis bank:

$$\begin{aligned} \left(\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}^T (\uparrow 2) \right)_{1 \times 2} &= \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T F_p(\mathbf{z}^2) (\uparrow 2) \\ &= \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T (\uparrow 2) F_p(\mathbf{z}) \end{aligned}$$

Overall Filterbank

$$\hat{X}(z) \neq (\text{Analysis})(\text{Synthesis})X(z) \quad \text{why?}$$

$$\begin{aligned} \hat{X}(z) &= (\text{Synthesis})(\text{Analysis})X(z) \\ &= \underbrace{\begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T (\uparrow 2) F_p(\mathbf{z})}_{\text{Synthesis}} \underbrace{H_p(\mathbf{z})(\downarrow 2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}}_{\text{Analysis}} X(z) \end{aligned}$$

Want: $F_p(\mathbf{z})H_p(\mathbf{z}) = z^{-\ell}I$. (why not $H_p(\mathbf{z})F_p(\mathbf{z})$?)