Part I: Review of Discrete-Time Signal Processing

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March 1, 2002

Abstract

Discrete-time signals and effects of linear systems, or filters, (probably time-varying) on them. Frequency-domain representations of signals and systems. Multirate systems.

Discrete-Time Signals

- A quantity $x[n]$ defined only for $n \in \mathbb{Z}$.
- Can be connected to *continuous-time* signals through sampling.
- Decomposition in terms of shifted unit impulses:

$$x[n] = \overbrace{\sum_{k=-\infty}^{\infty} x[k]\, \underbrace{\delta[n-k]}_{\text{func. of } n}}^{\text{function of } n}$$

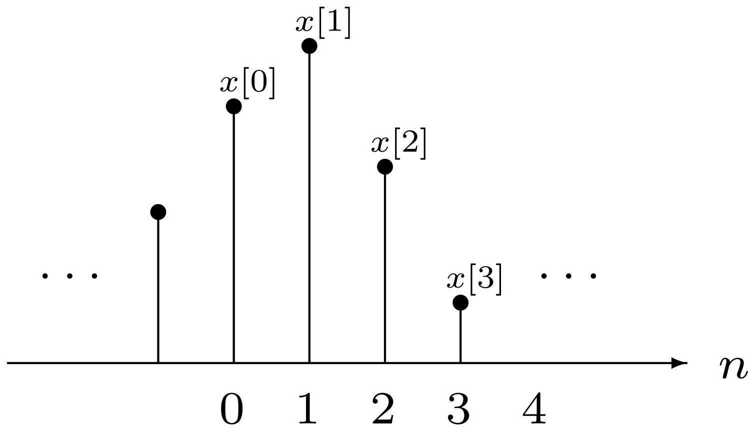

Figure 1: A discrete-time signal $x[n]$.

Linear Systems

Assume T is a linear system, or filter:

$$T\{x[\cdot]\}[n] = \sum_{k=-\infty}^{\infty} x[k]\ T\{\delta[\cdot - k]\}[n].$$

$T\{\delta[\cdot - k]\}[n]$ is the system response to a unit impulse located at time $n = k$. Call

$$h_k[n] \triangleq T\{\delta[\cdot - k]\}[n]$$

the k-th impulse response of T.

If T does not vary with time (time-invariant),

$$h_k[n] = h_0[n-k] \triangleq h[n-k] \tag{1}$$

$$\Downarrow$$

$$T\{x[\cdot]\}[n] = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k] \triangleq \underbrace{(x*h)[n]}_{\text{convolution}} \tag{2}$$

LTI Systems

- Finite-impulse-response (FIR) systems
- Infinite-impulse-response (IIR) systems
- output $y[n]$ = convolution between input $x[n]$ and inpulse response $h[n]$.
  - If we associate the signal $x[n]$ with a polynomial $P_x(z)$ where the coefficient of z^n is $x[n]$, i.e.,
  $$P_x(z) = \sum_n x[n]z^n.$$
  Similarly for $P_h(z)$ and $P_y(z)$.
  - The convolution formula suggests that $y[n]$ can be read off from the product (also polynomial) of $P_h(z)$ and $P_x(z)$:
  $$P_h(z)P_x(z) = \sum_n y[n]z^n.$$
- Causality, Stability, etc.

Frequency-Domain Representation of DT Signals

- To expand $x[n]$ in terms of sinusoids $e^{j\omega n}$ of different frequencies ω. $j = \sqrt{-1}$.
- If $x[n]$ is periodic with period N,

$$x[n] = \sum_{k=0}^{N-1} C_k \underbrace{e^{j\omega_k n}}_{\text{func of } n}, \quad \omega_k = \frac{2\pi k}{N}.$$

  C_k: strength of k-th frequency component in $x[n]$, forming the line spectrum of $x[n]$.
- If $x[n]$ is aperiodic,

$$x[n] = \int_{2\pi} C(\omega) e^{j\omega n} d\omega.$$

  $C(\omega)$: continuous spectrum of $x[n]$.
- Fourier Transform: the process of finding the frequency content/spectrum of a signal $x[n]$:
  Periodic

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi nk}{N}}, \quad \text{DFT}$$

Aperiodic

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}, \quad \text{DTFT}$$

- Inverse Fourier Transform: the process of reconstructing a signal $x[n]$ from its frequency content/spectrum:
  Periodic

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi nk}{N}}$$

Aperiodic

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(\omega) e^{j\omega n} d\omega$$

- The spectrum is periodic. Notion of "highest frequency" in discrete-time case.

z-Transform

- Extension of DTFT. Definition:

$$X(z) \triangleq \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

- DTFT is z-transform evaluated at

$$z = e^{j\omega}, \quad 0 \le \omega \le 2\pi,$$

the unit circle on the z-plane. Again, the DTFT spectrum is 2π-periodic.

- Properties of z-transform:
  1. $x[n-\ell] \longleftrightarrow z^{-\ell} X(z)$
  2. $x[-n] \longleftrightarrow X(z^{-1})$
  3. $z_0^n x[n] \longleftrightarrow X(z/z_0), \quad (-1)^n x[n] \longleftrightarrow X(-z)$
  4. $(\uparrow M) x[n] \longleftrightarrow X(z^M)$
  5. $(\downarrow M) x[n] \longleftrightarrow \frac{1}{M} \sum_{k=0}^{M-1} X\left(z^{\frac{1}{M}} e^{j\frac{2\pi k}{M}}\right)$
- Pole-zero plot (linear-ϕ, min-ϕ, real $h[n]$, etc.) If $h[n]$ is real, then the zeros and/or poles of $H(z)$ appear in complex-conjugate pairs (z_0, z_0^*). Furthermore, if $h[n]$ is

also linear-phase,

$$H(z^{-1}) = H(z) \text{ or } z^{-N+1}H(z^{-1}) = H(z)$$

they come in a pair of four: $(z_0, z_0^*, z_0^{-1}, (z_0^*)^{-1})$.

- Examples:
  1. $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}}, \quad |a/z| < 1$$

  2. $x[n] = -a^n u[-n-1]$

$$\begin{aligned} X(z) &= -\sum_{n=-\infty}^{-1} (az^{-1})^n = -\sum_{m=1}^{\infty} (a^{-1}z)^m \\ &= \frac{-a^{-1}z}{1 - a^{-1}z} = \frac{1}{1 - az^{-1}}, \quad |z/a| < 1 \end{aligned}$$

3. $y[n] + \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = 5x[n] + 2x[n-1] - \frac{1}{3}x[n-2]$

$$\begin{aligned} H(z) &\triangleq \frac{Y(z)}{X(z)} = \frac{5 + 2z^{-1} - \frac{1}{3}z^{-2}}{1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}} \\ &= 2 - \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{4}{1 - \frac{1}{3}z^{-1}} \end{aligned}$$

Can find $h[n]$ based on possible ROC's.

Halfband Condition

- Assume centered z-tranform $H(z)$.
- $H(z) + H(-z) = 2$
- $h[2n] = \delta[n]$, $h[2n+1]$ arbitrary.
- {0.5, 1, 0.5}
- {-1, 0, 9, 16, 9, 0, -1}/16
- {-1, 0, 3, 16, 5, 0, 3}/16
- {$\sqrt{2}$, 1, 0}

Polyphase Representation

- Useful when combined with multirate systems (rate change by $(\downarrow M)$ and/or $(\uparrow L)$). Noble Identities.
- $(\downarrow 2)x[n]$: keeps only even-numbered samples, $x[2n]$.
- $(\downarrow M)x[n]$: keeps only $x[Mn]$.
- Therefore, it is natural to separate a signal $x[n]$ into different "phases."
  - If $M = 2$, we can divide $x[n]$ into
    - even phase: $\{\cdots, x[-2], x[0], x[2], x[4], \cdots\}$
    - odd phase: $\{\cdots, x[-1], x[1], x[3], x[5], \cdots\}$
  - For general M, there are M phases.
    - k-th phase: $\{x[Mn+k] \mid n \in \mathbb{Z}\}$
    - $k = 0, 1, 2, \cdots, M-1$.
- In general,

  $$X(z) = \sum_{k=0}^{M-1} z^{-k} X_k(\boldsymbol{z^M}),$$

  where $X_k(z)$ is the k-th phase of $X(z)$.
- $(\downarrow M)x[n]$: only the 0-th phase, $\{x[Mn]\}$, survives.

- Examples:
  - $H(z) = 3 + 2z^{-1} - 2z^{-3} + 5z^{-4} + z^{-5}$

$$\begin{aligned} H(z) &= [3 + 5z^{-4}] + z^{-1}[2 - 2z^{-2} + z^{-4}] \\ &= H_{even}(z^2) + z^{-1}H_{odd}(z^2). \\ \therefore H_{even}(z) &= 3 + 5z^{-2} \\ H_{odd}(z) &= 2 - 2z^{-1} + z^{-2} \end{aligned}$$

  - $H(z) = \frac{b}{1-az^{-1}}$

$$\begin{aligned} H(z) &= \frac{b(1 + az^{-1})}{(1 - az^{-1})(1 + az^{-1})} \\ &= \frac{b + abz^{-1}}{1 - a^2z^{-2}} \end{aligned}$$

$$\therefore H_{even}(z) = \frac{b}{1 - a^2z^{-1}} \qquad H_{odd}(z) = \frac{ab}{1 - a^2z^{-1}}$$

Polyphase matrix for filter bank

Type 1 (analysis bank)

$$\begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} = \underbrace{\begin{bmatrix} H_{0,even}(\boldsymbol{z^2}) & H_{0,odd}(\boldsymbol{z^2}) \\ H_{1,even}(\boldsymbol{z^2}) & H_{1,odd}(\boldsymbol{z^2}) \end{bmatrix}}_{H_p(\boldsymbol{z^2})} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

Type 2 (synthesis bank) **[corrected]**

$$\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}^T = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T \underbrace{\begin{bmatrix} F_{0,odd}(\boldsymbol{z^2}) & F_{1,odd}(\boldsymbol{z^2}) \\ F_{0,even}(\boldsymbol{z^2}) & F_{1,even}(\boldsymbol{z^2}) \end{bmatrix}}_{F_p(\boldsymbol{z^2})}$$

Noble Identities & Filterbanks

Analysis bank:

$$\left((\downarrow 2) \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} \right)_{2\times 1} = (\downarrow 2) H_p(\boldsymbol{z}^{\boldsymbol{2}}) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$= H_p(\boldsymbol{z})(\downarrow 2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}$$

$$(\downarrow 2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} X(z)$$

$$= (\downarrow 2) \begin{bmatrix} X_{even}(z^2) + z^{-1} X_{odd}(z^2) \\ z^{-1} X_{even}(z^2) + z^{-2} X_{odd}(z^2) \end{bmatrix}$$

$$= \begin{bmatrix} X_{even}(z) \\ z^{-1} X_{odd}(z) \end{bmatrix}$$

$$\left((\downarrow 2) \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} \right) X(z) = H_p(z) \begin{bmatrix} X_{even}(z) \\ z^{-1} X_{odd}(z) \end{bmatrix}$$

Synthesis bank:

$$\left(\begin{bmatrix} F_0(z) \\ F_1(z) \end{bmatrix}^T (\uparrow 2) \right)_{1\times 2} = \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T F_p(\boldsymbol{z^2})(\uparrow 2)$$

$$= \begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T (\uparrow 2) F_p(\boldsymbol{z})$$

Overall Filterbank

$$\hat{X}(z) \neq \text{(Analysis)(Synthesis)} X(z) \qquad \text{why?}$$

$$\hat{X}(z) = \text{(Synthesis)(Analysis)} X(z)$$

$$= \underbrace{\begin{bmatrix} z^{-1} \\ 1 \end{bmatrix}^T (\uparrow 2) F_p(\boldsymbol{z})}_{Synthesis} \underbrace{H_p(\boldsymbol{z})(\downarrow 2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix}}_{Analysis} X(z)$$

Want: $F_p(\boldsymbol{z}) H_p(\boldsymbol{z}) = z^{-\ell} I$. (why not $H_p(\boldsymbol{z}) F_p(\boldsymbol{z})$?)