Part II: Basis Functions and Filters

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Ingredients for Multiresolution Analysis (MRA)

Hilbert Space: $L^2(\mathbb{R}) = \{g : \int |g(x)|^2 dx < \infty\}$.

Sequences of Nested Subspaces of $L^2(\mathbb{R})$: [coarse approximations]

$$\text{Primal} \quad \{0\} \longleftarrow \ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots \longrightarrow L^2(\mathbb{R})$$

$$\text{Dual} \quad \{0\} \longleftarrow \ldots \subset \tilde{V}_{-1} \subset \tilde{V}_0 \subset \tilde{V}_1 \subset \ldots \longrightarrow L^2(\mathbb{R})$$

Complementary Subspaces: [details]

$$V_j + W_j = V_{j+1} \quad \tilde{V}_j + \tilde{W}_j = \tilde{V}_{j+1}; \quad \sum_{j=-\infty}^{\infty} W_j = L^2(\mathbb{R}) = \sum_{j=-\infty}^{\infty} \tilde{W}_j$$

Basis Functions: [Use $g_n^j(x) \triangleq 2^{j/2} g(2^j x - n)$]

$$V_j = \text{span}\{\phi_n^j : n \in \mathbb{Z}\}, \quad W_j = \text{span}\{\psi_n^j : n \in \mathbb{Z}\}$$

$$\tilde{V}_j = \text{span}\{\tilde{\phi}_n^j : n \in \mathbb{Z}\}, \quad \tilde{W}_j = \text{span}\{\tilde{\psi}_n^j : n \in \mathbb{Z}\}$$

Shift Invariance: $f_n^j \in V_j \Longleftrightarrow f_{n-k}^j \in V_j$ [Not necessary in general]

Scale Invariance: $f(x) \in V_j \Longleftrightarrow f(2x) \in V_{j+1}$ [Not necessary in general]

Refinement Relations

$$
\begin{aligned}
\varphi(x) &= \sqrt{2}\sum_k h_0[k]\varphi(2x-k) \\
w(x) &= \sqrt{2}\sum_m h_1[m]\varphi(2x-m) \\
\tilde{\varphi}(x) &= \sqrt{2}\sum_k \tilde{h}_0[-k]\tilde{\varphi}(2x-k) \\
\tilde{w}(x) &= \sqrt{2}\sum_m \tilde{h}_1[-m]\tilde{\varphi}(2x-m)
\end{aligned}
$$

Note:

- Refinement relation is slightly different from the book.
- Signs in the dual refinement relations are reversed.
- The primary scaling function and wavelet correspond to the "Synthesis" side of FB.

Rescaled Bases

Define: $\varphi_{j,k}(x) = 2^{\frac{j}{2}}\varphi(2^j x - k)$, $w_{j,m}(x) = 2^{\frac{j}{2}} w(2^j x - m)$, etc.

Modified Refinement relations:

$$
\begin{aligned}
\varphi_{j,k}(x) &= \sum_l h_0[l]\varphi_{j+1,2k+l}(x) \\
w_{j,m}(x) &= \sum_l h_1[l]\varphi_{j+1,2m+l}(x) \\
\tilde{\varphi}_{j,k}(x) &= \sum_l h_0[-l]\tilde{\varphi}_{j+1,2k+l}(x) \\
\tilde{w}_{j,m}(x) &= \sum_l \tilde{h}_1[-l]\tilde{\varphi}_{j+1,2m+l}(x)
\end{aligned}
$$

No more $\sqrt{2}$:-).

Normalization for $h_0[n]$

Normalize scaling functions to have unit area:

$$\int \varphi(x)\,dx = 1$$

$$\begin{aligned} \int \varphi(x)\,dx &= 1 \\ \sqrt{2}\sum_k h_0[k] \int \varphi(2x-k)\,dx &= 1 \end{aligned}$$

But,

$$\int \varphi(2x-k)\,dx = \frac{1}{2}$$

Therefore,

$$\boxed{\sum_k h_0[k] = \sqrt{2}}$$

Biorthogonality and Perfect Reconstruction

Biorthogonality means:

$$\begin{aligned}
\langle \varphi_{j,k}(x), \tilde{\varphi}_{j,k'}(x) \rangle &= \delta_{k,k'} \\
\langle \varphi_{j,k}(x), \tilde{w}_{j,m'}(x) \rangle &= 0 \\
\langle w_{j,m}(x), \tilde{w}_{j,m'}(x) \rangle &= \delta_{m,m'} \\
\langle \tilde{\varphi}_{j,k}(x), w_{j,m}(x) \rangle &= 0,
\end{aligned}$$

for every level j.

Verify that these relations follow automatically:

$$\begin{aligned}
\langle w_{j,m}(x), \tilde{w}_{j',m'}(x) \rangle &= \delta_{m,m'}\delta_{j,j'} \\
\langle \varphi_{j',k}(x), \tilde{w}_{j,m}(x) \rangle &= 0 \, \forall j' < j.
\end{aligned}$$

What constraints does biorthogonality place on the filters?

Biorthogonality and PR - II

Consider

$$\langle \varphi_{j,k}(x), \tilde{\varphi}_{j,k'}(x) \rangle = \delta_{k,k'}.$$

Using refinement,

$$\begin{aligned} \left\langle \sum_l h_0[l]\varphi_{j+1,l+2k}(x), \sum_{l'} \tilde{h}_0[-l']\tilde{\varphi}_{j+1,l'+2k'}(x) \right\rangle &= \delta_{k,k'} \\ i.e., \sum_l \sum_{l'} h_0[l]\tilde{h}_0[-l']\delta_{l',l+2k-2k'} &= \delta_{k,k'} \end{aligned}$$

Finally, setting $k = 0$ without loss of generality and replacing k' by k,

$$\boxed{\sum_l h_0[l]\tilde{h}_0[2k-l] = \delta_k}$$

Double shift orthogonality!

Biorthogonality and PR - III

Next consider

$$\left\langle \varphi_{j,k}(x), \tilde{w}_{j,m'}(x) \right\rangle = 0$$

Again using the refinement relation,

$$\begin{aligned} \left\langle \sum_l h_0[l]\varphi_{j+1,l+2k}(x), \sum_{l'} \tilde{h}_1[-l']\tilde{\varphi}_{j+1,l'+2m'}(x) \right\rangle &= 0 \\ i.e., \sum_l \sum_{l'} h_0[l]\tilde{h}_1[-l']\delta_{l',l+2k-2m'} &= 0 \end{aligned}$$

Setting $k = 0$ WLG and replacing m' by m,

$$\boxed{\sum_l h_0[l]\tilde{h}_1[2m - l] = 0.}$$

Looks familiar?

Forward Wavelet Transform

$$
\begin{aligned}
f_{j+1}(x) &= \sum_l c_{j+1}[l]\varphi_{j+1,l}(x) \\
&= \sum_k c_j[k]\varphi_{j,k}(x) + \sum_m d_j[m]\varphi_{j,m}(x)
\end{aligned}
$$

Using b13y,

$$
\begin{aligned}
c_j[k] &= \left\langle \sum_l c_{j+1}[l]\varphi_{j+1,l}(x), \tilde{\varphi}_{j,k}(x) \right\rangle \\
d_j[m] &= \left\langle \sum_l c_{j+1}[l]\varphi_{j+1,l}(x), \tilde{w}_{j,m}(x) \right\rangle
\end{aligned}
$$

Use refinement for the duals and simplify as

$$
\boxed{
\begin{aligned}
c_j[k] &= \sum_l c_{j+1}[l]\tilde{h}_0[2k-l] &&= \sum_l \tilde{h}_0[l]c_{j+1}[2k-l] \\
d_j[m] &= \sum_l c_{j+1}[l]\tilde{h}_1[2m-l] &&= \sum_l \tilde{h}_1[l]c_{j+1}[2m-l]
\end{aligned}
}
$$

i.e., Filtering and Downsampling!

An Interesting Observation

Set

$$c_{j+1,l'} = \delta_{l,l'}$$

Then,

$$\begin{aligned} c_j[k] &= \tilde{h}_0[2k-l] \\ d_j[m] &= \tilde{h}_1[2m-l] \end{aligned}$$

Hence,

$$\boxed{\varphi_{j+1,l}(x) = \sum_k \tilde{h}_0[2k-l]\varphi_{j,k}(x) + \sum_m \tilde{h}_1[2m-l]w_{j,m}(x)}$$

The Un-Refinement Relation!

Inverse Wavelet Transform

$$\sum_l c'_{j+1}[l]\varphi_{j+1,l}(x) = \sum_k c_j[k](x)\varphi_{j,k}(x) + \sum_m d_j[m]w_{j,m}(x)$$

Use b13y and primary refinement relations:

$$\begin{aligned} c'_{j+1}[l] \quad = \quad & \sum_k \sum_{l'} h_0[l'] \left\langle \varphi_{j+1,l'+2k}(x), \tilde{\varphi}_{j+1,l}(x) \right\rangle + \\ & \sum_m \sum_{l'} h_1[l'] \left\langle \varphi_{j+1,l'+2m}(x), \tilde{\varphi}_{j+1,l}(x) \right\rangle \end{aligned}$$

But, $\left\langle \varphi_{j+1,l'+2k}(x), \tilde{\varphi}_{j+1,l}(x) \right\rangle = \delta_{l,l'+2k}$, etc. Hence,

$$\boxed{c'_{j+1}[l] = \sum_k c_j[k]h_0[l-2k] + \sum_m d_j[m]h_1[l-2m]}$$

Upsampling and Filtering !

Biorthogonality and Perfect Reconstruction

When is $c'_{j+1}[l] = c_{j+1}[l]$?

Substitute for $c_j[k]$ and $d_j[m]$ in the reconstruction equation:

$$\begin{aligned} c_{j+1}[l] \quad = \quad & \sum_{l'} \sum_{k} \tilde{h}_0[2k - l']h_0[l - 2k]c_{j+1}[l'] + \\ & \sum_{l'} \sum_{m} \tilde{h}_1[2m - l']h_1[l - 2m]c_{j+1}[l'] \end{aligned}$$

But $c_{j+1}[l] = \sum_{l'} c_{j+1}[l']\delta_{l,l'}$. Hence if LHS=RHS for every possible $c_{j+1}[l]$,

$$\sum_{k} \tilde{h}_0[2k - l']h_0[l - 2k] + \sum_{m} \tilde{h}_1[2m - l']h_1[l - 2m] = \delta_{l,l'}$$

But this is the condition for perfect reconstruction!

So, does b13y $\iff$ PR? We'll see in a moment...

Aside: Partition of Unity

PoU means that the primary (synthesis) scaling function can exactly represent constants.

$$\sum_k 1 \times \varphi(x-k) + \sum_m 0 \times w(x-m) = const.$$

From the expressions for the primary wavelet transforms, this means that the dual (analysis) wavelet must have at least one vanishing moment.

$$\int \tilde{w}(x-m)\, dx = 0.$$

Using the refinement relation and the fact that $\int \tilde{\varphi}(x)\, dx =$const., this in turn leads to:

$$\sum_l \tilde{h}_1[l] = 0 \rightarrow \sum_l (-1)^l h_0[l] = 0$$

The low-pass filter must have one zero at π.

Dual Transforms

$$\begin{aligned} f_{j+1}(x) &= \sum_l \tilde{c}_{j+1}[l]\tilde{\varphi}_{j+1,l}(x) \\ &= \sum_k \tilde{c}_j[k]\tilde{\varphi}_{j,k}(x) + \sum_m d_j[m]\tilde{w}_{j,m}(x) \end{aligned}$$

Analysis:

$$\begin{aligned} \tilde{c}_j[k] &= \sum_l h_0[l]\tilde{c}_{j+1}[l-2k] \\ \tilde{d}_j[m] &= \sum_l h_1[l]\tilde{c}_{j+1}[l-2m] \end{aligned}$$

Synthesis:

$$\tilde{c}_{j+1}[l] = \sum_k \tilde{c}_j[k]\tilde{h}_0[2k-l] + \sum_m \tilde{d}_j[m]\tilde{h}_1[2m-l]$$

Matrix Relations

Let,

$$\left[\frac{c_j}{d_j}\right] = \left[\frac{\mathbf{L}}{\mathbf{B}}\right][c_{j+1}] \qquad \left[\frac{\tilde{c}_j}{\tilde{d}_j}\right] = \left[\frac{\tilde{\mathbf{L}}}{\tilde{\mathbf{B}}}\right][\tilde{c}_{j+1}]$$

Then one can show that,

$$[c_{j+1}] = \tilde{\mathbf{L}}^T[c_j] + \tilde{\mathbf{B}}^T[d_j] \quad \text{and} \quad [\tilde{c}_{j+1}] = \mathbf{L}^T[\tilde{c}_j] + \mathbf{B}^T[\tilde{d}_j]$$

Hence for perfect reconstruction of the primary transform,

$$\boxed{\tilde{\mathbf{L}}^T\mathbf{L} + \tilde{\mathbf{B}}^T\mathbf{B} = \mathbf{I}}$$

But, by inverting the order of multiplication,

$$\boxed{\begin{array}{lll} \mathbf{L}\tilde{\mathbf{L}}^T = \mathbf{I} & & \mathbf{L}\tilde{\mathbf{B}}^T = \mathbf{0} \\ \mathbf{B}\tilde{\mathbf{L}}^T = \mathbf{0} & & \mathbf{B}\tilde{\mathbf{B}}^T = \mathbf{I} \end{array}}$$

These are the requirements for b13y! Hence b13y *does* imply PR.

Orthogonal Wavelets: Condition O

Orthogonal wavelet bases have $\varphi_{j,k}(x) = \tilde{\varphi}_{j,k}(x)$, $w_{j,m}(x) = \tilde{w}_{j,m}(x)$. Hence $h_0[n] = \tilde{h}_0[-n]$ and $h_1[n] = \tilde{h}_1[-n]$

$$\begin{aligned} \sum_l h_0[l]h_0[l-2k] &= \delta_k \\ \sum_l h_0[l]h_1[l-2m] &= 0 \end{aligned}$$

This is exactly Condition O! We can now derive all the familiar conditions ($\sum$ odd $= \sum$ even, etc.) directly from the above relations.