CHAPTER FIVE

MULTIPLE SCATTERING BY AN EXTENDED REGION OF INHOMOGENEITIES

1 Bragg scattering in a periodic medium

Longshore sand bars are often found along gentle beaches. The number of bars can range from a few to dozens and the spacing from tens to hundreds of meters. The bar amplitudes can be as high as a meter. Figure 1 shows a sample profile in Chesapeake Bay, Maryland, USA, recorded by accoustic sounding (Dolan, 1982).

Scientifically it is natural to ask how these sand bars are generated and how the bars affect the propagation of waves. In fact these two questions are coupled through the complex dynamics of sediment transport. It is easier to just consider the second question.

If the bar amplitudes are small, $D \ll h$, one might expect their effects on a train of progresseive waves to be small and apply a straightforward second-order analysis so that the effect on waves will appear at the order $kA(KD)$. The situation is different if however the incident waves are twice as long as the bar spacing, i.e., $K = 2k$ then the phenomenon of Bragg resonance occurs and the reflection by many small and periodic bars can be very strong. The source of this resonance is due to constructive interference of incident and reflected waves and is well known in x-ray diffraction by crystaline materials. Referring to Fig. 2 where a number of bars of wavelength λ_b are fixed on a horizontal bed, we consider the propagation of a train of waves incident from the left. Every wave crest passing over a bar will be mostly transmitted toward the next bar ahead and sends a weak reflected wave towards the bar behind. At any given bar crest, say $\mathcal{B}$, the total amplitude of the left-going wave is the sum of all left-going wave crests each of which is the consequence of reflection by the nth bar on the right. Therefore each of these crests has traveled the distance of $2n\lambda_b$. When

$2\lambda_b$ equals the surface wavelength, λ, all these reflected crests are in phase upon arival at $\mathcal{B}$, and reinforce one another, resulting in strong (resonant) reflection. Thus many small bars can give rise to strong reflection if the Bragg resonance condition is met.

Bragg resonance is of interest in many branches of physics. In crystalography, the phenomenon is used to study the structure of a crystaline solid by x-rays.

Let us use a one-dimensional example to describe the phenomenon. First, since many scatters must be involved in order for this phenomenon to be appreciable, the total region of disturbances must be much greater than the typical wavelength. The perturbation method of multiple scales can be used. Second, since reflection is strong, incident and reflected wave must be allowed to be comparable in order.

Let us consider the one-dimensional scattering of elastic waves in a rod with a slightly periodic elasticity,

$$\rho = \text{constant}, \quad E = E_o(1 + \epsilon D \cos Kx), \tag{1.1}$$

where D is of order unity, i.e.,

$$E_0 \frac{\partial}{\partial x}\left[(1 + \epsilon D \cos Kx)\frac{\partial u}{\partial x}\right] = \rho\frac{\partial^2 u}{\partial t^2} \tag{1.2}$$

We now assume that the spatial period of inhomogeneity $\ell \equiv 2\pi/K$ and the elastic wavelength $\ell' \equiv 2\pi/k = 2\pi\sqrt{E_o/\rho}/\omega$ are comparable. As a consequence, wave reflection can be significant.

Let us first try a naive expansion, $u = u_0 + \epsilon u_1 + \cdots$. The crudest solution is easily found to be

$$u_0 = \frac{A}{2}e^{ikx-i\omega t} + \text{c.c.}, \tag{1.3}$$

where c.c. signifies the complex conjugate of the preceding term, and

$$\frac{2\pi}{k} \equiv \sqrt{\frac{E_o}{\rho}}\frac{2\pi}{\omega}, \quad \text{or} \quad \frac{\omega}{k} = C = \sqrt{\frac{E_0}{\rho}}. \tag{1.4}$$

At the next order the governing equation is

$$\frac{\partial}{\partial x}\left(E_o\frac{\partial u_1}{\partial x}\right) - \rho\frac{\partial^2 u_1}{\partial t^2} = \frac{-E_oD}{2}\frac{\partial}{\partial x}\left[\left(e^{iKx} + e^{-iKx}\right)\frac{\partial u_0}{\partial x}\right]$$

$$= \frac{-E_oD}{2}\frac{\partial}{\partial x}\left[\left(e^{iKx} + e^{-iKx}\right)\left(\frac{ikA}{2}e^{ikx-i\omega t} - \frac{ikA_*}{2}e^{-ikx+i\omega t}\right)\right]. \tag{1.5}$$

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Clearly, when

$$K = 2k + \delta, \quad \delta \ll k, \tag{1.6}$$

some of the forcing terms on the right will be close to a natural mode $\exp(\pm i(kx + \omega t))$. Resonance of the reflected waves must be expected. It sufices to illustrate the response to one of these terms,

$$E_o \frac{\partial^2 u_1}{\partial x^2} - \rho \frac{\partial^2 u_1}{\partial t^2} = A e^{i\phi_o} e^{i\delta x}, \quad \text{with} \quad \phi_o = kx + \omega t.$$

Combining homogeneous and inhomogeneous solutions and requiring that $u_1(0, t) = 0$, we find

$$u_1 = \frac{A e^{i\phi_o} \left(1 - e^{i\delta x}\right)}{E_o((k + \delta)^2 - k^2)}.$$

Clearly if $\delta = O(\epsilon)$, $\epsilon u_1 \sim O(\epsilon/\delta)$ and is not small compared to u_0 except for $\delta x \ll 1$. Furthermore as x increases, u_1 grows as ϵx. This implies that the reflected waves are resonated and is no longer much smaller that the incident waves in the distance $\epsilon x = O(1)$. The relation $2K = k$ (cf. (1.6) is the well-known condition for Bragg resonance.

Let us now focus attention on the case of Bragg resonance. To render the solution uniformly valid for all x, we introduce fast and slow variables in space

$$x, \bar{x} = \epsilon x \tag{1.7}$$

To allow slight detuning from exact resonance, we assume that the incident wave frequency is $\omega + \epsilon \omega'$, where $\epsilon \omega'$ represents the small detuning and gives rise to a very slow variation in time. Therefore two time variables are needed,

$$t, \bar{t} = \epsilon t \tag{1.8}$$

The following multiple scale expansion is then proposed,

$$u = u_0(x, \bar{x}; t, \bar{t}) + \epsilon u_1(x, \bar{x}; t, \bar{t}) + \cdots . \tag{1.9}$$

After making the changes

$$\frac{\partial}{\partial x} \to \frac{\partial}{\partial x} + \epsilon \frac{\partial}{\partial \bar{x}}, \quad \frac{\partial}{\partial t} \to \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial \bar{t}} \tag{1.10}$$

and substituting (1.9), (1.10) into (1.2), we get

$$\frac{\partial}{\partial x}\left(E_o\frac{\partial u_0}{\partial x}\right) - \rho\frac{\partial^2 u_0}{\partial t^2} = 0 \tag{1.11}$$

at $O(1)$. Anticipating strong but finite reflection, we take the solution to be

$$u_0 = \frac{A}{2}e^{ikx-i\omega t} + * + \frac{B}{2}e^{-ikx-i\omega t} + \text{c.c.}. \tag{1.12}$$

where $A(x_1, t_1)$ and $B(x_1, t_1)$ vary slowly in space and time. At the order $O(\epsilon)$ we have

$$\begin{aligned}
\frac{\partial}{\partial x}&\left(E_o\frac{\partial u_1}{\partial x}\right) - \rho\frac{\partial^2 u_1}{\partial t^2} = -2E_o\frac{\partial^2 u_0}{\partial x\partial\bar{x}} + 2\rho\frac{\partial^2 u_0}{\partial t\partial\bar{t}} \\
&-\frac{E_o D}{2}\frac{\partial}{\partial x}\left[\left(e^{2ikx} + e^{-2ikx}\right)\frac{\partial u_0}{\partial x}\right] \\
= &-E_o\left[\frac{\partial A}{\partial\bar{x}}(ik)e^{ikx-i\omega t} + \text{c.c.} + \frac{\partial B}{\partial\bar{x}}(-ik)e^{-ikx-i\omega t} + \text{c.c.}\right] \\
&+\rho\left[\frac{\partial A}{\partial\bar{t}}(-i\omega)e^{ikx-i\omega t} + \text{c.c.} + \frac{\partial B}{\partial\bar{t}}(-i\omega)e^{-ikx-i\omega t} + \text{c.c.}\right] \\
&-\frac{E_o D}{4}\frac{\partial}{\partial x}\left\{\left(e^{2ikx} + \text{c.c.}\right)\frac{\partial}{\partial x}\left[Ae^{ikx-i\omega t} + \text{c.c.} + Be^{-ikx-i\omega t} + \text{c.c.}\right]\right\} \tag{1.13}
\end{aligned}$$

The last line can be reduced to

$$\begin{aligned}
-\frac{E_o D}{4}\Big(&k^2 Be^{ikx-i\omega t} + \text{c.c.} + k^2 Ae^{-ikx-i\omega t} + \text{c.c.} \\
&-3k^2 Ae^{3ikx-i\omega t} + \text{c.c.} - 3k^2 Be^{-3ikx-i\omega t} + \text{c.c.}\Big)
\end{aligned}$$

To avoid unbounded resonance of u_1, i.e., to ensure the solvability of u_1, we equate to zero the coefficients of terms $e^{\pm i(kx-\omega t)}$ and $e^{\pm i(kx+\omega t)}$ on the right of (1.13). The following equations are then obtained:

$$\frac{\partial A}{\partial\bar{t}} + c\frac{\partial A}{\partial\bar{x}} = \frac{ikCD}{4}B \tag{1.14}$$

$$\frac{\partial B}{\partial\bar{t}} - c\frac{\partial B}{\partial\bar{x}} = \frac{ikCD}{4}A, \tag{1.15}$$

where $\sqrt{E_o/\rho_o} = C = \omega/k$ denotes the phase speed. These equations govern the macroscale variation of the envelopes of the incident and reflected waves, and can be combined to give the Klein-Gordon equation

$$\frac{\partial^2 A}{\partial\bar{t}^2} - C^2\frac{\partial^2 A}{\partial\bar{x}^2} + \left(\frac{kCD}{4}\right)^2 A = 0. \tag{1.16}$$

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Note that

$$\frac{kCD}{4} = \frac{\omega D}{4} \equiv \Omega_0 \qquad (1.17)$$

has the dimension of frequency.

With suitable initial and boundary conditions on the macro scale, one finds the slow variation of these wave envelopes, hence the global behaviour of wave motion.

Let the inhomogeneity of wavenumber $2k$ be confined in $0 < \bar{x} < L$ and the incident wave train be slightly detuned from resonance, so that the wave frequency is $\omega + \epsilon\Omega$ and the wavenumber is $k + \epsilon K$, where $\Omega = O(\omega)$ and $K = O(k)$. Since $\omega + \epsilon\Omega$ and $k + \epsilon K$ must be related by the dispersion relation (1.4),

$$\Omega = KC. \qquad (1.18)$$

The detuned incident wave

$$\zeta = A_o \exp[i(k + \epsilon K)x - (\omega + \epsilon\Omega)t] + *, \quad \bar{x} < 0, \qquad (1.19)$$

can be alternatively written as

$$\zeta = A(\bar{x}, \bar{t})e^{ikx - i\omega t}, \quad \bar{x} < 0, \qquad (1.20)$$

where

$$A(\bar{x}, \bar{t}) = A_o e^{iK(\bar{x} - C\bar{t})}, \quad \bar{x} < 0. \qquad (1.21)$$

When such a wavetrain passes a patch of periodic bars, A and B must vary with $\bar{x}$ and $\bar{t}$ according to (1.14) and (1.15).

To the left and to the right of the bars, the governing equations are simply

$$A_{\bar{t}} + CA_{\bar{x}} = 0, \quad B_{\bar{t}} - CB_{\bar{x}} = 0, \quad \bar{x} < 0, \quad \text{and} \quad \bar{x} > L. \qquad (1.22)$$

We shall assume further that $B = 0$ for $\bar{x} > L$. Over the bars (1.14) and (1.15), or (1.16) hold. In order that displacement and stress and horizontal velocity be continuous at $x = 0, L$, A and B must be continuous at $x = 0, L$. Since the solutions must be of the form,

$$(A, B) = A_0(T(\bar{x}), R(\bar{x}))e^{-i\Omega\bar{t}}, \quad 0 < \bar{x} < L.$$

T and R are governed by

$$T_{\bar{x}\bar{x}} + \frac{(\Omega^2 - \Omega_0^2)}{C}T = 0, \quad 0 < \bar{x} < L.$$

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Several cases can be distinguished according to the sign of $\Omega^2 - \Omega_0^2$:

Subcritical detuning: $0 < \Omega < \Omega_0$.

Let

$$Qc = (\Omega_0^2 - \Omega^2)^{1/2} \tag{1.23}$$

then

$$T(x) = \frac{iQC \cosh Q(L - \bar{x}) + \Omega \sinh Q(L - \bar{x})}{iQC \cosh QL + \Omega \sinh QL} \tag{1.24}$$

and

$$R(x) = \frac{Q \sinh Q(L - \bar{x})}{iQC \cosh QL + \Omega \sinh QL} . \tag{1.25}$$

On the incidence side the reflection coefficient is just $R(0)$ and on the transmission side the transmission coefficient is $T(L)$. Clearly the dependence on L and $\bar{x}$ is monotonic. In the limit of $L \to \infty$, it is easy to find that

$$T(x) = e^{-Q\bar{x}}, \quad R(x) = \frac{Q}{iQC + \Omega} e^{-Q\bar{x}} . \tag{1.26}$$

Thus all waves are localized in the range $\bar{x} < O(1/Q)$.

Supercritical detuning: $\Omega > \Omega_0$.

Let

$$Pc = (\Omega^2 - \Omega_0^2)^{1/2} \tag{1.27}$$

then the transmission and reflection coifficients are:

$$T(x) = \frac{PC \cos P(L - \bar{x}) - i\Omega \sin P(L - \bar{x})}{PC \cos PL - i\Omega \sin PL} \tag{1.28}$$

and

$$R(x) = \frac{-iQ_0 \sin P(L - \bar{x})}{PC \cos PL - i\Omega \sin PL} . \tag{1.29}$$

The dependence on L and $\bar{x}$ is clearly oscillatory. Thus Ω_0 is the cut-off frequency marking the transition of the spatial variation. For subcritical detuning complete reflection can occur for sufficinetly large L. For super-critical detuning there can be windows of strong transsission.

In the special case of perfect resonance, we get from (1.24) and (1.25) that

$$T(\bar{x}) = \frac{A}{A_o} = \frac{\cosh \frac{\Omega_o(L - \bar{x})}{C}}{\cosh \frac{\Omega_o L}{C}} \quad R(\bar{x}) = \frac{B}{A_o} = -\frac{i \sinh \frac{\Omega_o(L - x)}{C}}{\cosh \frac{\Omega_o L}{C}} . \tag{1.30}$$

In a laboratory experiment for water waves, Heathershaw(1982) installed 10 sinusoidal bars of amplitude $D = 5$ cm and wavelength 100 cm on the bottom of a long wave flume. Incident waves of length $2\pi/k = 200$ cm were sent from one side of the bar patch. On the transmission side, waves are essentially absorbved by breaking on a gentle beach. Sizable reflection coefficients were measured along many stations over the bar patch. This experiment gives the first observed evidence of strong reflection by periodic bars. Let us apply the present theory to a more general case where the normally incident wave is slightly detuned from perfect resonance.

Clearly they both decrease monotonically from $\bar{x} = 0$ to $\bar{x} = L$. These results agree quite well with the experiments of Heathershaw, as shown in Fig. 3, therefore confirm that enough small bars can generate strong reflection, especially in very shallow water.

Exercise 5.1: Bragg resonance by a corrugated river bank.

An infinitely long river has contant depth h and contant averaged width $2a$. In the stretch $0 < x < L$, the banks are slightly sinusoidal about the mean so that

$$y = \pm a \pm B \sin Kx, \quad KB \equiv \epsilon \ll 1. \tag{1.31}$$

See Fig. 4. Let a train of monochromatic waves be incident from $x \sim -\infty$,

$$\zeta = \frac{A}{2} e^{i(kx - \omega t)} \tag{1.32}$$

where $kh, ka = O(1)$. Develop a uniformly valid linearized theory to predict Bragg resonance. Can the corrugated boundary be used to reflect waves as a breakwater? Discuss your results for various parameters that can affect the function as a breakwater.

2 Wave localization in a random medium

[Ref]: Mei & Pihl Localization of nonlinear dispersive waves in weakly random media, *Proc. Roy. Soc. Lond.* 2002, 458, 119-134.

There are numerous situations where one needs to know how waves propagate through a medium with random impurities: light through sky with dust particles, sound through water with bubbles, elastic waves through a solid with cracks, fibers, cavities, hard or soft grains. Sea waves over a irregular topography, etc. In these situations several kinds

of questions can be of physical interest : determinstic (sinusoidal or impulsive) waves through a random medium, random waves through a deterministically irregular medium, and random waves through a random medium.

There is an extensive literature on the propagation of infinitesimal sinusoidal waves in random media. Based on linearized field equations, perturbation theories have been developed for cases where the random inhomogeneity is weak and the fluctuation length scale is comparable to the typical wave length (see, Chernov, 1960; Keller, 1964, Karal & Keller, 1964; Chen & Soong, 1972). Diagrammatic techniques have also been employed (Frisch, 1968; Elter & Molyneux, 1972). If the inhomogeneities extend over a large spatial region, multiple scattering yields a change in the wavenumber (or phase velocity) as well as an amplitude attenuation over a large distance. These changes amount to a shift of the complex propagation constant with the real part corresponding to the wavenumber and the imaginary part to attenuation. In particular, the spatial attenuation (localization) is a distinctive feature of randomness and is effective for a broad range of incident wave frequencies.This is in sharp contrast to periodic inhomogeneities which cause strong scattering only for certain frequency bands (Bragg scattering, see e.g., Nayfeh). Phillip W. Anderson (1958) was the first to show that the quantum- mechanical motion of a particle in a random potential can be localized in space, turning a conductor to an insulator. This phenomoena, now called *Anderson localization*, is now known to be important in classical mechanical systems too. A survey of localization in many types of classical waves based on linearized theories can be found in the monograph by Sheng (1998). For surface water waves, localization by strong inhomogeneities have also been treated by semi-numerical means for surface water waves over randomly rough seabed where the height of the roughness is comparable to the mean depth (Devillard et al, 1988; Nachbin & Papanicolaou, 1992; Nachbin, 1995). Experimental confirmation has been reported by Belzons et al (1988).

For weak inhomogeneties, the shift of propagation constant amounts to slow spatial modulations with a length scale much longer than the wavelnegth by a factor inversely proportional to the correlation of the fluctuations. In this section we use the *method of multiple scales* to examine weak random inhomogeneities. The simple case of an elastically supported string is used as an example, while extensions to other waves

can be anticipated. After deriving the envelope equation, physcial implications will be explored.

We begin with the equation for the lateral displacement of a taut string, which is buried in a linear elastic medium,

$$\rho\frac{\partial^2 V}{\partial t^2} - T\frac{\partial^2 V}{\partial x^2} + K(1 + \epsilon M(x))V = 0 \tag{2.1}$$

V denotes the lateral displacement, ρ the mass per unit length, T tension in the string, K the mean spring constant of the surrounding medium, $\epsilon K M(x)$ the random fluctuations of the linear spring force. We assume that M has zero mean and the typical length scale of $O(1/k)$. For the sake of demonstration we have chosen to let the linear part of the spring to contain random irregularities. In principle the randomness can appear in the density ρ also.

Since the correlation of a random function is proportional to the square of the amplitude of random fluctuations, the length scale of modulation due to randomness must be of the order $O(1/k\epsilon^2$. Let us introduce fast and slow variables $x, x_2 = \epsilon^2 x$ and further assume two-vairiable expansions,

$$V = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots, \quad \text{with} \;\; V_n = V_n(x, x_2, t), n = 0, 1, 2, ... \tag{2.2}$$

In the multiplescale formalism, we first pretend the two variables to independent, then use the definition of the slow variable x_2. In particular, we must make the following replacement:

$$\frac{\partial F(x,t)}{\partial x} \rightarrow \frac{\partial F(x, x_2, t)}{\partial x} + \frac{\partial F(x, x_2, t)}{\partial x_2}\frac{\partial x_2}{\partial x} = \frac{\partial F}{\partial x} + \epsilon^2\frac{\partial F}{\partial x_2} \tag{2.3}$$

The following perturbation equations result:

$O(\epsilon^0)$:

$$\rho\frac{\partial^2 V_0}{\partial t^2} - T\frac{\partial^2 V_0}{\partial x^2} + KV_0 = 0 \tag{2.4}$$

$O(\epsilon)$:

$$\rho\frac{\partial^2 V_1}{\partial t^2} - T\frac{\partial^2 V_1}{\partial x^2} + KV_1 + KMV_0 = 0 \tag{2.5}$$

$O(\epsilon^2)$:

$$\rho\frac{\partial V_2}{\partial t^2} - T\frac{\partial^2 V_2}{\partial x^2} + KV_2 + KMV_1 - T\left(2\frac{\partial^2 V_0}{\partial x\partial x_2}\right) = 0. \tag{2.6}$$

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Let us take the leading-order solution to be a progressive wave

$$V_0 = A(x_2) e^{i(kx-\omega t)} \tag{2.7}$$

with the dispersion relation

$$\omega = \left(\frac{Tk^2 + K}{\rho}\right)^{1/2}, \quad C = \frac{\omega}{k} = \left(\frac{T}{\rho} + \frac{K}{\rho k^2}\right)^{1/2}. \tag{2.8}$$

At the order $O(\epsilon)$ Eqn. (2.5) can be written

$$\rho \frac{\partial^2 V_1}{\partial t^2} - T \frac{\partial^2 V_1}{\partial x^2} + KV_1 = -KM(x, x_2)Ae^{ikx-i\omega t}. \tag{2.9}$$

where the forcing term on the right is a random function of x and x_2. Let

$$V_1 = \overline{V}_1 e^{-i\omega t} \tag{2.10}$$

then

$$\frac{\partial^2 \overline{V}_1}{\partial x^2} + k^2 \overline{V}_1 = \frac{K}{T} M(x, x_2) A(x_2) e^{ikx}, \tag{2.11}$$

where use is made of

$$k = \left(\frac{\rho\omega^2 - K}{T}\right)^{1/2}. \tag{2.12}$$

From here on we consider the frequency to be above cutoff $\sqrt{K/T}$ so that k is real and positive. Equation (2.11) can be solved by using the Green function,

$$G = \frac{-i}{2k} e^{ik|x-\xi|} \tag{2.13}$$

which satisfies the radiation condition at infinities. The result is

$$\overline{V}_1 = \frac{-iKA}{2kT} \int_{-\infty}^{\infty} e^{ik|x-\xi|} M(x) e^{ik\xi} d\xi \tag{2.14}$$

so that

$$V_1 = \frac{-iKA}{2kT} e^{ikx-i\omega t} \int_{-\infty}^{\infty} d\xi e^{ik|x-\xi|} e^{-ik(x-\xi)} M(x) \tag{2.15}$$

which behaves as outgoing waves as $x - \xi \to \pm\infty$.

Since the ensemble average of $M(x)$ vanishes, i.e., $\langle M(x) \rangle = 0$, we have,

$$\langle V_1 \rangle = \langle \overline{V}_1 \rangle e^{-i\omega t} = 0. \tag{2.16}$$

The ensemble average of Eq. (2.6) becomes

$$\rho\frac{\partial^2\langle V_2\rangle}{\partial t^2} - T\frac{\partial^2\langle V_2\rangle}{\partial x^2} + K\langle V_2\rangle - T\left(2ik\frac{\partial V_0}{\partial x_2}\right)$$
$$+ \left[K\left\{\frac{-iK}{2kT}\int e^{ik|x-\xi|}\langle M(x)M(\xi)\rangle e^{-ik(x-\xi)}d\xi\right\}e^{ikx-i\omega t}\right] = 0 \qquad (2.17)$$

where $\langle M(x)M(\xi)\rangle$ is the correlation function of the irregularities. Equating the sum of all secular terms to zero, we get the evolution equation for A,

$$-2i\omega\rho\left(c\frac{\partial A}{\partial x_2}\right) - i\frac{K^2 A}{2kT}\int_{-\infty}^{\infty}\langle M(x)M(\xi)\rangle e^{ik|x-\xi|}e^{-k(x-\xi)}d\xi = 0 \qquad (2.18)$$

where where

$$c = \frac{\partial\omega}{\partial k} = \frac{Tk}{\rho\omega} \qquad (2.19)$$

is the group velocity.

As a specific example we take

$$\langle M(x)M(\xi)\rangle = \sigma^2 e^{-\alpha|x-\xi|}, \qquad (2.20)$$

thus $\epsilon\sigma$ corresponds to the root-mean-square of the fluctuation amplitude. The correlation length scale is $1/\alpha$. small α means a high degree of randomness. It can be shown that

$$\int_{-\infty}^{\infty}e^{ik|x-\xi|}\sigma^2 e^{-\alpha|x-\xi|}e^{-ik(x-\xi)}d\xi = \sigma^2\left[\frac{2(\alpha^2+2k^2)}{\alpha(\alpha^2+4k^2)} + \frac{2ik}{\alpha^2+4k^2}\right]. \qquad (2.21)$$

(Chen & Soong, 1972). Let

$$2\beta = 2(\beta_r + i\beta_i) \qquad (2.22)$$

with

$$\beta_r = -\left(\frac{K^2\sigma^2}{2\rho\omega T}\right)\frac{1}{\alpha^2+4k^2}, \quad \beta_i = \left(\frac{K^2\sigma^2}{2\rho\omega T}\right)\frac{(\alpha^2+2k^2)}{k\alpha(\alpha^2+4k^2)}. \qquad (2.23)$$

we get from (2.17)

$$2i\left(c\frac{\partial A}{\partial x_2}\right) + 2\beta A = 0. \qquad (2.24)$$

If we write

$$A = a(x_2)e^{i\theta(x_2)} \qquad (2.25)$$

where a is the magnitude and θ the phase of A. From the real part we get

$$\theta = \beta_r x_2/c \qquad (2.26)$$

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From the imaginary part we get

$$2c\frac{\partial a}{\partial x_2} + 2\beta_i a = 0,\tag{2.27}$$

hence

$$a(x_2) = A_0 \exp\left(-\frac{\beta_i x_2}{c}\right).\tag{2.28}$$

showing that the incident waves are attenuated (localized) by random irregularities. The ratio of attenuation (localization) distance is

$$L = \frac{c}{\epsilon^2 \beta_i} = \frac{2\rho\omega cTk}{\epsilon^2 K^2 \sigma^2}\frac{\alpha(\alpha^2+4k^2)}{\alpha^2+2k^2} = \frac{2T^2k^2}{\epsilon^2 K^2\sigma^2}\frac{\alpha(\alpha^2+4k^2)}{\alpha^2+2k^2}.\tag{2.29}$$

For fixed α and k, L is small (strong attenuation) if the fluctuation amplitude $\epsilon\sigma$ is large. For fixed $\epsilon\sigma$, L is also large for large k (short waves) or large α, corresponding to small correlation distance (very random).

The total change in wave number due to randomness is

$$\Delta k = \frac{\epsilon^2 \beta_r}{c} = -\frac{\epsilon^2}{c}\frac{K^2\sigma^2}{2\rho\omega kT}\frac{1}{\alpha^2+4k^2} = -\frac{\epsilon^2 K^2\sigma^2}{2T^2k}\frac{2k}{\alpha}\frac{1}{\alpha^2+4k^2}\tag{2.30}$$

It is negative, hences contributes to the lengthening of waves or increase in phase velocity. The magnitude of the wavenumber shift increases with increasing $\epsilon^2\sigma^2$ and descreasing α (decreasing randomness).

For a discussion of numercal results, see Mei and Pihl, 2002.

IAP (challenge) Project : Scattering of elastic waves by random distribution of hard grains or cavities.

References

[1] Belzons, M., Guazzelli, E., & Parodi, O., 1988, *J. Fluid Mech.* 186: 539-.

[2] Chernov, L. A. 1960, *Wave Propagation in a Random Medium* Dover. 168 pp.

[3] Devillard, P., Dunlop, F. & Souillard, J. 1988. *J. Fluid Mech.* Localization of gravity waves on a channel with a random bottom, *J. Fluid Mech.* 186: 521-538.

[4] Elter, J. F. & Molyneux, J. E., 1972. The long-distance propagation of shallow water waves over an ocean of random depth. *J. Fluid Mech.* 53: 1-15.

[5] Frisch, U. Wave propagation in random media, in *Probabilistic Methods in Applied Mathematics*, v. 1. Academic.

[6] Karal, F.C., & Keller, J. B., 1964, Elastic, electromagnatic and other waves in a random medium. *J. Math. Phys.* 5(4): 537-547.

[7] Keller, J. B., 1964, Stochastic eqaiton and wave propogation in readom media, *Proc. 16th Symp. Appl. Math.*, 145-170. Amer. Math. Soc. Rhode Island.

[8] Nachbin, A., & Papanicolaou, G.C., 1992, Water waves in shallow channels of rapidly varying depth. *J. Fluiid Mech.* 241: 311-332.

[9] Nachbin, A., 1997. The localization length of randomly scattered water waves. *J. Fluid Mech.* 296: 353-372.

[10] Mei, C. C., 1985, Resonant reflection of surface waves by periodic bars, *J. Fluid Mech.*

[11] Mei. C. C., 1989, *Applied Dynamics of Ocean Surface Waves*, World Scieitific, Singapore. 700 pp.

[12] Rosales, Rodolfo R., & Papanicolaou, G. C. 1983, Gravity waves in a channel with a rough bottom. *Stud. Appl. Math.* 68: 89-102.

[13] Sheng, Ping, 1995. *Introduction of Wave Scattering, Localization, and Mesoscopic Phenomena*, Academic, 339 pp.

[14] Sheng, Ping,(ed), 1990. *Scattering and Localization of Classical Waves in Random Media,* World Scientific.

[15] Soong, T. T., 1973, *Random Differential Equations in Science and Engineering*, 327pp.Academic.

R. Buckminster Fuller, *Intuition: Metaphysical Mosaic.*

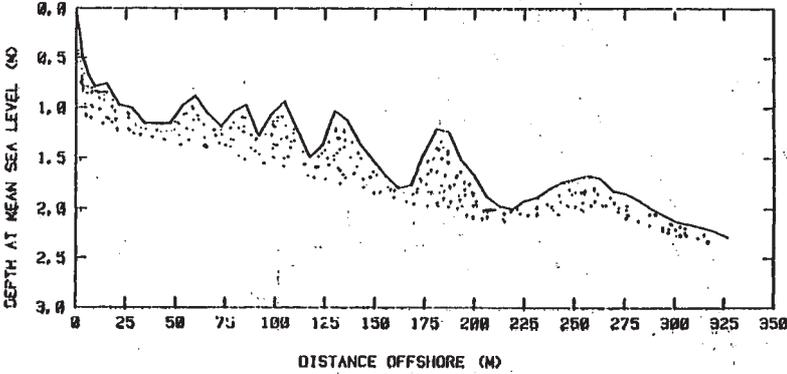**Chesapeake Bay, Scientists Cliffs (Dolan 1982)**

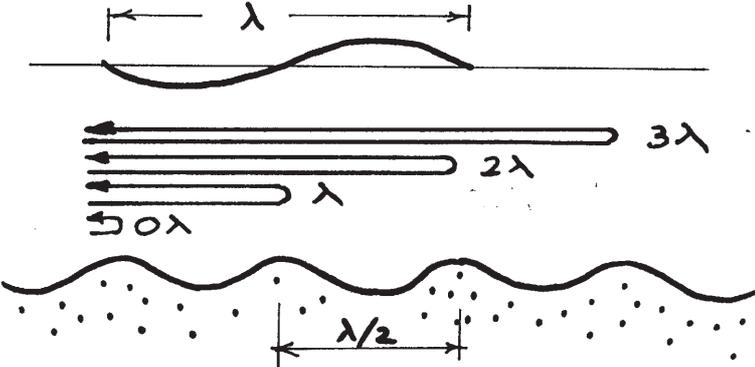Figure 1: Bar profile at Scientists Cliff, Chesapeake Bay, by Dolan.

Figure 2: Bragg resonance as the result of constructive interference.
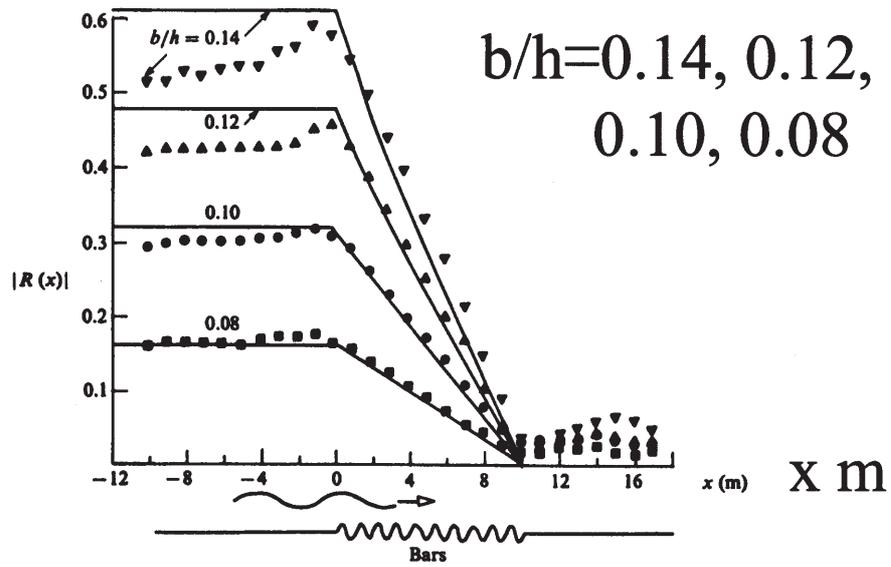
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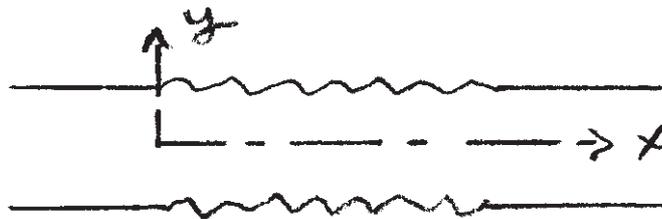Figure 3: Bragg scattering of surface water waves by periodic bars. Comparison of theory with measurements by Heathershaw.

Figure 4: Can wavy banks serve as a breakwater?

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