18.327/1.130: Wavelets, Filter Banks and Applications

Solutions to Problem Set 2

Mark Distribution

Question	Marks
Matlab Exercise	4
Problem Set 3.4	1
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Problem Set 4.3	1
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MATLAB **Exercise**

This exercise can be solved in four steps. First, given p, we determine the polynomial $Q(z)$ of degree $2p-2$ using the `prodfilt` routine. Then, we compute all its roots within the unit circle using the `roots` function in Matlab. Next, we form two polynomials B(z) consisting of the binomial terms with p zeros at π and $Q_0(z)$ composed of the roots of $Q(z)$ within the unit circle (this can be done using the `poly` command in Matlab). Finally, we convolve $B(z)$ and $Q_0(z)$ to get the coefficients of the filter $H_0(z)$. The coefficients of $H_1(z)$ can then be obtained using the alternating flip construction. This approach is more robust than the náively computing the roots of the half-band filter. The spectral factors of the degree 10 product filter which is of the form $4p-2$ can be obtained by calling `SpecFact` with

$p = 3$. The resulting low and highpass filters are then obtained as (compare with output from `daub(6)`):

$$
\begin{aligned}
h_0 &= \{0.47046720778416, 1.14111691583144, 0.65036500052623, \\
&\quad -0.19093441556833, -0.12083220831040, 0.04981749973688\} \\
h_1 &= \{0.04981749973688, 0.12083220831040, -0.19093441556833, \\
&\quad -0.65036500052623, 1.14111691583144, -0.47046720778416\}
\end{aligned}
$$

```
% SPECFACT  Spectral factorization.
%    [H0,H1] = SPECFACT(P) computes the low
%    and highpass orthogonal filters with P zeros at
%    Pi by computing the roots of the product filter
%    of degree 4P-2.

function [h0,h1] = SpecFact(p)

  % Compute P0(z), B(z), Q(z)
  [p0,b,q] = prodfilt(p);

  % Compute the roots of Q(z)
  r = roots(q);
  % .. within the unit circle
  r0 = r(abs(r)<1);

  % Compute polynomial with roots r0
  q0 = poly(r0);

  % Find binomial term with p zeros at Pi
  b0 = poly(-ones(1,p));

  % Compute and normalize the lowpass filter
  h0 = conv(b0,q0);
  h0 = 2*h0./sum(h0); l = length(h0);

  % Compute the highpass filter
  h1 = ((-1).^(0:l-1)).*h0(l:-1:1);
```

Figure 1: MATLAB code for computing spectral factors.

Textbook Problems

1. Problem Set 3.4, pp. 102.

   (a) Problem 3. Figure 2 shows the simplification of the system given in the problem. It essentially involves repeated application of the Noble identities and fractional sampling rules. From the last step, we have in the Z domain,

   $$Y(z) = \frac{z^{-3}}{4}\left[X(z^{\frac{3}{4}}) + X(iz^{\frac{3}{4}}) + X(-z^{\frac{3}{4}}) + X(-iz^{\frac{3}{4}})\right]$$

   For the input signals given in the problem, the corresponding outputs can be easily determined as:

   $$\begin{aligned} x[n] &= \delta[n] & y[n] &= \delta[n-3] \\ x[n] &= \{\ldots,1,1,1,1,\ldots\} & y[n] &= \{\ldots,1,0,0,1,0,0,1,\ldots\} \\ x[n] &= \{\ldots,-1,1,-1,1,\ldots\} & y[n] &= \{\ldots,1,0,0,1,0,0,1,\ldots\} \quad \text{or} \\ & & & \{\ldots,-1,0,0,-1,0,0,-1,\ldots\} \end{aligned}$$

   (b) Problem 6. Writing $H(z)$ and $X(z)$ in terms of the polyphase components we have

   $$\begin{aligned} H(z) &= z^{-1}H_{odd}(z^2) \\ X(z) &= X_{\text{even}}(z^2) + z^{-1}X_{\text{odd}}(z^2) \end{aligned}$$

   From the First Noble Identity,

   $$(\downarrow 2)\, H(z)X(z) = z^{-1}H_{odd}(z)X_{odd}(z) = H_{odd}(z)\,(\downarrow 2)\left(z^{-1}X(z)\right).$$

   Therefore, we first downsample a delayed version of the signal and then convolve it with the non-zero components of the filter.

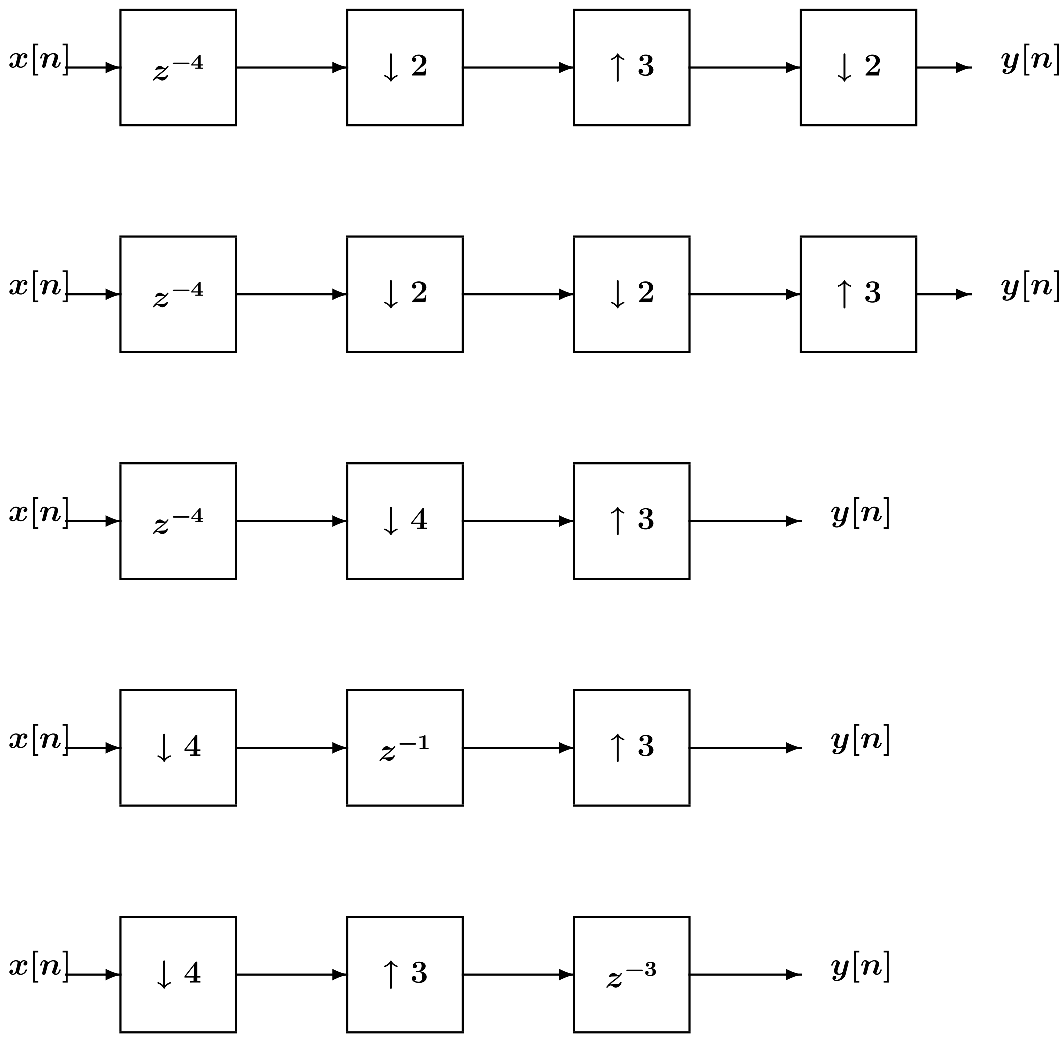


Figure 2: Simplification of the system for Problem 3

2. Problem Set 4.2, pp. 121.

(a) Problem 2. $X(z) = X_{\text{even}}(z^2) + z^{-1}X_{\text{odd}}(z^2)$. Therefore,

$$(\uparrow 2)\,(\downarrow 2)\left(X_{\text{even}}(z^2) + z^{-1}X_{\text{odd}}(z^2)\right) = (\uparrow 2)\,X_{\text{even}}(z) = X_{\text{even}}(z^2).$$

The operation that will produce $X_1(z^2) = X_{\text{odd}}(z^2)$ is the following:

$$(\uparrow 2)\,(\downarrow 2)\,(zX(z)) = (\uparrow 2)\,(\downarrow 2)\left(zX_{\text{even}}(z^2) + X_{\text{odd}}(z^2)\right) = (\uparrow 2)\,X_{\text{odd}}(z) = X_{\text{odd}}(z^2).$$

(b) Problem 4.

a. Note that the denominator $1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}$ can be factored as $\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)$. Hence we have

$$H(z) = \frac{\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)}{\left(1-\frac{1}{4}z^{-2}\right)\left(1-\frac{1}{9}z^{-2}\right)} = \frac{1+\frac{1}{6}z^{-2}}{\left(1-\frac{1}{4}z^{-2}\right)\left(1-\frac{1}{9}z^{-2}\right)} + \frac{\frac{5}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-2}\right)\left(1-\frac{1}{9}z^{-2}\right)}$$

Hence the polyphase components are $H_{\text{even}}(z) = \frac{1+\frac{1}{6}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{9}z^{-1}\right)}$ and $H_{\text{odd}}(z) = \frac{\frac{5}{6}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{9}z^{-1}\right)}$.

b. As in part(a), we multiply the numerator and denominator by $\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right)$ so that the denominator consists of only even powers of z^{-1}. The numerator can then be expanded as

$$\left(1+2z^{-1}+5z^{-2}\right)\left(1+\frac{1}{2}z^{-1}\right)\left(1+\frac{1}{3}z^{-1}\right) = 1+\frac{41}{6}z^{-2}+\frac{5}{6}z^{-4}+z^{-1}\left(\frac{17}{6}+\frac{9}{2}z^{-2}\right)$$

Hence the polyphase components in this case are $H_{\text{even}}(z) = \frac{1+\frac{41}{6}z^{-1}+\frac{5}{6}z^{-2}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{9}z^{-1}\right)}$ and $H_{\text{odd}}(z) = \frac{\frac{17}{6}+\frac{9}{2}z^{-1}}{\left(1-\frac{1}{4}z^{-1}\right)\left(1-\frac{1}{9}z^{-1}\right)}$

(c) Problem 5.

a. By simple factorization, we have

$$1-a^4z^{-4} = \left(1-az^{-1}\right)\left(1+az^{-1}\right)\left(1+a^2z^{-2}\right) = \left(1-az^{-1}\right)\left(1+az^{-1}+a^2z^{-2}+a^3z^{-3}\right).$$

Hence the required polynomial is $(1 + az^{-1} + a^2z^{-2} + a^3z^{-3})$.

b. Using the previous result,

$$\frac{1}{(1-az^{-1})} = \frac{1}{1-a^4z^{-4}} + z^{-1}\frac{a}{1-a^4z^{-4}} + z^{-2}\frac{a^2}{1-a^4z^{-4}} + z^{-3}\frac{a^3}{1-a^4z^{-4}}.$$

Hence,

$$\begin{array}{ll} H_0(z) = \frac{1}{1-a^4z^{-1}} & H_1(z) = \frac{a}{1-a^4z^{-1}} \\ H_2(z) = \frac{a^2}{1-a^4z^{-1}} & H_3(z) = \frac{a^3}{1-a^4z^{-1}}. \end{array}$$

Similarly,

$$\begin{aligned} \frac{1+2z^{-1}+5z^{-2}}{\left(1-\frac{1}{2}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} &= \frac{\left(1+2z^{-1}+5z^{-2}\right)\left(1+\frac{1}{2}z^{-1}+\frac{1}{4}z^{-2}+\frac{1}{8}z^{-3}\right)\left(1+\frac{1}{3}z^{-1}+\frac{1}{9}z^{-2}+\frac{1}{27}z^{-3}\right)}{\left(1-\frac{1}{16}z^{-4}\right)\left(1-\frac{1}{81}z^{-4}\right)} \\ &= \frac{1}{\left(1-\frac{1}{16}z^{-4}\right)\left(1-\frac{1}{81}z^{-4}\right)} \left(\begin{array}{cccc} \left(1+\frac{719}{216}z^{-4}+\frac{5}{216}z^{-8}\right) & + & z^{-1}\left(\frac{17}{16}+\frac{46}{27}z^{-4}\right) & + \\ z^{-2}\left(\frac{259}{36}+\frac{53}{108}z^{-4}\right) & + & z^{-3}\left(\frac{1193}{216}+\frac{1}{8}z^{-4}\right) & \end{array} \right) \end{aligned}$$

Hence,

$$\begin{aligned} H_0(z) &= \frac{1}{\left(1-\frac{1}{16}z^{-1}\right)\left(1-\frac{1}{81}z^{-1}\right)}\left(1+\frac{719}{216}z^{-1}+\frac{5}{216}z^{-2}\right) \\ H_1(z) &= \frac{1}{\left(1-\frac{1}{16}z^{-1}\right)\left(1-\frac{1}{81}z^{-1}\right)}\left(\frac{17}{16}+\frac{46}{27}z^{-1}\right) \\ H_2(z) &= \frac{1}{\left(1-\frac{1}{16}z^{-1}\right)\left(1-\frac{1}{81}z^{-1}\right)}\left(\frac{259}{36}+\frac{53}{108}z^{-1}\right) \\ H_3(z) &= \frac{1}{\left(1-\frac{1}{16}z^{-1}\right)\left(1-\frac{1}{81}z^{-1}\right)}\left(\frac{1193}{216}+\frac{1}{8}z^{-1}\right) \end{aligned}$$

3. Problem Set 4.3, pp. 126.

(a) Problem 2. From Equation 4.48 we have

$$\mathbf{H}_p(z) = \frac{1}{2}\mathbf{H}_m(z^{\frac{1}{2}}) \begin{bmatrix} 1 & z^{\frac{1}{2}} \\ 1 & -z^{\frac{1}{2}} \end{bmatrix}$$

Hence, $\mathbf{H}_p^T(z^{-1})\mathbf{H}_p(z)$ can be written as

$$\frac{1}{4}\begin{bmatrix} 1 & 1 \\ z^{-\frac{1}{2}} & -z^{-\frac{1}{2}} \end{bmatrix} \underbrace{\mathbf{H}_m^T(z^{-\frac{1}{2}})\mathbf{H}_m(z^{\frac{1}{2}})}_{2\mathbf{I}} \begin{bmatrix} 1 & z^{\frac{1}{2}} \\ 1 & -z^{\frac{1}{2}} \end{bmatrix} = \mathbf{I}$$

(b) Problem 12. The equivalence between the two representations is shown below:

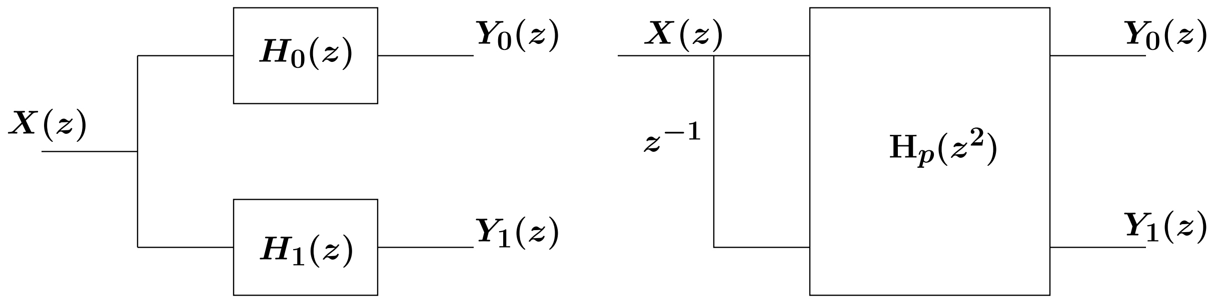


Figure 3: Two equivalent representations.

$$\begin{aligned}
\begin{bmatrix} Y_0(z) \\ Y_1(z) \end{bmatrix} &= \begin{bmatrix} H_0(z) \\ H_1(z) \end{bmatrix} X(z) && \text{(Left representation)} \\
&= \begin{bmatrix} H_{00}(z^2) + z^{-1}H_{01}(z^2) \\ H_{10}(z^2) + z^{-1}H_{11}(z^2) \end{bmatrix} X(z) \\
&= \begin{bmatrix} H_{00}(z^2) & H_{01}(z^2) \\ H_{10}(z^2) & H_{11}(z^2) \end{bmatrix} \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} X(z) \\
&= \mathbf{H}_p(z^2) \begin{bmatrix} 1 \\ z^{-1} \end{bmatrix} X(z). && \text{(Right representation)}
\end{aligned}$$

(c) Problem 17. b.

$$\mathbf{H}_p(z) = \begin{bmatrix} 3 + 3z^{-2} & 1 + 3z^{-2} - z^{-3} + z^{-4} \\ 4 - 3z^{-1} - z^{-2} & 1 - z^{-1} + z^{-2} - 3z^{-3} - z^{-4} \end{bmatrix} = \begin{bmatrix} H_{0,\text{even}}(z) & H_{0,\text{odd}}(z) \\ H_{1,\text{even}}(z) & H_{1,\text{odd}}(z) \end{bmatrix}$$

Therefore,

$$\begin{aligned}
H_0(z) &= H_{0,\text{even}}(z^2) + z^{-1}H_{0,\text{odd}}(z^2) &&= 3 + z^{-1} + 3z^{-4} + 3z^{-5} - z^{-7} + z^{-9} \\
H_1(z) &= H_{1,\text{even}}(z^2) + z^{-1}H_{1,\text{odd}}(z^2) &&= 4 + z^{-1} - 3z^{-2} - z^{-3} - z^{-4} + z^{-5} - 3z^{-7} - z^{-9}
\end{aligned}$$

c. On multiplying out the three terms in the lattice factorization (see Sec. 4.5, pp. 134 for details), we get

$$\mathbf{H}_p(z) = \begin{bmatrix} c_1c_2 - s_1s_2z^{-1} & c_1s_2 + s_1c_2z^{-1} \\ -s_1c_2 - c_1s_2z^{-1} & -s_1s_2 + c_1c_2z^{-1} \end{bmatrix}$$

Therefore,

$$\begin{aligned} H_0(z) &= H_{0,\text{even}}(z^2) + H_{0,\text{odd}}(z^2) &= c_1c_2 + c_1s_2z^{-1} - s_1s_2z^{-2} + s_1c_2z^{-3} \\ H_1(z) &= H_{1,\text{even}}(z^2) + H_{1,\text{odd}}(z^2) &= -s_1c_2 - s_1s_2z^{-1} - c_1s_2z^{-2} + c_1c_2z^{-3} \end{aligned}$$

4. Problem Set 4.4, pp. 133.

(a) Problem 9. For the filter bank shown in the figure, we have

$$\begin{aligned} H_0(z) &= z^{-2} \qquad & H_1(z) &= z^{-1} \\ F_0(z) &= 1 \qquad & F_1(z) &= z^{-1} \end{aligned}$$

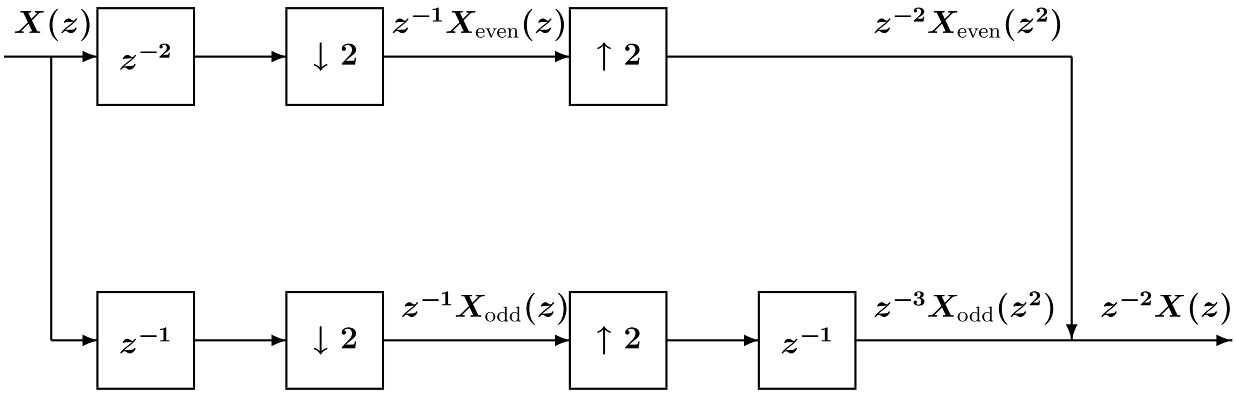
Therefore, the Type 1 polyphase matrix for analysis and the Type 2 polyphase matrix for synthesis can be written as:

$$\mathbf{H}_p(z) = \begin{bmatrix} H_{0,\text{even}}(z) & H_{0,\text{odd}}(z) \\ H_{1,\text{even}}(z) & H_{1,\text{odd}}(z) \end{bmatrix} = \begin{bmatrix} z^{-1} & 0 \\ 0 & 1 \end{bmatrix}$$

$$\mathbf{F}_p(z) = \begin{bmatrix} F_{0,\text{odd}}(z) & F_{1,\text{odd}}(z) \\ F_{0,\text{even}}(z) & F_{1,\text{even}}(z) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

The following figure shows the action of the filter on a signal $X(z)$. It is clear that $\hat{X}(z) = z^{-2}X(z)$. Hence the filter bank is PR albeit with a two sample delay.

5. Problem Set 5.2, pp. 152.



(a) Problem 2. On the unit circle, $H(\omega) = h_0 + h_1 e^{-i\omega} + h_2 e^{-2i\omega} + h_3 e^{-3i\omega}$, $H_{\text{even}}(\omega) = h_0 + h_2 e^{-i\omega}$ and $H_{\text{odd}}(\omega) = h_1 + h_3 e^{-i\omega}$. Now,

$$H_{\text{even}}(\omega)H^*_{\text{even}}(\omega) + H_{\text{odd}}(\omega)H^*_{\text{odd}}(\omega) = h_0^2 + h_1^2 + h_2^2 + h_3^2 + 2\cos(\omega)(h_0h_2 + h_1h_3)$$

$$\frac{1}{2}\left(H(\omega)H^*(\omega) + H(\omega+\pi)H^*(\omega+\pi)\right) = h_0^2+h_1^2+h_2^2+h_3^2+2\cos(2\omega)(h_0h_2+h_1h_3)$$

If the filter with coefficients $\{h_0, h_1, h_2, h_3\}$ satisfies Condition O, then the required result follows.

(b) Problem 6. If two low-pass filters satisfy Condition O, then their product does not necessarily satisfy Condition O. A simple counterexample is provided by taking $C(z) = H(z) = \frac{1}{\sqrt{2}}(1 + z^{-1})$. Clearly, the coefficients in $C(z)H(z) = \frac{1}{2}(1 + 2z^{-1} + z^{-2})$ do not satisfy the double-shift orthogonality condition.

(c) Problem 7. If $\mathbf{H}_p(z)$ and $\mathbf{K}_p(z)$ are paraunitary matrices then,

$$\mathbf{H}_p(z)\mathbf{H}_p^{\mathsf{T}}(z^{-1}) = \mathbf{K}_p(z)\mathbf{K}_p^{\mathsf{T}}(z^{-1}) = \mathbf{I} \qquad \forall z$$

Therefore, we clearly have that the product of the two, $\mathbf{H}_p(z)\mathbf{K}_p(z)$ is also paraunitary since

$$\mathbf{H}_p(z)\mathbf{K}_p(z)\left(\mathbf{H}_p(z^{-1})\mathbf{K}_p(z^{-1})\right)^{\mathsf{T}} = \mathbf{H}_p(z)\mathbf{K}_p(z)\mathbf{K}_p^{\mathsf{T}}(z^{-1})\mathbf{H}_p^{\mathsf{T}}(z^{-1}) = \mathbf{I}$$

6. Problem Set 5.5, pp. 172.

(a) Problem 1. $P(\omega) = 1 + \sum_{n=2k+1} p[n]e^{-i\omega n}$, therefore,

$$P(\pi) = 1 + \sum_{n=2k+1} p[n]e^{-i\pi n} = 1 - \sum_{n=2k+1} p[n] = 0 \Rightarrow \sum_{n=2k+1} p[n] = 1$$

Hence,

$$P(0) = 1 + \sum_{n=2k+1} p[n] = 1 + 1 = 2$$