18.327/1.130: Wavelets, Filter Banks and Applications

Solutions to Problem Set 1

Mark Distribution

Question	Marks	Distribution
Matlab Exercise	3	1+1+1
Problem Set 1.1	1	1/2+1/2
Problem Set 1.2	1	1/2+1/2
Problem Set 1.3	1	1
Problem Set 2.1	1	1/2+1/2
Problem Set 3.1	1	1
Problem Set 3.2	1	1/2+1/2
Problem Set 4.1	1	1/2+1/2
Total	10	

Grading Policy

1. Any problem (or sub-problem) not attempted fetched 0 marks.
2. A reasonable effort to answer a problem (or sub-problem) fetched half the total marks for that problem (or sub-problem).
3. Anything close to the correct answer fetched full marks.

Matlab Exercises

1. (a) We have,

$$H_0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}z^{-1} \quad H_1(z) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}z^{-1}$$

Hence, choosing

$$F_0(z) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}z^{-1} \quad F_1(z) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}z^{-1}$$

will give perfect reconstruction with a one sample delay, whereas choosing an anti-causal set of reconstruction filters,

$$F_0(z) = \frac{1}{\sqrt{2}}z + \frac{1}{\sqrt{2}} \quad F_1(z) = -\frac{1}{\sqrt{2}}z + \frac{1}{\sqrt{2}}$$

would give perfect reconstruction with no delay.

(b) The three signal are shown in Figure 1.

(c) The zeros of the four filters are shown in Figure 2.

(d) The spectra of $v_0[n]$ and $v_1[n]$ are shown in Figure 3. Note that because of the short length of the original signal, the spectra shown in Figure 3 would depend on how the filtered signals were downsampled. The signals $v_0[n]$ and $v_1[n]$ were obtained using the `dyaddown` command in the wavelet toolbox which removes the *even* components of a vector (i.e., the odd-indexed elements since Matlab arrays are 1-based). However, note that perfect reconstruction is still achieved as is evident from Figure 1.

2. For the second set of analysis filters, the synthesis filters for perfect reconstuction are given as:

$$\begin{aligned} F_0(z) &= \frac{1}{2\sqrt{2}}\left(1 + 2z^{-1} + z^{-2}\right) \\ F_1(z) &= \frac{1}{4\sqrt{2}}\left(1 + 2z^{-1} - 6z^{-2} + 2z^{-3} + z^{-4}\right) \end{aligned}$$

Note that making the synthesis filters causal leads to perfect reconstruction with a three-sample delay. Figure 4 shows the signals $v_0[n]$, $v_1[n]$ and $\hat{x}[n]$; Figure 5 shows the poles and zeros of the four filters and and Figure 6 shows the frequency spectra of

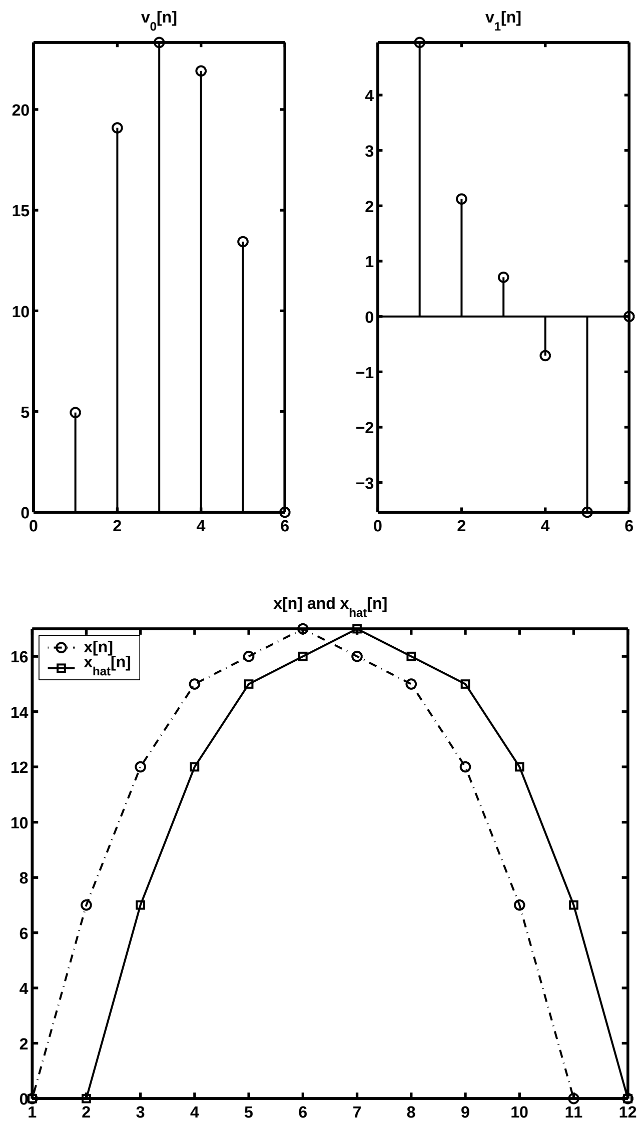

Figure 1:

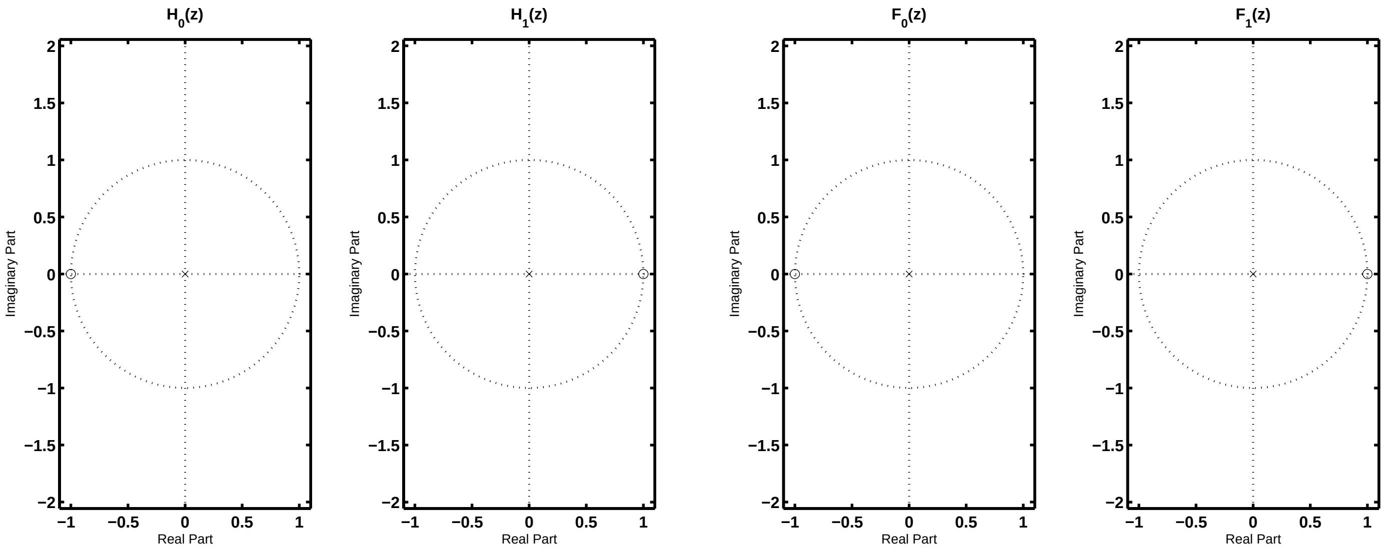

Figure 2:

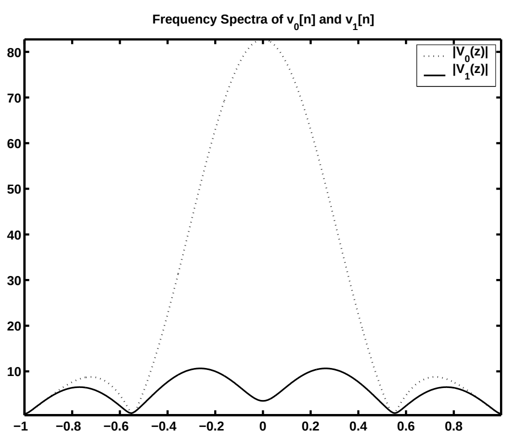

Figure 3:

the signals $v_0[n]$ and $v_1[n]$. The Matlab code is listed on Page 7.

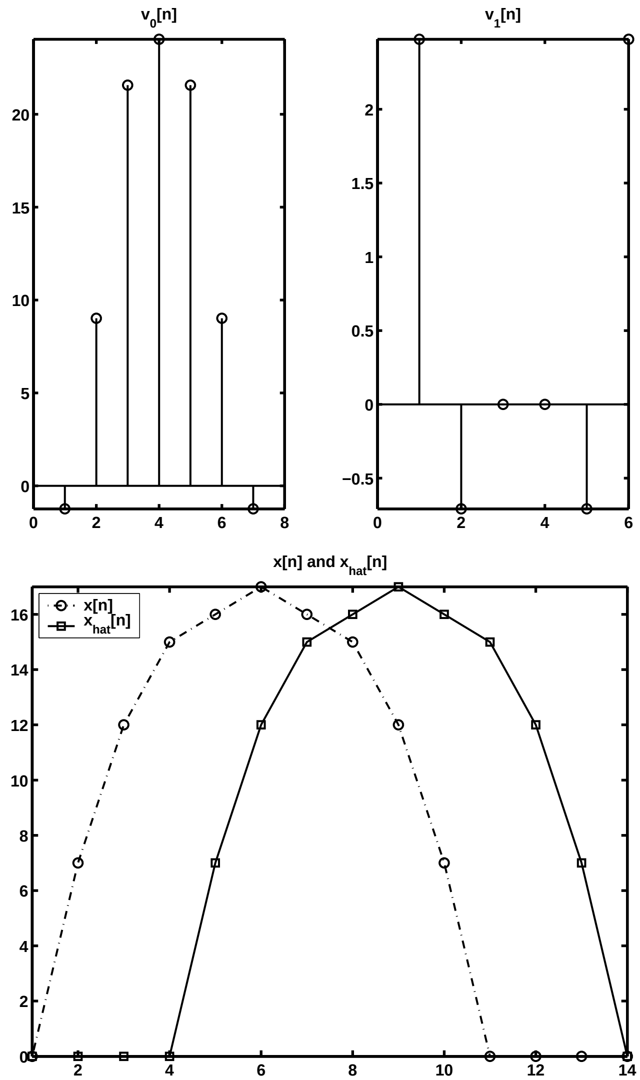

Figure 4:

3. For the first high-pass filter, the output is given as $y[n] = \frac{1}{\sqrt{2}}\left(x[n] - x[n-1]\right)$ whereas for the second high-pass filter the output is given as $y[n] = \frac{1}{2\sqrt{2}}\left(x[n] - 2 \cdot x[n-1] + x[n-2]\right)$. Therefore, a linear signal of the form $x[n] = a \cdot n + b$ with $a \neq 0$ will produce non-zero detail coefficients when filtered through the first high-pass filter, but zero details when filtered through the second high-pass filter.

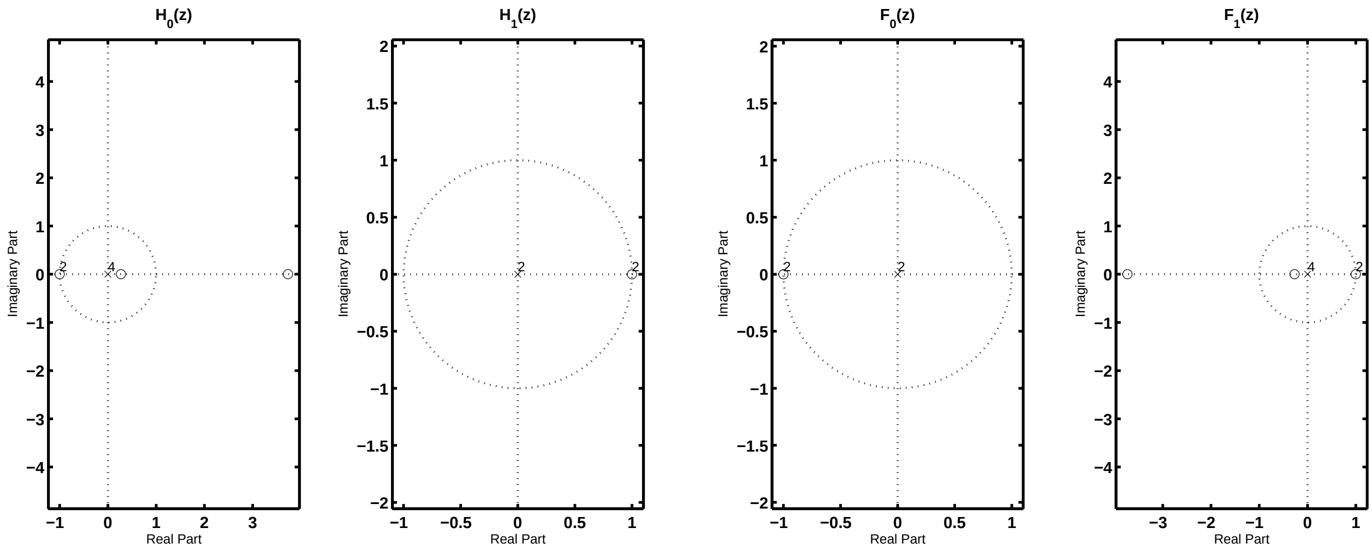

Figure 5:

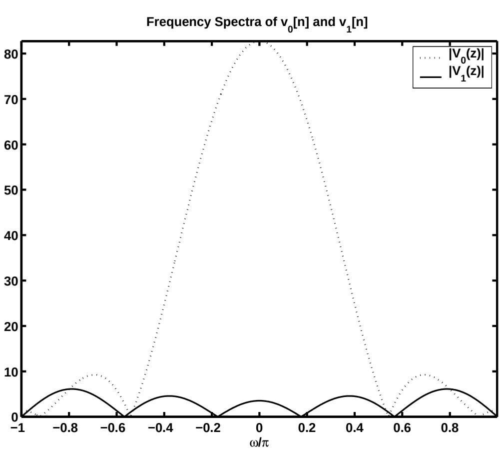

Figure 6:

```matlab
% Code for the Matlab Exercises

N = 1024;
w = (-N/2:N/2-1) / (N/2);

%%%%%%%%%%%
% Part 1
%%%%%%%%%%%
% h0 = [1/sqrt(2), 1/sqrt(2)];
% h1 = [1/sqrt(2), -1/sqrt(2)];
% f0 = [1/sqrt(2), 1/sqrt(2)];
% f1 = [-1/sqrt(2), 1/sqrt(2)];
% delay = 1;

%%%%%%%%%%%
% Part 2
%%%%%%%%%%%
h0 = 1/(4*sqrt(2))*[-1,2,6,2,-1];
h1 = 1/(2*sqrt(2))*[1,-2,1];
f0 = 1/(2*sqrt(2))*[1,2,1];
f1 = 1/(4*sqrt(2))*[1,2,-6,2,1];
delay = 3;

x = [0,7,12,15,16,17,16,15,12,7,0];
v0  = dyaddown(conv(x,h0));
v1  = dyaddown(conv(x,h1));
xhat = conv(dyadup(v0),f0)+conv(dyadup(v1),f1);

X  = fftshift(fft(x,N));
V0 = fftshift(fft(v0,N));
V1 = fftshift(fft(v1,N));

figure(1);
subplot(1,2,1);
h = stem(v0,'k');
set(h,'LineWidth',1.5);
set(h,'MarkerSize',7);
axis tight;
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('v_0[n]');
subplot(1,2,2);
h = stem(v1,'k');
set(h,'LineWidth',1.5);
set(h,'MarkerSize',7);
axis tight;
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('v_1[n]');

figure(2);
h = plot([1:length(x)+delay],[x zeros(1,delay)],'ko-.',[1:length(x)+delay],xhat(
1:(length(x)+delay)),'ks-');
set(h(1),'LineWidth',1.5);
set(h(1),'MarkerSize',7);
set(h(2),'LineWidth',1.5);
set(h(2),'MarkerSize',7);
axis tight;
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('x[n] and x_{hat}[n]');
legend('x[n]','x_{hat}[n]',2);

figure(3);
subplot(1,2,1);
[hz,hp,hl] = zplane(h0);
set(hz,'MarkerSize',7);
set(hp,'MarkerSize',7);
set(hl,'LineWidth',1.5);
set(hl,'Color','k');
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('H_0(z)');
subplot(1,2,2);
[hz,hp,hl] = zplane(h1);
set(hz,'MarkerSize',7);
set(hp,'MarkerSize',7);
set(hl,'LineWidth',1.5);
set(hl,'Color','k');
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('H_1(z)');

figure(4);
subplot(1,2,1);
[hz,hp,hl] = zplane(f0);
set(hz,'MarkerSize',7);
set(hp,'MarkerSize',7);
set(hl,'LineWidth',1.5);
set(hl,'Color','k');
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('F_0(z)');
subplot(1,2,2);
[hz,hp,hl] = zplane(f1);
set(hz,'MarkerSize',7);
set(hp,'MarkerSize',7);
set(hl,'LineWidth',1.5);
set(hl,'Color','k');
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
title('F_1(z)');

figure(5);
h = plot(w,abs(V0),'k:',w,abs(V1),'k-');
set(h(1),'LineWidth',1.5);
set(h(2),'LineWidth',1.5);
axis tight;
set(gca,'FontSize',12);
set(gca,'FontWeight','bold');
set(gca,'LineWidth',2);
set(gca,'Box','on');
xlabel('\omega/\pi');
title('Frequency Spectra of v_0[n] and v_1[n]');
legend('|V_0(z)|','|V_1(z)|');
```

Textbook Problems

1. Problem Set 1.1, pp. 6–7.

   (a) Problem 5. In the time domain we have,

   $$\hat{v}[n] = \sum_k \delta[k]v[n-k] = v[n].$$

   In the frequency domain,

   $$\begin{aligned} \hat{V}(\omega) &= \left(\sum_k \delta[k]e^{-i\omega k}\right) V(\omega) \\ &= V(\omega) \end{aligned}$$

   Hence convolving a signal with the Kronecker delta filter leaves the signal unchanged.

   (b) Problem 11. We have $p[n] = \sum_k h[k]\tilde{h}[n-k]$, where $\tilde{h}[n] = h[-n]$. Hence, $p[n] = \sum_k h[k]h[k-n]$.

2. Problem Set 1.2, pp. 11.

   (a) Problem 9.

      i. Let $h[n] = \left\{\frac{1}{2}, 0, \frac{1}{2}\right\}$. Then $H(\omega) = \frac{1}{2}\left(1 + e^{-2i\omega}\right) = \cos(\omega)e^{-i\omega}$. Hence $|H(\omega)| = \cos(\omega)$.

      ii. Let $h[n] = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$. Then $H(\omega) = \frac{1}{4}e^{-i\omega}\left(e^{i\omega} + 2 + e^{-i\omega}\right) = \frac{1}{2}e^{-i\omega}\left(1 + \cos(\omega)\right)$. Hence $|H(\omega)| = \cos^2(\omega/2)$.

   In both cases, the filters are non-unique because we obtain the same magnitude response by changing the phase of the filters.

   (b) Problem 13. Let $\tilde{G}(z) = H(z^{-1})$. Clearly, $\tilde{g}[n] = h[-n]$. Now, $G(z) = z^{-N}\tilde{G}(z)$, hence $g[n] = \tilde{g}[n-N] = h[N-n]$. The filter $g[n]$ is therefore a causal flip of the filter $h[n]$. If the filter $h[n]$ is symmetric, so is $g[n]$and hence $G(\omega)$ will be a linear-phase filter. In the frequency domain, we have $G(\omega) = e^{-i\omega N}H(-\omega)$. Hence if $H(\pi) = 0$, so is $G(\pi)$. Therefore, $g[n]$ will be a low-pass filter.

3. Problem Set 1.3, pp. 15.

   (a) Problem 5. If phase changes are not permissible, then the equation

$$H(\omega)X(\omega) = \frac{1}{2}X(\omega)$$

   is satisfied only by the trivial signal $X(\omega) = 0$, i.e., $x[n] = 0$. On the other hand, permitting phase delays but requiring that the magnitude of the signal be cut in half, we have

$$|H(\omega)|\,|X(\omega)| = \frac{1}{2}\,|X(\omega)|\,,$$

   or

$$\left|\sin\left(\frac{\omega}{2}\right)\right|\,|X(\omega)| = \frac{1}{2}\,|X(\omega)|\,,$$

   which is satisfied by choosing $|X(\omega)| = A\delta(\omega - \pi/3) + B\delta(\omega + \pi/3)$. If phase changes are permissible and the signal is not required to have finite energy then

$$x[n] = 2^n\ x[0]$$

   also gets cut in half when passed through the moving difference filter.

4. Problem Set 2.1, pp. 44–45.

   (a) Problem 7. From pp. 40, the delay matrix $\mathbf{S}$ is a unit lower-triangular matrix. The product $\mathbf{H} = \mathbf{S}^2$ can therefore be represented as:

$$\begin{bmatrix} . & . & . & . & . & . \\ . & 0 & 0 & 0 & 0 & . \\ . & 1 & 0 & 0 & 0 & . \\ . & 0 & 1 & 0 & 0 & . \\ . & 0 & 0 & 1 & 0 & . \\ . & . & . & . & . & . \end{bmatrix} \times \begin{bmatrix} . & . & . & . & . & . \\ . & 0 & 0 & 0 & 0 & . \\ . & 1 & 0 & 0 & 0 & . \\ . & 0 & 1 & 0 & 0 & . \\ . & 0 & 0 & 1 & 0 & . \\ . & . & . & . & . & . \end{bmatrix} = \begin{bmatrix} . & . & . & . & . & . \\ . & 0 & 0 & 0 & 0 & . \\ . & 0 & 0 & 0 & 0 & . \\ . & 1 & 0 & 0 & 0 & . \\ . & 0 & 1 & 0 & 0 & . \\ . & . & . & . & . & . \end{bmatrix}$$

   Alternatively, in component form, $\mathbf{S}_{ij} = \delta_{i,j+1}$, where $\delta_{i,j}$ is the Kronecker delta.

Hence,

$$\mathbf{H}_{i,j} = \sum_k S_{i,k} S_{k,j} = \sum_k \delta_{i,k+1}\delta_{k+1,j+2} = \delta_{i,j+2}.$$

Observe therefore that the filter $\mathbf{H}$ delays a signal $x[n]$ by two samples. The coefficient vector can therefore be written as $h[n] = \delta[n-2]$ in the time domain, or $H(\omega) = e^{-2i\omega}$ in the frequency domain.

(b) Problem 13.

i. $H_1(z) = (1-\alpha z^{-1})(1-\beta z^{-1})$.

$$H_1^{-1}(z) = \frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})} = \frac{A}{(1-\alpha z^{-1})} + \frac{B}{(1-\beta z^{-1})}$$

where $A = \frac{\alpha}{\alpha-\beta}$ and $B = -\frac{\beta}{\alpha-\beta}$. The filter is therefore invertible when $|z| > |\alpha|$ and $|z| > |\beta|$. The inverse can be written down explicitly as:

$$H_1^{-1}(z) = A\sum_{n=0}^{\infty}\left(\alpha z^{-1}\right)^n + B\sum_{n=0}^{\infty}\left(\beta z^{-1}\right)^n$$

hence,

$$h_1^{-1}[n] = A\alpha^n u[n] + B\beta^n u[n].$$

Clearly, the inverse is <u>causal</u> and <u>IIR</u>. Moreover, since $H_1^{-1}(z) \neq 1 \forall z$, the inverse is <u>not all-pass</u>. **Note:**. The above inverse is not unique. It is possible to derive an anti-causal inverse as follows:

$$H_1^{-1}(z) = \tilde{A}\frac{1}{1-\alpha^{-1}z} + \tilde{B}\frac{1}{1-\beta^{-1}z}$$

where $\tilde{A} = -\frac{Az}{\alpha}$ and $\tilde{B} = -\frac{Bz}{\beta}$. Clearly, the region of convergence in this case is <u>$|z| < |\alpha|$ and $|z| < |\beta|$</u>. The inverse filter can then be written as:

$$h_1^{-1}[n] = -A\alpha^n u[-n-1] - B\beta^n u[-n-1].$$

Hence, the inverse is <u>anticausal</u> and <u>IIR</u>, but not <u>all-pass</u>.

ii. $H_2(z) = 1 + \beta z^{-1} + \beta^2 z^{-2} + \ldots$. Hence,

$$H_2^{-1}(z) = \frac{1}{1 + \beta z^{-1} + \beta^2 z^{-2} + \ldots} = 1 - \beta z^{-1}$$

provide $|z| > |\beta|$. The filter is therefore invertible for $|z| > |\beta|$; since $H_2^{-1}(z) = 0$ at $z = \beta$, the inverse is causal when $|\beta| < 1$; the inverse is FIR since the z-transform consists of only a finite-number of non-zero coefficients; the inverse is not all-pass since $|H_2^{-1}(z)| \neq 1$ in general.

iii. Notice that there is a pole-zero cancelletion in $H_3(z)$,

$$H_3(z) = \frac{z - \beta}{1 - \beta z^{-1}} = \frac{z(1 - \beta z^{-1})}{1 - \beta z^{-1}} = z$$

Hence the inverse $H_3^{-1}(z) = z^{-1}$ is simply the one-sample delay filter. The given filter $H_3(z)$ is therefore invertible in the entire z plane; the inverse is clearly causal and FIR as it contains only a finite number of negative powers of z; since $|H_3^{-1}(\omega)| = 1$, the inverse is all-pass.

iv. $H_4(z) = 1 - \beta z^{-1} + z^{-2} = (k_1 - z^{-1})(k_2 - z^{-1})$, where $k_1 = \frac{\beta + \sqrt{\beta^2 - 4}}{2}$ and $k_2 = \frac{\beta - \sqrt{\beta^2 - 4}}{2}$. Inverting this filter (and using partial fraction expansion) we obtain

$$H_4^{-1}(z) = \frac{1}{(k_1 - z^{-1})(k_2 - z^{-1})} = \frac{1}{k_2 - k_1}\left(\frac{1}{k_1} \cdot \frac{1}{\left(1 - \frac{z^{-1}}{k_1}\right)} - \frac{1}{k_2} \cdot \frac{1}{\left(1 - \frac{z^{-1}}{k_2}\right)}\right)$$

The filter $H_4^{-1}(z)$ is therefore invertible when $|z| > \frac{1}{k_1}$ and $|z| > \frac{1}{k_2}$. We now have

$$H_4^{-1}(z) = \frac{1}{k_1\,(k_2 - k_1)} \sum_{n=0}^{\infty} (k_1 z)^{-n} - \frac{1}{k_2\,(k_2 - k_1)} \sum_{n=0}^{\infty} (k_2 z)^{-n}$$

or,

$$h_4^{-1}[n] = A k_1^{-n} u[n] + B k_2^{-n} u[n]$$

where $A = \frac{1}{k_1(k_2 - k_1)}$ and $B = -\frac{1}{k_2(k_2 - k_1)}$. The inverse is therefore causal and IIR.

Moreover, it is not all-pass since $H_4^{-1}(z) \neq 1$ for all values of ω.

5. Problem Set 3.1, pp. 90–91.

   (a) Problem 2. The easiest approach in solving this problem is to observe that $\downarrow 2 \uparrow 2 \;\; u[n] = u[n]$. Hence,

$$\begin{aligned} \hat{x}[n] &= \uparrow 2 \downarrow 2 \uparrow 2 \downarrow 2 \; x[n] \\ &= \uparrow 2 \downarrow 2 \; x[n] \\ &= \tfrac{1}{2}\left(1 + (-1)^n\right) x[n] \end{aligned}$$

   In the frequency domain, one has

$$\begin{aligned} \hat{X}(z) &= \tfrac{1}{2} \sum_{n=-\infty}^{\infty} x[n]z^{-n} + x[n](-z)^{-n} \\ &= \tfrac{1}{2}\left(X(z) + X(-z)\right) \end{aligned}$$

6. Problem Set 3.2, pp. 95–96.

   (a) Problem 3. Following the notation used in Equations 3.10 – 3.16, let

$$\begin{aligned} u[n] &= \begin{cases} x\left[\frac{n}{3}\right] & n = 3k, k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \\ &= \tfrac{1}{3}\left(1 + e^{-2\pi i n/3} + e^{-4\pi i n/3}\right) x[n] \end{aligned}$$

   Hence,

$$U(\omega) = \frac{1}{3}\left(X(\omega) + X(\omega + 2\pi/3) + X(\omega + 4\pi/3)\right)$$

   and

$$V(\omega) = U(\omega/3) = \frac{1}{3}\left(X(\omega/3) + X((\omega + 2\pi)/3) + X((\omega + 4\pi)/3)\right)$$

   (b) Problem 7. An M-channel filter bank without any filters is illustrated in Figure 7. Notice that it is composed only of delays and hence the reconstructed signal is delayed by $M-1$ samples. A filter bank without any delays is illustrated in 8.

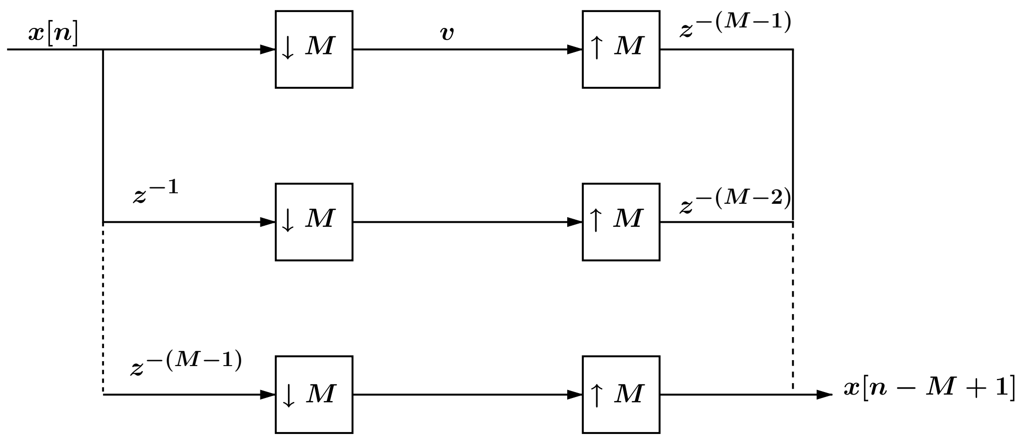

Figure 7: An M-channel filter bank without any filters leading to output delayed by $M-1$ samples.

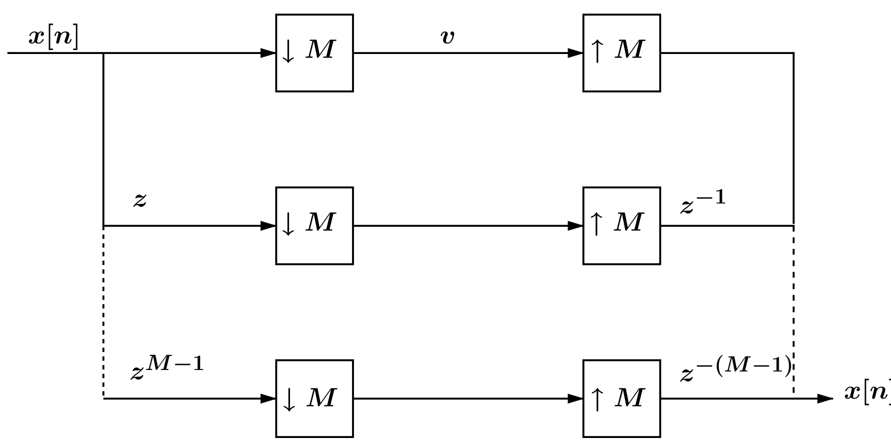

Figure 8: An M-channel filter bank without any filters leading to output with no delay.

7. Problem Set 4.1, pp. 113–114.

(a) Problem 9. The formula for $P_0(z)$ of any order is given in Section 5.5 (Eq. 5.75). The 10th degree half-band polynomial is,

$$P_0(z) = \frac{1}{256}(3-25z^{-2}+150z^{-4}+256z^{-5}+150z^{-6}-25z^{-8}+3z^{-10}) = (1+z^{-1})^6 Q(z)$$

where $Q(z) = \frac{1}{256}(3 - 18z^{-1} + 38z^{-2} - 18z^{-3} + 3z^{-4})$. To solve for the roots of $Q(z)$, we first multiply by z^2 to get the following quadratic equation for $z + \frac{1}{z}$

$$3\left(z+\frac{1}{z}\right)^2 - 18\left(z+\frac{1}{z}\right) + 32 = 0$$

which gives

$$z + \frac{1}{z} = 3 \pm i\frac{\sqrt{15}}{3}$$

which can then be solved for the four roots of $Q(z)$ of the form $r, \overline{r}, r^{-1}$ and $\overline{r^{-1}}$, where

$$r = \frac{1}{2}\left(3 - i\frac{\sqrt{15}}{3} - \sqrt{\frac{10}{3} - 2i\sqrt{15}}\right)$$

Here, r and $\overline{r}$ are within the unit circle and their reciprocals are outside the unit circle. The 10 roots of the polynomial P(z) are plotted in Figure 9. To obtain the Daubechies 6, we use a minimum phase/maximum phase factorization, by splitting 3 zeros at π and the roots of Q(z) within (resp. outside) the unit circle between $H_0(z)$ (resp. $H_0(z^{-1})$)

(b) Problem 12. For the original system of filters we have

$$\begin{aligned} H_0(z)F_0(z) + H_1(z)F_1(z) &= 2z^{-l} \\ H_0(-z)F_0(z) + H_1(-z)F_1(z) &= 0 \end{aligned}$$

By interchanging H_i and F_i we observe that the PR conditions are still satisfied (since the alias cancellation condition for the new system is obtained simply by replacing z with -z in the original equation and the no-distortion condition remains unchanged).

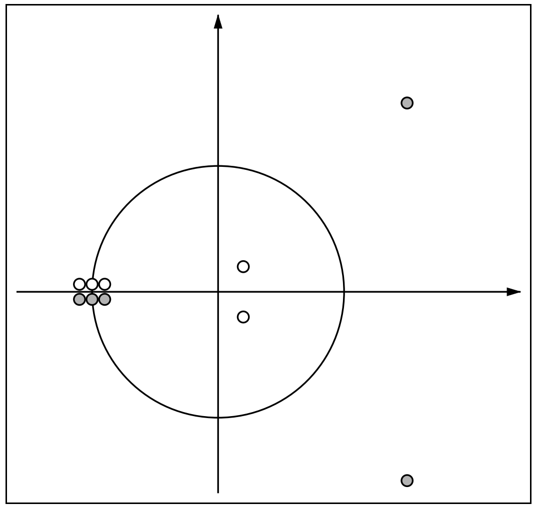

Figure 9: Distribution of roots to give the D_6 filters. The shaded roots go to the high-pass filter.

Now, replacing $H_i(z)$ by $H_i(-z)$ and $F_i(z)$ by $F_i(-z)$, we observe that the alias cancellation condition is still satisfied (since it is obtained by replacing z by $-z$ in the original equation). However, the no-distortion condition becomes

$$H_0(-z)F_0(-z) + H_1(-z)F_1(-z) = 2(-z)^{-l} \qquad \Rightarrow \quad \text{No Longer PR}$$