18.327/1.130: Wavelets, Filter Banks and Applications

Solutions to Problem Set 3

MARK DISTRIBUTION

PROBLEM	MARKS
6.1.3	1.0
6.1.5	0.5
6.2.7	1.0
6.2.8	1.0
6.3.2	0.5
6.3.3	0.5
6.4.4	1.0
6.5.4	0.5
6.5.7	BONUS
6.5.9	1.0
7.1.8	1.0
7.2.2	0.5
7.3.7	0.5
7.3.8	0.5
7.5.4	0.5
TOTAL	10.0

GRADING POLICY

- A reasonable attempt to answer a one-mark question fetched half-marks
- Problems not attempted fetched no marks.
- Since lifting was not covered before the problem set was due, Problem 6.5.7 was converted into a one-mark bonus question.

1. PROBLEM SET 6.1, PP. 186.

   (a) PROBLEM 3. A piecewise constant function is of the form

   $$f(t) = \sum_{n\in\mathbb{Z}} f[n]\varphi(t-n)$$

   where $\varphi(x)$ is the box function. Therefore,

   $$\int_{-\infty}^{\infty} f^2(t)\,dt = \int_{-\infty}^{\infty} \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} f[m]f[n]\varphi(t-m)\varphi(t-n)\,dt = \sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\int_{-\infty}^{\infty} f[m]f[n]\varphi(t-m)\varphi(t-n)\,dt$$

   Since the box function forms an orthogonal basis on the real-line,

   $$\int_{-\infty}^{\infty} \varphi(t-m)\varphi(t-n)\,dt = \delta_{m,n}.$$

   Hence

   $$\int_{-\infty}^{\infty} f^2(t)\,dt = \sum_{n\in\mathbb{Z}} f^2[n].$$

   Therefore if $f(t) \in L^2$ then $f[n] \in \ell^2$ and vice-versa.

   (b) PROBLEM 5. Since $V_0 \subset V_1$ and $f \in V_0$ and $g \in V_1$, $g - f \in V_1$. Clearly, $g - f \notin W_1$ in general since $V_1 \oplus W_1 = V_2$ which means that the only element common between V_1 and $W_1$is the 0 function. However, it is also true in general that $g - f \notin W_0$. This is because any element $g \in V_1$ can be *uniquely* written as

   $$g = \underbrace{g_1}_{\in V_0} + \underbrace{g_2}_{\in W_0} \Rightarrow g - f = \underbrace{g_1 - f}_{\in V_0} + \underbrace{g_2}_{\in W_0}.$$

   Hence $g - f$ does not lie in W_0 unless $f = g_1$ is the projection of g in V_0.

2. PROBLEM SET 6.2, PP. 193.

   (a) PROBLEM 7. The first two steps of the cascade algorithm are plotted in Figure 1.
   The ith iteration of the cascade algorithm can be easily verified to be:

   $$\varphi^{(i)}(x) = \sum_{k=0}^{2^i-1} \varphi^{(0)}(2^i x - k),$$

   from which we obtain

   $$\varphi^{(i)}(x) = \begin{cases} \frac{1}{2} & 0 \le x < 2^{-i} \\ 1 & 2^{-i} \le x < 1 \\ \frac{1}{2} & 1 \le x < 1 + 2^{-i} \\ 0 & \text{elsewhere} \end{cases}.$$

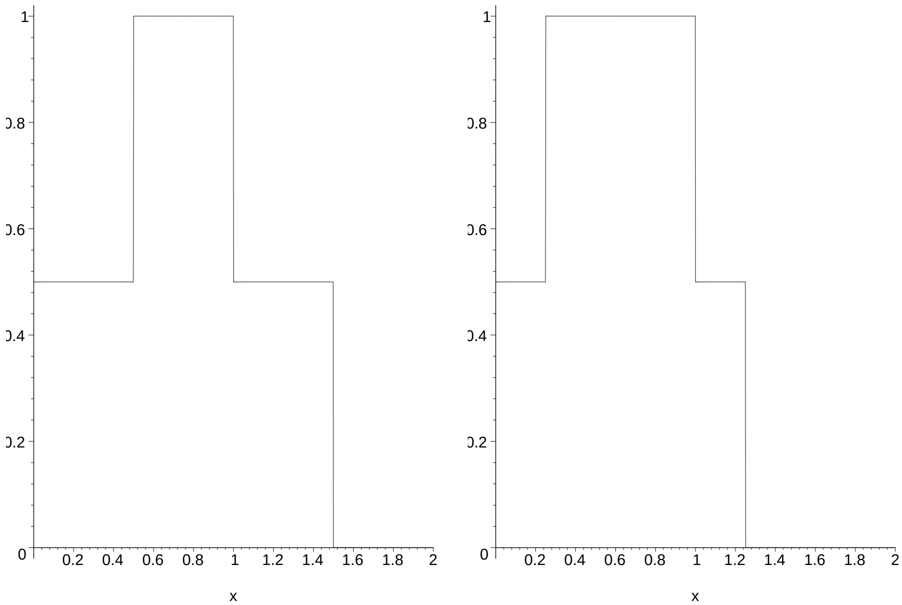

Figure 1: First two steps in the cascade algorithm: $\varphi^{(1)}(x)$ and $\varphi^{(2)}(x)$

The sixth iteration of the cascade algorithm is shown in Figure 2. It can be seen that the algorithm eventually converges to the box function on the unit interval.

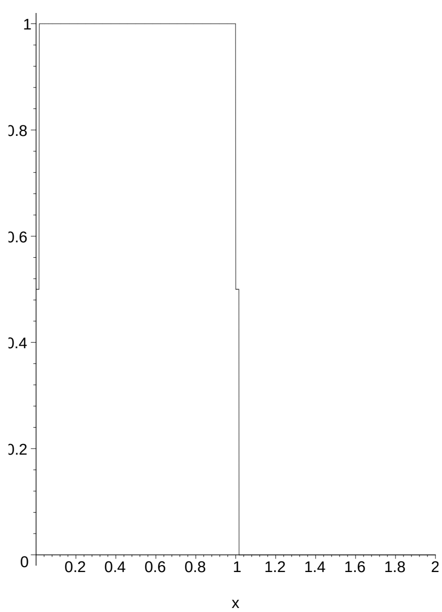

Figure 2: Sixth step of cascade algorithm: $\varphi^{(6)}(x)$.

(b) PROBLEM 8. The first two steps in the cascade algorithm are plotted in Figure 3. At the ith iteration step, we have

$$\varphi^{(i)}(x) = \begin{cases} 1 & \frac{k}{2^i} < x \leq \frac{k+\frac{1}{2}}{2^i}, k = 0, \ldots, 2^i - 1 \\ -1 & \frac{k+\frac{1}{2}}{2^i} < x \leq \frac{k+1}{2^i}, k = 0, \ldots 2^i - 1. \\ 0 & \text{elsewhere} \end{cases}$$

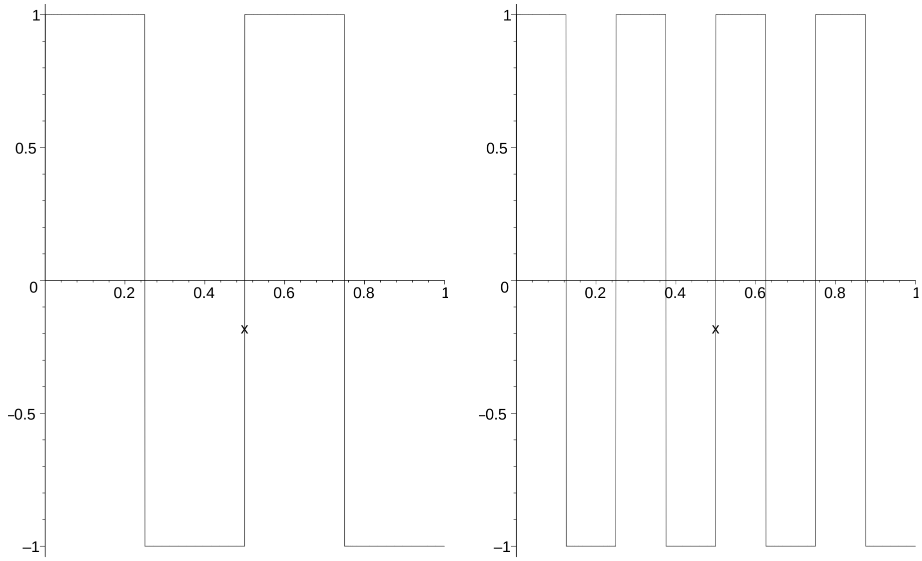

Figure 3: First two step of cascade algorithm: $\varphi^{(1)}(x)$ and $\varphi^{(2)}(x)$.

For an integrable function we have

$$\lim_{i\to\infty} \int \varphi^{(i)}(x)\ f(x)\ \ dx = 0 = 0 \cdot \int \varphi(x)\ f(x)\ \ dx = 0 \cdot \int f(x)\ \chi_{[0,1]}\ \ dx.$$

The required constant is therefore 0.

3. PROBLEM SET 6.3, PP. 201.

(a) PROBLEM 2. Let the two scaling functions satisfy refinement equations of the form:

$$\begin{aligned} \varphi_1(x) &= 2\sum_k h_1[k]\varphi_1(2x-k) \\ \varphi_2(x) &= 2\sum_l h_2[l]\varphi_2(2x-l) \end{aligned} \quad . \tag{1}$$

Their product can be expanded out as

$$\begin{aligned} P(x) &= 4\sum_k \sum_l h_1[k]h_2[l]\ \varphi_1(2x-k)\varphi_2(2x-l) \\ &= 2\sum_k \left(2h_1[k]h_2[k]\right) P(2x-k) + 4\sum_k \sum_{l\neq k} h_1[k]h_2[l]\varphi_1(2x-k)\varphi_2(2x-l), \end{aligned}$$

which implies that the product $P(x)$ does not satisfy a dilation equation in general since the second term in the summation need not vanish for any two scaling functions.

(b) PROBLEM 3. For the two scaling functions in Eq.(1),

$$(\varphi_1 * \varphi_2)(x) = 4\int \varphi_1(X)\varphi_2(x-X)\,dX = 4\sum_k\sum_l h_1[k]h_2[l]\int \varphi_1(2X-k)\varphi_2(2x-2X-l)\,dX$$

Let $2X-k=Y$ and $k+l=n$. The above expression can now be simplified as:

$$(\varphi_1 * \varphi_2)(x) \quad = \quad 2\sum_n\sum_l h_1[n-l]h_2[l]\int \varphi_1(Y)\varphi_2(2x-Y-n)\,dY$$

Hence the filter in the refinement relation is $\mathbf{h}_1 \star \mathbf{h}_2$.
This problem can also be solved by taking the Fourier transforms of both sides of Eq.(1).

4. PROBLEM SET 6.4, PP. 208.

(a) PROBLEM 4. By differentiating the infinite product formula, Equation (6.45) of the text we have

$$\frac{d^i}{d\omega^i}\hat{\Phi}(\omega) = \sum_{k=1}^{\infty}\frac{1}{2^{k+i}}\frac{d^i}{d\omega^i}H\left(\frac{\omega}{2^k}\right)\prod_{k\neq j=1}^{\infty}H\left(\frac{\omega}{2^j}\right)$$

Substituting $\omega = 2^l(2m+1)\pi, l \geq 0$,

$$\left.\frac{d^i}{d\omega^i}\hat{\Phi}(\omega)\right|_{2^l(2m+1)\pi} = \sum_{k=1}^{\infty}\frac{1}{2^{k+i}}\left.\frac{d^i}{d\omega^i}H(\omega)\right|_{2^{l-k}\pi}\prod_{k\neq j=1}^{\infty}H(\omega)|_{2^{l-j}\pi}$$

Now, for $0 \leq i \leq p$, all terms in the summation on the RHS vanish since $H(\omega)$ has p zeros at π, i.e., $\hat{\Phi}(\omega)$ has p zeros at $\omega = 2\pi n, n > 0$.

5. PROBLEM SET 6.5, PP. 218.

(a) PROBLEM 4. Note that the problem specifies two different normalizations for the analysis and synthesis refinement and wavelet equations. To avoid confusion, we may renormalize the filters and adjust the delays to give a new set of filters:

$$\begin{array}{rclcrcl} H_0(z) & = & \sqrt{2} & \quad & F_0(z) & = & \sqrt{2}\left(\frac{1}{4}z+\frac{1}{2}+\frac{1}{4}z^{-1}\right) \\ H_1(z) & = & \sqrt{2}\left(-\frac{1}{4}+\frac{1}{2}z-\frac{1}{4}z^2\right) & & F_1(z) & = & \sqrt{2}z^{-1} \end{array},$$

which can be verified to satisfy both the conditions (Alias Cancellation and No Distortion) for perfect reconstruction. Recall that the refinement and wavelet relations for biorthogonal filter-banks can be written as:

$$\begin{array}{rclcrcl} \tilde{\varphi}(x) & = & \sqrt{2}\sum h_o[-n]\tilde{\varphi}(2x-n) & \quad & \varphi(x) & = & \sqrt{2}\sum f_0[n]\varphi(2x-n) \\ \tilde{w}(x) & = & \sqrt{2}\sum h_1[-n]\tilde{\varphi}(2x-n) & & w(x) & = & \sqrt{2}\sum f_1[n]\varphi(2x-n) \end{array}$$

Now by iterating on the cascade algorithm, it is easy to see that the analysis scaling function corresponds to the Dirac distribution, $\delta(x)$ while the synthesis scaling function corresponds to the hat function centered

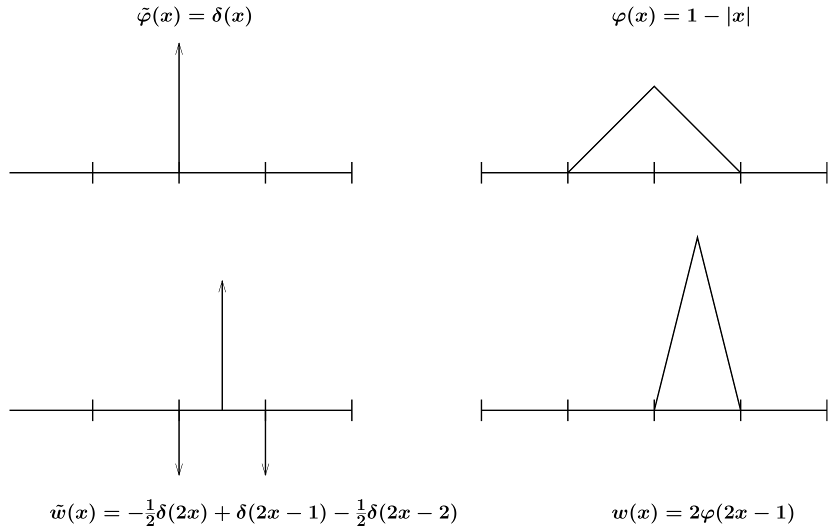

Figure 4: Scaling functions and wavelets for Problem 6.5.4.

at $x = 0$. The scaling functions and wavelets generated by the above filter-bank are shown in Figure 4. We can now directly verify the biorthogonality conditons:

$$\begin{aligned} \int \varphi(x)\tilde{\varphi}(x-k)\, dx &= \delta_{k,0} \\ \int w(x)\tilde{w}(x-k)\, dx &= \delta_{k,0} \\ \int \varphi(x)\tilde{w}(x-k) &= 0 \\ \int w(x)\tilde{\varphi}(x-k) &= 0 \end{aligned}$$

(b) PROBLEM 7. From the primary lifting equation,

$$H_0^{\#}(z) = H_0(z) + S(z^2)H_1(z).$$

Note that the second term in the the RHS involves upsampling $S(z)$ and convolving the result with $H_1(z)$. Hence, the time-domain representation of the above equation can be given as:

$$h_0^{\#}[k] = h_0[k] + \sum_l s[l]h_1[k-2l]$$

Now,

$$\begin{aligned} a_j^{\#}[n] &= \sum_k a_{j+1}[k]h_0^{\#}[k-2n] \\ &= \sum_k a_{j+1}[k]\left(h_0[k-2n] + \sum_m s[m]h_1[k-2m-2n]\right) \\ &= a_j[n] + \sum_m s[m]b_j[m+n] \\ &= a_j[n] + \sum_l s[l-n]b_j[l] \end{aligned}$$

Since the lifting step does not change the highpass analysis filter, the coefficients $b_j[n]$ remain unaltered.

This can be represented in matrix notation as

$$\begin{bmatrix} \mathbf{a}_j^{\#} \\ \mathbf{b}_j^{\#} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} \end{bmatrix}}_{\mathbf{L}} \begin{bmatrix} \mathbf{a}_j \\ \mathbf{b}_j \end{bmatrix}$$

Hence, the coefficients corresponding to the initial (unlifted) set of filters can be recovered as

$$\begin{bmatrix} \mathbf{a}_j \\ \mathbf{b}_j \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{I} & -\mathbf{S} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}}_{\mathbf{L}^{-1}} \begin{bmatrix} \mathbf{a}_j^{\#} \\ \mathbf{b}_j^{\#} \end{bmatrix}$$

At this point, the reconstruction equations for the original set of filters holds good.

(c) PROBLEM 9. A linear interpolating subdivision requires

$$\begin{aligned} y[2n] &= x[n] \\ y[2n+1] &= \frac{1}{2}\left(x[n] + x[n+1]\right). \end{aligned}$$

Hence

$$\begin{aligned} Y(z) &= \sum_{n\in\mathbb{Z}} y[2n]z^{-2n} + \sum_{n\in\mathbb{Z}} y[2n+1]z^{-(2n+1)} \\ &= \sum_{n\in\mathbb{Z}} x[n]z^{-2n} + \frac{z^{-1}}{2}\sum_{n\in\mathbb{Z}} x[n]z^{-2n} + \frac{z}{2}\sum_{n\in\mathbb{Z}} x[n+1]z^{-(2n+2)} \\ &= X\left(z^2\right)\left(\frac{1}{2}z + 1 + \frac{1}{2}z^{-1}\right) = X\left(z^2\right)H_0(z). \end{aligned}$$

Therefore the linear half-band filter gives rise to linear-interpolating subdivision. The corresponding scaling function is the hat-function.

6. PROBLEM SET 7.1, PP. 233.

(a) PROBLEM 8. By assumption

$$\int t^k \, w(t) \, dt = 0 \quad k = 0, \ldots p-1.$$

Now, the Fourier transform of $w(t)$ can be written as

$$W(\omega) = \int w(t)e^{-i\omega t} \, dt.$$

Letting $\theta = -i\omega$ and expanding the exponential term as

$$e^{-i\omega t} = e^{\theta t} = \sum_{k=0}^{\infty} \frac{\theta^k t^k}{k!}.$$

Hence

$$W(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!}\theta^k \int w(t)\, t^k \; dt = \theta^p \sum_{k=0}^{\infty} \frac{1}{(k+p)!}\theta^k \int w(t)\, t^{k+p} \; dt.$$

Note that interchanging the integral and the infinite sum can be justified by a careful application of Fubini's theorem. We can therefore observe that $W(\omega)$ has p zeros at $\omega = 0$.

7. PROBLEM SET 7.2, PP. 242.

(a) PROBLEM 2. For $h[n] = \frac{1}{3}\left\{1, 1, 1\right\}$, $t[n] = h[n] \star h[-n] = \frac{1}{9}\left\{1, 2, 3, 2, 1\right\}$. Hence the transition matrix is:

$$\mathbf{T}_3 = \frac{2}{9}\begin{bmatrix} 2 & 1 & 0 \\ 2 & 3 & 2 \\ 0 & 1 & 2 \end{bmatrix}.$$

For $h[n] = \left\{c, \frac{1}{2}, \frac{1}{2}, -c\right\}$, $t[n] = \left\{-2c^2, 0, \frac{1}{2}, 4c^2+1, \frac{1}{2}, 0, -2c^2\right\}$. In this case, the transition matrix can be written as:

$$\mathbf{T}_5 = \begin{bmatrix} 0 & -2c^2 & 0 & 0 & 0 \\ 4c^2+1 & \frac{1}{2} & 0 & -2c^2 & 0 \\ 0 & \frac{1}{2} & 4c^2+1 & \frac{1}{2} & 0 \\ 0 & -2c^2 & 0 & \frac{1}{2} & 4c^2+1 \\ 0 & 0 & 0 & -2c^2 & 0 \end{bmatrix}$$

8. PROBLEM SET 7.3, PP. 250.

(a) PROBLEM 7. The product filter $P^{\#}(z)$ is given as

$$P^{\#}(z) = F(z)H^{\#}(z) = \frac{1}{1024}\left(-1 - 14z^{-2} + 255z^{-4} + 512z^{-5} + 380z^{-6} - 95z^{-8} + 18z^{-10} - z^{-12}\right).$$

The product filter is clearly half-band and the system delay is $l = 5$.
The transition matrix $\mathbf{T}^{\#}$ may now be written as:

$$\mathbf{T}^{\#} = \frac{1}{512}\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 32 & 14 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 224 & -193 & 32 & 14 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 224 & 868 & 224 & -193 & 32 & 14 & 0 & 1 & 0 & 0 & 0 \\ 32 & -193 & 224 & 868 & 224 & -193 & 32 & 14 & 0 & 1 & 0 \\ 0 & 14 & 32 & -193 & 224 & 868 & 224 & -193 & 32 & 14 & 0 \\ 0 & 1 & 0 & 14 & 32 & -193 & 224 & 868 & 224 & -193 & 32 \\ 0 & 0 & 0 & 1 & 0 & 14 & 32 & -193 & 224 & 868 & 224 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 14 & 32 & -193 & 224 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 14 & 32 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

whose eigenvalues are given as

$$\lambda = \left\{1.2598, 1.0000, -0.5213, -0.4691, 0.5000, -0.0028, -0.0026, 0.0761, 0.0667, 0.0434, 0.0460\right\}.$$

The smoothness, $s = -\log_4 \lambda_{\max} = -\log_4 1.2598 = 0.1666$.
The scaling function $\tilde{\varphi}(x)$ is interpolating since all the odd powers of $H^{\#}(z)$ are zero except z^{-3}.

(b) PROBLEM 8. The Dirac distribution satisfies a refinement relation with $h_0[n] = \{1\}$. Hence, the transition matrix, $\mathbf{T} = 2$ and the smoothness, $s = -\log_4(2) = -\frac{1}{2}$. Hence the smoothness of the Dirac distribution is $-\frac{1}{2}$.

9. PROBLEM SET 7.5, PP. 262.

(a) PROBLEM 4. The functions $\{\varphi_1(x), \varphi_2(x)\}$ shown in Figure 7.7 are piecewise cubic and Hermite interpolating in nature. That is, they satisfy,

$$\begin{array}{rclcrclcrcl} \varphi_1(x)|_{x=1} & = & 1 & \text{and} & \varphi_1(x)|_{x=0,2} & = & 0 & \text{with} & \frac{d}{dx}\,\varphi_1(x)|_{x=0,1,2} & = & 0 \\ \frac{d}{dx}\,\varphi_2(x)|_{x=1} & = & 1 & \text{and} & \frac{d}{dx}\,\varphi_2(x)|_{x=0,2} & = & 0 & \text{with} & \varphi_2(x)|_{x=0,1,2} & = & 0 \end{array}$$

The closed form expressions for the two scaling functions can be determined as:

$$\begin{aligned} \varphi_1(x) &= \begin{cases} -2x^3 + 3x^2 & 0 \le x \le 1 \\ 2x^3 - 9x^2 + 12x - 4 & 1 \le x \le 2 \end{cases} \\ \varphi_2(x) &= \begin{cases} x^3 - x^2 & 0 \le x \le 1 \\ x^3 - 5x^2 + 8x - 4 & 1 \le x \le 2 \end{cases} \end{aligned}$$

To derive the matrix refinement relation for the scaling functions, note that $\varphi_1(2x)$ and $\varphi_2(2x)$ are non-zero between 0 and 1. Hence the interval from 0 to 2 can be spanned by three pairs of scaling functions, $\{\varphi_1(2x), \varphi_2(2x)\}$, $\{\varphi_1(2x-1), \varphi_2(2x-1)\}$ and $\{\varphi_1(2x-2), \varphi_2(2x-2)\}$. Observe that the dilated version of the scaling functions are Hermite interpolating on half intervals.

Let the refinement relation be

$$\left\{\begin{array}{c} \varphi_1(x) \\ \varphi_2(x) \end{array}\right\} = \mathbf{C}[0]\left\{\begin{array}{c} \varphi_1(2x) \\ \varphi_2(2x) \end{array}\right\} + \mathbf{C}[1]\left\{\begin{array}{c} \varphi_1(2x-1) \\ \varphi_2(2x-1) \end{array}\right\} + \mathbf{C}[2]\left\{\begin{array}{c} \varphi_1(2x-2) \\ \varphi_2(2x-2) \end{array}\right\}$$

At $x = 1$, the only contribution to the refinement relation comes from $\varphi_1(2x-1)$ and $\varphi_2(2x-1)\}$.

$$\begin{array}{rclcl} \varphi_1(x)|_{x=1} & = & 1 & = & \varphi_1(2x-1)|_{x=1} \\ \frac{d}{dx}\,\varphi_1(x)|_{x=1} & = & 0 & & \\ \varphi_2(x)|_{x=1} & = & 0 & & \\ \frac{d}{dx}\,\varphi_2(x)|_{x=1} & = & 1 & = & \frac{1}{2}\frac{d}{dx}\,\varphi_2(2x-1)|_{x=1} \end{array}$$

Hence, $\mathbf{C}[1] = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$. At $x = \frac{1}{2}$ the only contribution to the refinement relation comes from $\varphi_1(2x)$ and $\varphi_2(2x)$. Hence, by comparing the values and derivatives of the scaling functions at the coarser and finer levels, we obtain $\mathbf{C}[0] = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix}$. Similarly, by comparing the value and derivative of the scaling functions at $x = \frac{3}{2}$ gives $\mathbf{C}[2] = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix}$