I-campus project

School-wide Program on Fluid Mechanics

Modules on Waves in fluids

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CHAPTER TWO

ONE-DIMENSIONAL PROPAGATION

Since the equation

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \nabla^2 \Phi$$

governs so many physical phenomena in nature and technology, its properties are basic to the understanding of wave propagation. This chapter is devoted to its analysis when the extent of the medium is infinite and the motion is one dimensional. To be be specific, physical discussions are made for shallow-water waves in the sea. The results are however readily transferable or modified for sound, waves in blood vessels and other types of waves.

1 General solution to wave equation

Recall that for waves in an artery or over shallow water of constant depth, the governing equation is of the classical form

$$\frac{\partial^2 \Phi}{\partial t^2} = c^2 \frac{\partial^2 \Phi}{\partial x^2} \tag{1.1}$$

It is easy to verify by direct substitution that the most general solution of the one dimensional wave equation (1.1) is

$$\Phi(x,t) = \mathcal{F}(x - ct) + \mathcal{G}(x + ct) \tag{1.2}$$

where $\mathcal{F}$ and g are arbitrary functions of their arguments. In the x, t (space,time) plane $\mathcal{F}(x - ct)$ is constant along the straight line $x - ct =$ constant. Thus to the observer (x, t) who moves at the steady speed c along the positive x-axis, the function $\mathcal{F}$ is stationary. Thus to an observer moving from left to right at the speed c, the signal described initially by $\mathcal{F}(x)$ at $t = 0$ remains unchanged in form as t increases, i.e., $\mathcal{F}$ is a wave propagating to the right at the speed c. Similarly $\mathcal{G}$ propagates to the left at the speed c. The lines $x - ct$ =constant and $x + ct =$ constant are called the *characteristic* curves (lines) along which signals propagate. Note that another way of writing (1.2) is

$$\Phi(x,t) = \hat{\mathcal{F}}(t - x/c) + \hat{\mathcal{G}}(t + x/c) \tag{1.3}$$

Let us illustrates an application of this simple result.

2 Branching of arteries

References: Y C Fung : *Biomechanics, Circulation.* Springer1997
M.J. Lighthill : *Waves in Fluids*, Cambridge 1978.

Recall that (1.1) governs both the pressure and the velocity in the blood

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{2.1}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2.2}$$

The two unknowns are related by the momentum equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{2.3}$$

The general solutions are :

$$p = p_+(x - ct) + p_-(x + ct) \tag{2.4}$$

$$u = u_+(x - ct) + u_-(x + ct) \tag{2.5}$$

Since

$$\frac{\partial p}{\partial x} = p'_+ + p'_-,$$

and

$$\rho \frac{\partial u_-}{\partial t} = -\rho c u'_+ + \rho c u'_-$$

where primes indicated ordinary differentiation with respect to the argument. Equation (2.3) can be satisfied if

$$p_+ = \rho c u_+, \quad p_- = -\rho c u_- \tag{2.6}$$

Denote the discharge by $Q = uA$ then

$$Q_\pm = u_\pm A = \pm Z p_\pm \tag{2.7}$$

where

$$Z = \frac{\rho c}{A} \tag{2.8}$$

is the property of the tube and is call the *impedance*.

Now we examine the effects of branching; Referring to figure 1, the parent tube, characterized by wave speed c and impedance Z, branches into two characterized by c_1 and c_2 and Z_1 and Z_2. An incident wave approaching the junction will cause reflection

$$p = p_i(t - x/c) + p_r(t + x/c), \quad x > 0 \tag{2.9}$$

and transmitted waves in the branches are $p_1(t - x/c_1)$ and $p_2(t - x/c_2)$ in $x > 0$. At the junction $x = 0$, continuity of pressure and fluxes requires

$$p_i(t) + p_r(t) = p_1(t) = p_2(t) \tag{2.10}$$

and

$$\frac{p_i - p_r}{Z} = \frac{p_1}{Z_1} + \frac{p_2}{Z_2} \tag{2.11}$$

Define the reflection coefficient R to be the amplitude ratio of reflected wave to incident wave, then

$$R = \frac{p_r(t)}{p_i(t)} = \frac{\frac{1}{Z} - \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)}{\frac{1}{Z} + \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)} \tag{2.12}$$

Similarly the tranmission coefficients are

$$T = \frac{p_1(t)}{p_i(t)} = \frac{p_2(t)}{p_i(t)} = \frac{\frac{2}{Z}}{\frac{1}{Z} + \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)} \tag{2.13}$$

Note that both coefficients are constants depending only on the impedances. Hence the transmitted waves propagate in the direction of increasing x and are similar in form to the incident waves except smaller by the factor T. On the incidence side waves the incident and reflected waves propagate in opposite directions.

Figure 1: Branching of arteries

3 Shallow water waves in an infinite sea due to initial disturbances

Recall for one-dimensional long waves in a shallow sea of depth $h(x)$, the linearized conservation laws of mass and momentum are

$$\frac{\partial \zeta}{\partial t} + \frac{\partial (uh)}{\partial x} = 0 \tag{3.1}$$

and

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} \tag{3.2}$$

where $\zeta(x,t)$ is the vertical displacement of the free surface and $u(x,t)$ the horizontal velocity. The atmospheric pressure over the entire free surface is uniform and constant. By cross-differentiation, ζ is seen to be governed by

$$\frac{\partial^2 \zeta}{\partial t^2} = g\frac{\partial}{\partial x}\left(h\frac{\partial \zeta}{\partial x}\right) \tag{3.3}$$

In the limit of constant depth (h =constant), the above equation reduces to the classical wave equation

$$\frac{\partial^2 \zeta}{\partial t^2} = c^2\frac{\partial^2 \zeta}{\partial x^2}, \quad \text{where} \quad c = \sqrt{gh}. \tag{3.4}$$

Consider now a sea of infinite extent, $-\infty < x < \infty$. Let the initial surface displacement and velocity be prescribed along the entire surface

$$\zeta(x,0) = \mathcal{F}(x) \tag{3.5}$$

$$\frac{\partial \zeta}{\partial t}(x,t) = \mathcal{G}(x), \tag{3.6}$$

where $\mathcal{F}(x)$ and $\mathcal{G}(x)$ are non-zero only in the finite domain of x. At infinities $x \to \pm\infty$, ζ and $\partial\zeta/\partial t$ are zero for any finite t. In (3.4) the highest time derivative is of the second order and initial data are prescribed for ζ and $\partial\zeta/\partial t$. Initial conditions that specify all derivatives of all orders less than the highest in the differential equation are called the Cauchy initial conditions. These conditions are best displayed in the space-time diagram as shown in Figure 2.

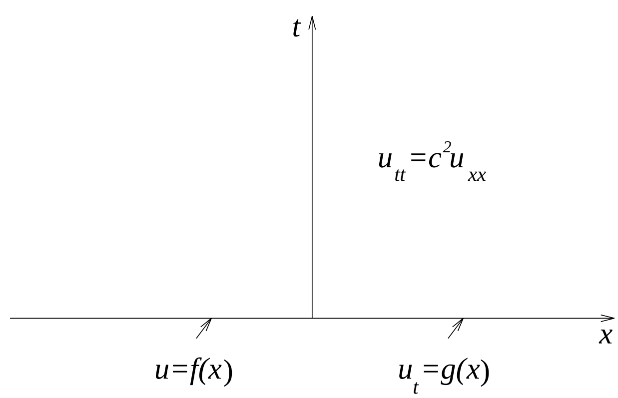

Figure 2: Summary of the initial-boundary-value problem

The present initial-boundary-value problem has a famous solution due to d'Alembert, which can be derived from (1.3), i.e.,

$$\zeta = \phi(\xi) + \psi(\eta) = \phi(x+ct) + \psi(x-ct), \tag{3.7}$$

where ϕ and ψ are so far arbitrary functions of the characteristic variables $\xi = x - ct$ and $\eta = x + ct$ respectively.

From the initial conditions we get

$$\begin{aligned} \zeta(x,0) &= \phi(x)+\psi(x)=f(x) \\ \frac{\partial\zeta}{\partial t}(x,0) &= c\phi'(x)-c\psi'(x)=g(x). \end{aligned} \tag{3.8}$$

The last equation may be integrated with respect to x

$$\phi-\psi=\frac{1}{c}\int_{x_0}^{x} g(x')dx'+K, \tag{3.9}$$

where K is an arbitrary constant. Now ϕ and ψ can be solved from (3.8) and (3.8 as functions of x,

$$\begin{aligned} \phi(x) &= \frac{1}{2}[f(x)+K]-\frac{1}{2c}\int_{x_0}^{x} g(x')dx' \\ \psi(x) &= \frac{1}{2}[f(x)-K]+\frac{1}{2c}\int_{x_0}^{x} g(x')dx', \end{aligned}$$

where K and x_0 are some constants. Replacing the arguments of ϕ by $x+ct$ and of ψ by $x-ct$ and substituting the results in u, we get

$$\begin{aligned} \zeta(x,t) &= \frac{1}{2}f(x-ct)-\frac{1}{2c}\int_{x_0}^{x-ct} g\,dx' \\ &\quad +\frac{1}{2}f(x+ct)+\frac{1}{2c}\int_{x_0}^{x+ct} g\,dx' \\ &= \frac{1}{2}[f(x-ct)+f(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,dx', \end{aligned} \tag{3.10}$$

which is d'Alembert's solution to the homogeneous wave equation subject to general Cauchy initial conditions.

To see the physical meaning, let us draw in the space-time diagram a triangle formed by two characteristic lines passing through the observer at x,t, as shown in Figure 3. The base of the triangle along the initial axis $t=0$ begins at $x-ct$ and ends at $x+ct$. The solution (3.1.9) depends on the initial displacement at just the two corners $x-ct$ and $x+ct$, and on the initial velocity only along the segment from $x-ct$ to $x+ct$. Nothing outside the triangle matters. Therefore, to the observer at x,t, the *domain of dependence* is the base of the characteristic triangle formed by two characteristics passing through x,t. On the other hand, the data at any point x on the initial line $t=0$ must influence all observers in the wedge formed by two characteristics drawn from $x,0$ into the region of $t>0$; this characteristic wedge is called the *range of influence*.

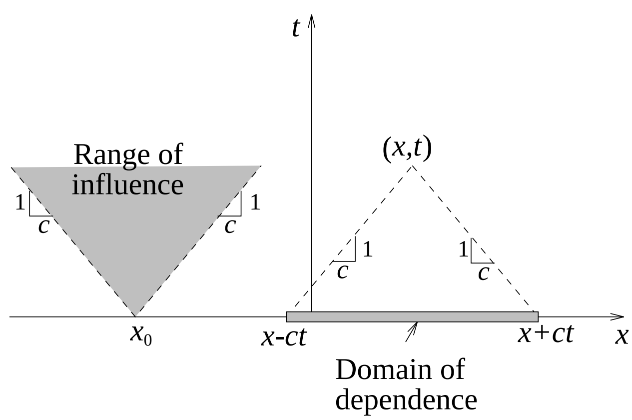

Figure 3: Domain of dependence and range of influence

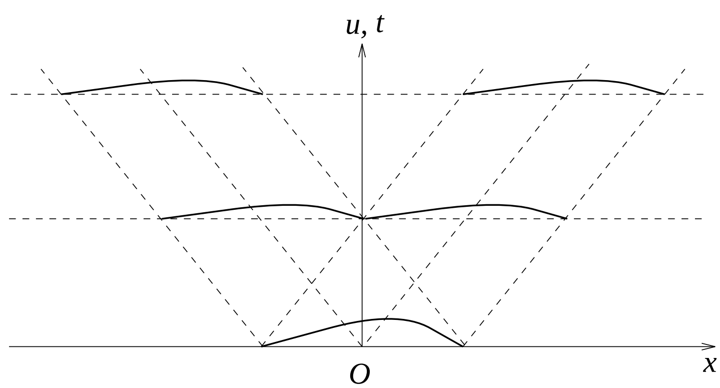

Figure 4: Waves due to initial displacement

Let us illustrate the physical effects of initial displacement and velocity separately.

Case (i): Initial displacement only: $f(x) \neq 0$ and $g(x) = 0$. The solution is

$$\zeta(x,t) = \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct)$$

and is shown for a simple $f(x)$ in Figure 4 at successive time steps. Clearly, the initial disturbance is split into two equal waves propagating in opposite directions at the speed c. The outgoing waves preserve the initial profile, although their amplitudes are reduced by half.

Case (ii): Initial velocity only: $f(x) = 0$, and $g(x) \neq 0$. Consider the simple example where

$$g(x) \quad = \quad g_0 \quad \text{when} \quad |x| < b, \quad \text{and}$$

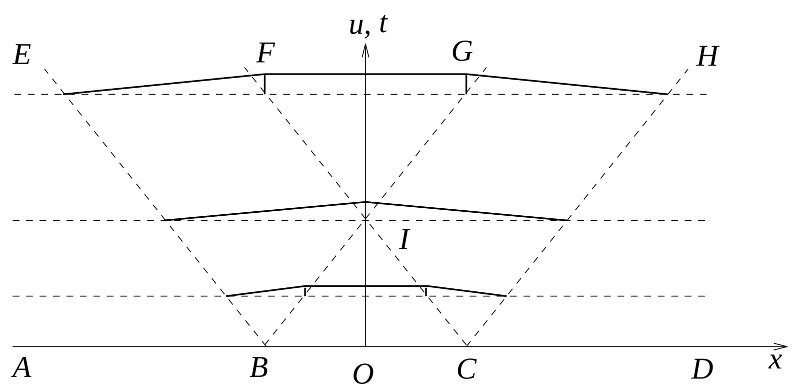

Figure 5: Waves due to initial velocity

$$= \quad 0 \quad \text{when} \quad |x| > 0.$$

Referring to Figure 5, we divide the $x \sim t$ diagram into six regions by the characteristics with B and C lying on the x axis at $x = -b$ and $+b$, respectively. The solution in various regions is:

$$\zeta = 0$$

in the wedge ABE;

$$\zeta = \frac{1}{2c} \int_{-b}^{x+ct} g_0 \, dx' = \frac{g_o}{2c}(x + ct + b)$$

in the strip $EBIF$;

$$\zeta = \frac{1}{2c} \int_{x-ct}^{x+ct} g_o dx' = g_o t$$

in the triangle BCI;

$$\zeta = \frac{1}{2c}\int_{-b}^{b} g_0\,dx' = \frac{g_o b}{c}$$

in the wedge FIG;

$$\zeta = \frac{1}{2c}\int_{x-ct}^{b} g_0\,dx' = \frac{g_o}{2c}(b - x + ct)$$

in the strip $GICH$; and

$$\zeta = 0$$

in the wedge HCD. The spatial variation of u is plotted for several instants in Figure 5. Note that the wave fronts in both directions advance at the speed c. In contrast to Case (i), disturbance persists for all time in the region between the two fronts.

4 Reflection of shallow water waves from a cliff

Let us use the d'Alembert solution to a problem in a half infinite domain $x > 0$. Let the sea be on the positive side of a cliff along $x = 0$ and extend to infinity. How do disturbances generated near the coast propagate as the result of initial displacement and velocity?

At the left boundary $x = 0$ must now add the condition of zero horizontal velocity which implies

$$\frac{\partial \zeta}{\partial x} = 0, \quad x = 0,\ t > 0. \tag{4.1}$$

In the space-time diagram let us draw two characteristics passing through x, t. For an observer in the region $x > ct$, the characteristic triangle does not intersect the time axis because t is still too small. The observer does not feel the presence of the fixed end at $x = 0$, hence the solution (3.10) for an infinite domain applies,

$$\zeta = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, \quad x > ct. \tag{4.2}$$

But for $x < ct$, this result is no longer valid. To ensure that the boundary condition is satisfied we employ the idea of mirror reflection. Consider a fictitious extension of the sea to $-\infty < x \le 0$. If on the side $x < 0$ the initial data are imposed such that $f(x)$ and $g(x)$ are even in x, then $\zeta(0, t) = 0$ is assured by symmetry. We now have initial conditions stated over the entire x axis

$$\zeta(x, 0) = F(x) \quad \text{and} \quad \zeta_t(x, 0) = G(x) \quad -\infty < x < \infty,$$

Figure 6: Initial-boundary-value problem and the mirror reflection

where

$$F(x) = \begin{cases} f(x) & \text{if } x > 0 \\ f(-x) & \text{if } x < 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & \text{if } x > 0 \\ g(-x) & \text{if } x < 0. \end{cases}$$

These conditions are summarized in Figure 6. Hence the solution for $0 < x < ct$ is

$$\begin{aligned} \zeta &= \frac{1}{2}\left[F(x+ct)+F(x-ct)\right]+\frac{1}{2c}\left(\int_{x-ct}^{0}+\int_{0}^{x+ct}\right)G(x')dx' \\ &= \frac{1}{2}\left[f(x+ct)+f(ct-x)\right]+\frac{1}{2c}\left(\int_{0}^{ct-x}+\int_{0}^{x+ct}\right)g(x')dx' \\ &= \frac{1}{2}\left[f(x+ct)+f(ct-x)\right]+\frac{1}{2c}\left(2\int_{0}^{ct-x}g(x')dx'+\int_{ct-x}^{ct+x}g(x')dx'\right)\text{‘}. \end{aligned} \quad (4.3)$$

Note that the point $(ct-x, 0)$ on the x axis is the mirror reflection (with respect to the cliff $x = 0$) of left tip $(x - ct, 0)$ of the characteristic triangle . The effect of the initial velocity in the region $(0, ct - x)$ is doubled.

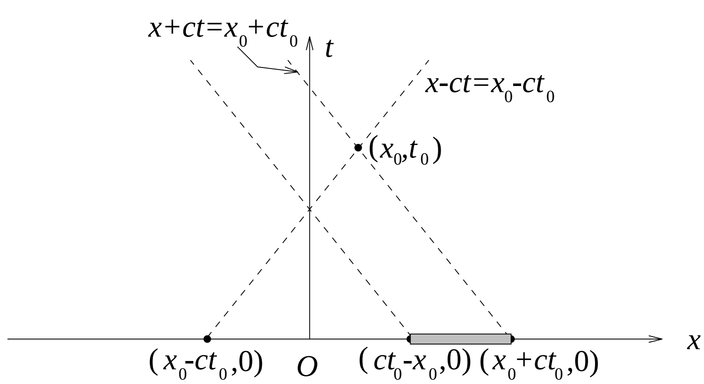

Figure 7: Reflection of long water waves from a cliff

5 Forced waves in an infinite domain

If there is a nonuniform distribution of atmospheric pressure $P(x,t)$ on the free surface, the fluid pressure is $p = P + g(\zeta - z)$ and momentum conservation should read

$$\frac{\partial u}{\partial t} = -g\frac{\partial \zeta}{\partial x} - g\frac{\partial P}{\partial x} \tag{5.1}$$

The wave equation is now inhomogeneous

$$\frac{\partial^2 \zeta}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + q(x,t) \qquad t > 0,\ |x| < \infty, \tag{5.2}$$

with the forcing term equaling

$$q(x,t) = gh\frac{\partial P}{\partial x}$$

Because of linearity, we can treat the effects of initial data separately. Let us therefore focus attention only to the effects of persistent forcing and let the initial data be zero,

$$\zeta(x,0) = 0, \qquad \left[\frac{\partial \zeta}{\partial t}\right]_{t=0} = 0, \tag{5.3}$$

The boundary conditions are

$$\zeta \to 0, \quad |x| \to \infty. \tag{5.4}$$

The inhomogeneous initial-boundary-value problem can be solved by Fourier transform. Let the transform of any function $f(x)$ be defined by

$$\bar{f}(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \tag{5.5}$$

and the inverse transform by

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{i\alpha x}\, d\alpha \tag{5.6}$$

The transformed wave equation is now an ordinary differential equation for $u(x,t)$, i.e., $\bar{\zeta}(\alpha,t)$,

$$\frac{d^2\bar{\zeta}}{dt^2} + c^2\alpha^2\bar{\zeta} = \bar{q} \qquad t > 0$$

where $\bar{q}(\alpha,t)$ denotes the transform of the forcing function. The initial conditions for $\bar{\zeta}$ are:

$$\bar{\zeta}(\alpha,0) = \bar{f}(\alpha), \qquad \frac{d\bar{\zeta}(\alpha,0)}{dt} = \bar{g}(\alpha).$$

Let us hide the parametric dependence on α for the time being. The general solution to the the inhomogeneous second-order ordinary differential equation is

$$\bar{\zeta} = C_1\bar{\zeta}_1(t) + C_2\bar{\zeta}_2(t) + \int_0^t \frac{\bar{q}(\tau)}{W}\left[\bar{\zeta}_1(\tau)\bar{\zeta}_2(t) - \bar{\zeta}_2(\tau)\bar{\zeta}_2(t)\right] d\tau, \tag{5.7}$$

where $\bar{u}_1$ and $\bar{u}_2$ are the homogeneous solutions

$$\bar{\zeta}_1 = e^{-i\alpha ct} \qquad \bar{\zeta}_2 = e^{i\alpha ct}$$

and W is the Wronskian

$$W = \bar{\zeta}_1\bar{\zeta}_2' - \bar{\zeta}_2\bar{\zeta}_1' = 2i\alpha c = \text{constant}.$$

The two initial conditions require that $C_1 = C_2 = 0$, hence

$$\bar{\zeta} = \int_0^t \frac{\bar{q}(\alpha,\tau)}{2i\alpha c}\left[e^{i\alpha c(t-\tau)} - e^{-i\alpha c(t-\tau)}\right] d\tau. \tag{5.8}$$

To get the inverse transform of the integral in (5.8), observe that

$$\begin{aligned}
\int_a^b d\xi\, q(\xi,\tau) &= \frac{1}{2\pi}\int_a^b d\xi \int_{-\infty}^{\infty} d\alpha\, \bar{q}\, e^{i\alpha\xi} \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, \bar{q}(\alpha,\tau)\frac{e^{i\alpha b} - e^{i\alpha a}}{i\alpha}
\end{aligned}$$

after changing the order of integration. If we let $b = x + c(t-\tau)$ and $a = x - c(t-\tau)$, the following

$$\frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi\, h(\xi,\tau)$$

is easily seen to be the inverse transform of the double integral. The final result if the inverse transform is

$$\zeta(x,t) = \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi\, h(\xi,\tau), \tag{5.9}$$

Thus the observer is affected only by the forcing inside the characteristic triangle defined by the two characteristics passing through (x,t).

For non-zero initial data $\zeta(x,0) = f(x)$ and $\zeta_t(x,0) = g(x)$, we get by linear superposition the full solution of D'Alambert

$$\begin{aligned}
\zeta(x,t) &= \frac{1}{2}[f(x+ct) + f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi g(\xi) \\
&\quad + \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi\, q(\xi,\tau),
\end{aligned} \tag{5.10}$$

The domain of dependence is entirely within the characteristic triangle.

Homework

6 Scattering of monochromatic waves by an obstacle

If the sea depth changes significantly, an incoming train of waves will be partly reflected and partly transmitted. In wave physics the determination of the scattering properties for a known scatterer is an important task. Various mathematical techniques are needed for different cases: (i) Strong scatterer if it height is comparable to the sea depth and the length to the wave length. (ii) Weak scatterers characterized by small amplitude relative to the wavelength, or slow variation within a wavelength.

Consider an ocean bottom with a step-wise variation of depth.

$$h = \begin{cases} h_1, & x < -a; \\ h_2, & -a < x < a; \\ h_3 = h_1, & x > a \end{cases} \tag{6.1}$$

If a sinusoidal wave train of frequency ω arrives from $x \sim -\infty$, how does the step change the propagation ?

In each zone of constant depth $(i = 1, 2, 3)$, the shallow water equations read:

$$\frac{\partial \zeta_i}{\partial t} + h_i \frac{\partial u_i}{\partial x} = 0 \tag{6.2}$$

$$\frac{\partial u_i}{\partial t} + g \frac{\partial \zeta_i}{\partial x} = 0 \tag{6.3}$$

A monochromatic wave of frequency ω can be written in the form

$$\zeta_i = \eta_i e^{-i\omega t}, \quad u_i = U_i e^{-i\omega t} \tag{6.4}$$

therefore,

$$-i\omega\eta_i + h_i \frac{\partial U_i}{\partial x} = 0 \tag{6.5}$$

$$-i\omega U_i + g \frac{\partial \eta_i}{\partial x} = 0 \tag{6.6}$$

which can be combined to

$$\frac{d^2\eta_i}{dx^2} + k_i^2 \eta_i = 0, \quad \text{where} \quad k_= \frac{\omega}{\sqrt{gh_i}} \tag{6.7}$$

The most general solution is a linear combination of terms proportional to

$$e^{ikx} \quad \text{and} \quad e^{-ikx},$$

Together with the time factor $e^{-i\omega t}$, the first term is a wave train propagating from left to right, while the second from right to left. The free-surface displacement of the incident wave therefore can be written as

$$\zeta_I = e^{ik_1 x - i\omega t} \tag{6.8}$$

where the amplitude is taken to be unity for brevity. At a junction, the pressure and the flux must be equal, hence we impose the following boundary conditions,

$$\eta_1 = \eta_2, \quad \text{and} \quad h_1 \frac{d\eta_1}{dx} = h_2 \frac{d\eta_2}{dx},, \quad x = -a; \tag{6.9}$$

$$\eta_2 = \eta_3, \quad \text{and} \quad h_2 \frac{d\eta_2}{dx} = h_1 \frac{d\eta_3}{dx},, \quad x = a. \tag{6.10}$$

Far from the step, sinusoidal disturbances caused by the presence of the step must be outgoing waves. Physically, this so-called *radiation condition* implies that, to the left of the step, there must be a reflected wavetrain traveling from right to left. To the right of the step, there must be a transmitted wavetrain traveling from left to right. Accordingly, the wave heights in each zone of constant depth are:

$$\eta_1 = e^{ik_1(x+a)} + Re^{-ik_1(x+a)}, \quad x < -a; \tag{6.11}$$

$$\eta_2 = Ae^{ik_2 x} + Be^{-ik_2 x}, \quad -a < x < a \tag{6.12}$$

$$\eta_3 = Te^{ik_1(x-a)}, \quad x > a \tag{6.13}$$

The reflection and transmission coefficients R and T as well as A and B are yet unknown.

Applying the matching conditions at the left junction, we get two relations

$$1 + R = Ae^{ik_2 a} + Be^{ik_2 a} \tag{6.14}$$

$$k_1 h_1 (1 - R) = k_2 h_2 (Ae^{-ik_2 a} - Be^{ik_2 a}). \tag{6.15}$$

Similarly the matching conditions at $x = a$ gives

$$Ae^{-ik_2 a} + Be^{-ik_2 a} = T \tag{6.16}$$

$$k_2 h_2 (Ae^{-ik_2 a} - Be^{ik_2 a}) = k_1 h_1 T. \tag{6.17}$$

These four equations can be solved to give

$$T = \frac{4s}{(1+s)^2 e^{2ik_2a} - (1-s)^2 e^{-2ik_2a}} \tag{6.18}$$

$$R = \frac{-(1-s)^2 (e^{-2ik_2a} - e^{2ik_2a}}{(1+s)^2 e^{2ik_2a} - (1-s)^2 e^{-2ik_2a}} \tag{6.19}$$

$$A = \frac{T}{2} e^{-ik_2a} (1+s) \tag{6.20}$$

$$B = \frac{T}{2} e^{ik_2a} (1-s) \tag{6.21}$$

where

$$s = \frac{k_1 h_1}{k_2 h_2} = \sqrt{\frac{h_1}{h_1}} = \frac{c_1}{c_2} \tag{6.22}$$

The energy densities associated with the transmitted and reflected waves are :

$$|T|^2 = \frac{4s^2}{4s^2 + (1-s^2)^2 \sin^2 2k_2a} \tag{6.23}$$

$$|R|^2 = \frac{(1-s^2) \sin^2 2k_2a}{4s^2 + (1-s^2)^2 \sin^2 2k_2a} \tag{6.24}$$

It is evident that $|R|^2 + |T|^2 = 1$, meaning that the total energy of the scattered waves is equal to that of the incident wave.

Over the shelf the free surface is given by

$$\eta_2 = \frac{2s\left[(1+s)e^{ik_2(x-a)} + (1-s)e^{-ik_2(x-a)}\right]}{(1+s)^2 e^{-ik_2a} - (1-s)^2 e^{ik_2a}} \tag{6.25}$$

Recalling the time factor $e^{-i\omega t}$, we see that the free surface over the shelf consists of two wave trains advancing in oppopsite directions. Therefore along the shelf the two waves can interfere each other constructively, with the crests of one coinciding with the crests of the other at the same moment. At other places the interference is destructive, with the crests of one wave train coinciding with the troughs of the other. The envelope of energy on the shelf is given by

$$|\eta|^2 = \frac{4s^2\left[\cos^2 k_2(x-a) + s^2 \sin^2 k_2(x-a)\right]}{4s^2 + (1-s)^2 \sin^2 2k_2a} \tag{6.26}$$

At the downwave edge of the shelf, $x = a$, the envelope is

$$|\eta|^2 = \frac{4s^2}{4s^2 + (1-s)^2 \sin^2 2k_2a} \tag{6.27}$$

Note that the reflection and transmission coefficients are oscillatory in $k_2 a$. In particular for $2k_2 a = n\pi, n = 1, 2, 3...$, that is, $4a/\lambda = n$, $|R| = 0$ and $|T| = 1$; the shelf is transparent to the incident waves. It is the largest when $2k_2 a = n\pi$, corresponding to the most constructive interference and the strongest transmission Minimum transmission and maximum reflection occur when $2k_2 a = (n - 1/2)\pi$, or $4a/\lambda = n - 1/2$, when the interference is the most destructive. The corresponding transmission and reflection coefficients are

$$\min|T|^2 = \frac{4s^2}{(1+s^2)^2}, \quad \max|R|^2 = \frac{(1-s^2)^2}{(1+s^2)^2}. \tag{6.28}$$

See figure 8.

The features of interference can be explained physically. When a crest first strikes the left edge at $x = a$, part of the it is transmitted onto the shelf and part is reflected towards $x \sim -\infty$. After reaching the right edge at $x = a$, the tranmitted crest has a part reflected to the left and re-reflected by the edge $x = -a$ to the right. When the remaining crest arrives at the right edge the second time, its total travel distance is an integral multiple of the wave length λ_2, hence is in phase with all the crests entering the shelf either before or after. Thus all the crests reinforce one another at the right edge. This is constructive interference, leading to the strongest tranmission to the right $x \sim \infty$. On the other hand if $2k_2 a = (n - 1/2)\pi$ or $4a/\lambda = n - 1/2$, some crests will be in opposite phase to some other crests, leading to the most destructive interference at the right edge, and smallest transmission.

7 Refraction by a slowly varying seabed

For time-harmonic waves over a seabed of variable depth, the governing equation can be derived from (3.3),

$$\frac{d}{dx}\left(h\frac{d\eta}{dx}\right) + \frac{\omega^2}{g}\eta = 0 \tag{7.1}$$

Consider a sea depth which varies slowly within a wavelength, i.e.,

$$\frac{1}{kh}\frac{dh}{dx} = O(\mu) \ll 1 \tag{7.2}$$

Earlier analysis suggests and will be demonstrated below reflection is negligibly small. Thus the solution is expected to be a locally progressive wave with both the wavenumber

Figure 8: Scattering coefficients for a step

and amplitude varying much more slowly than the wave phase in x . Hence we try the solution

$$\eta = A(x)e^{i\theta(x)} \tag{7.3}$$

where $\theta(x) - \omega t$ is the phase function and

$$k(x) = \frac{d\theta}{dx} \tag{7.4}$$

is the local wave number. Let us calculate the first derivative:

$$\frac{d\eta}{dx} = \left(ikA + \frac{dA}{dx}\right)e^{i\theta}$$

and assume

$$\frac{\frac{dA}{dx}}{kA} = O(kL)^{-1} \ll 1$$

In fact we shall assume each derivative of h, A or k is μ times smaller than kh, kA or k^2. Furthermore,

$$\frac{d}{dx}\left(h\frac{d\eta}{dx}\right) + \frac{\omega^2}{g}\eta = \left[ik\left(ikh + h\frac{dA}{dx}\right) + \frac{d}{dx}\left(h\frac{dA}{dx}\right) + i\frac{d(khA)}{dx} + \frac{\omega^2 A}{g}\right]e^{i\theta} = 0$$

Now let us expand

$$A = A_0 + A_1 + A_2 + \cdots \tag{7.5}$$

with $A_1/A_0 = O(\mu), A_2/A_0 = O(\mu^2), \cdots$. From $O(\mu^0)$ the dispersion relation follows:

$$\omega^2 = ghk^2, \quad \text{or} \quad k = \frac{\omega}{\sqrt{gh}} \tag{7.6}$$

Thus the local wave number and the local depth are related to frequency according to the well known dispersion relation for constant depth. As the depth decreases, the wavenumber increases. Hence the local phase velocity

$$c = \frac{\omega}{k} = \sqrt{gh} \tag{7.7}$$

also decreases.

From $O(\mu)$ we get,

$$ikh\frac{dA_0}{dx} + i\frac{d(khA_0)}{dx} = 0$$

or

$$\frac{d}{dx}(khA_0^2) = 0 \tag{7.8}$$

which means

$$khA_0^2 = C^2 = \text{constant}$$

or,

$$\sqrt{gh}A_0^2 = \text{constant} = \sqrt{gh_\infty}A_\infty^2 \tag{7.9}$$

Since in shallow water the group velocity equals the phase velocity, the above result means that the rate of energy flux is the same for all x and is consistent with the original assumption of unidirectional propagation. Furthermore, the local amplitude increases with depth as

$$\frac{A_0(x)}{A_\infty} = \left(\frac{h_\infty}{h}\right)^{1/4} \tag{7.10}$$

This result is called Green's law.

In summary, the leading order solution is

$$\zeta = A_\infty \left(\frac{h_\infty}{h}\right)^{1/4} e^{i\theta - i\omega t} = A_\infty \left(\frac{h_\infty}{h}\right)^{1/4} \exp\left(i\int^x k(x')dx' - i\omega\right) \tag{7.11}$$

We also provide an example to illustrate that the wave reflected by a slowly varying topography is negligibly small, actually exponentially small. The depth function $h(x)$ is taken in the form

$$h(x) = h_0 - A_b \tanh(x/\Lambda), \tag{7.12}$$

where Λ is the length scale of the bottom variation, h_0 is the average bottom depth and A_b is the bottom variation amplitude. We discretized the shallow water equation (7.1) using the finite difference method. We computed numerically the reflection coefficient as a function of the ratio between the wavelength of the incident wave and the length scale of the bottom variation (parameter Λ). We label this ratio as β. Figures 9 and 10 show the modulus of the reflection and transmission coefficients as a function of the parameter β for the bottom topography with depth function $h(x)$ defined by equation (7.12). The modulus of the reflection and transmission coefficients displayed in figures 9 and 10 are, respectively, for the cases of wave incidence from the deep side of the bottom (wave incidence from left) and the shallow side of the bottom (wave incidence from right).

Notice that for the case of wave incidence from the deep region (wave incidence from left) we have a transmission coefficient with modulus greater than one, as illustrated in figure 9. For wave incidence from the shallow region (wave incidence from right), the modulus of the transmission coefficient is less than one (see figure 10), and the reflection coefficient is the same for both cases.

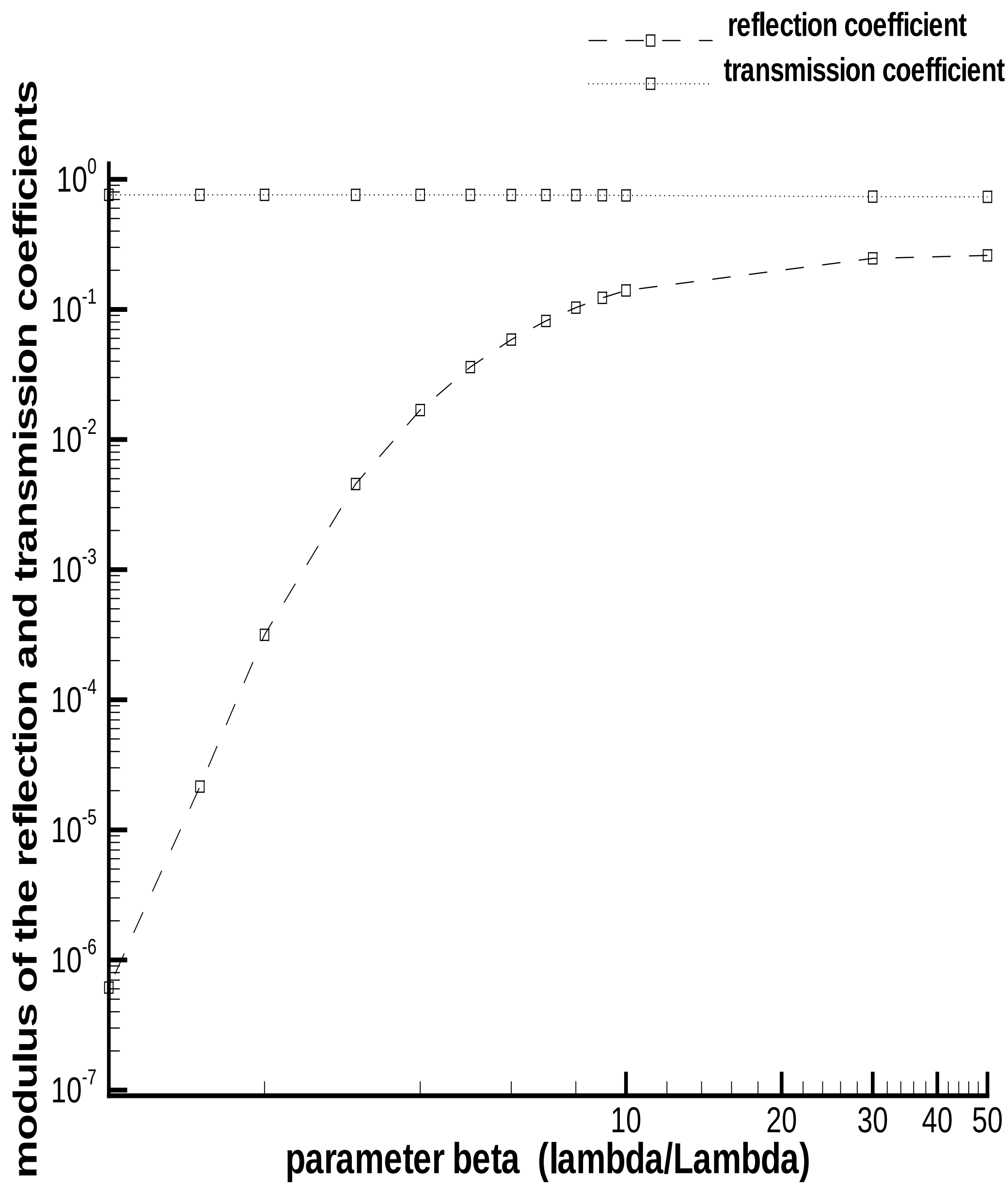

Figure 9: Modulus of the reflection and transmission coefficients as a function of the parameter $\beta = \frac{\lambda}{\Lambda}$ (λ - wavelength) for wave incidence from the deep side of the bottom (wave incidence from left). $A_b = 0.5$ in equation (7.12) for the depth function $h(x)$.

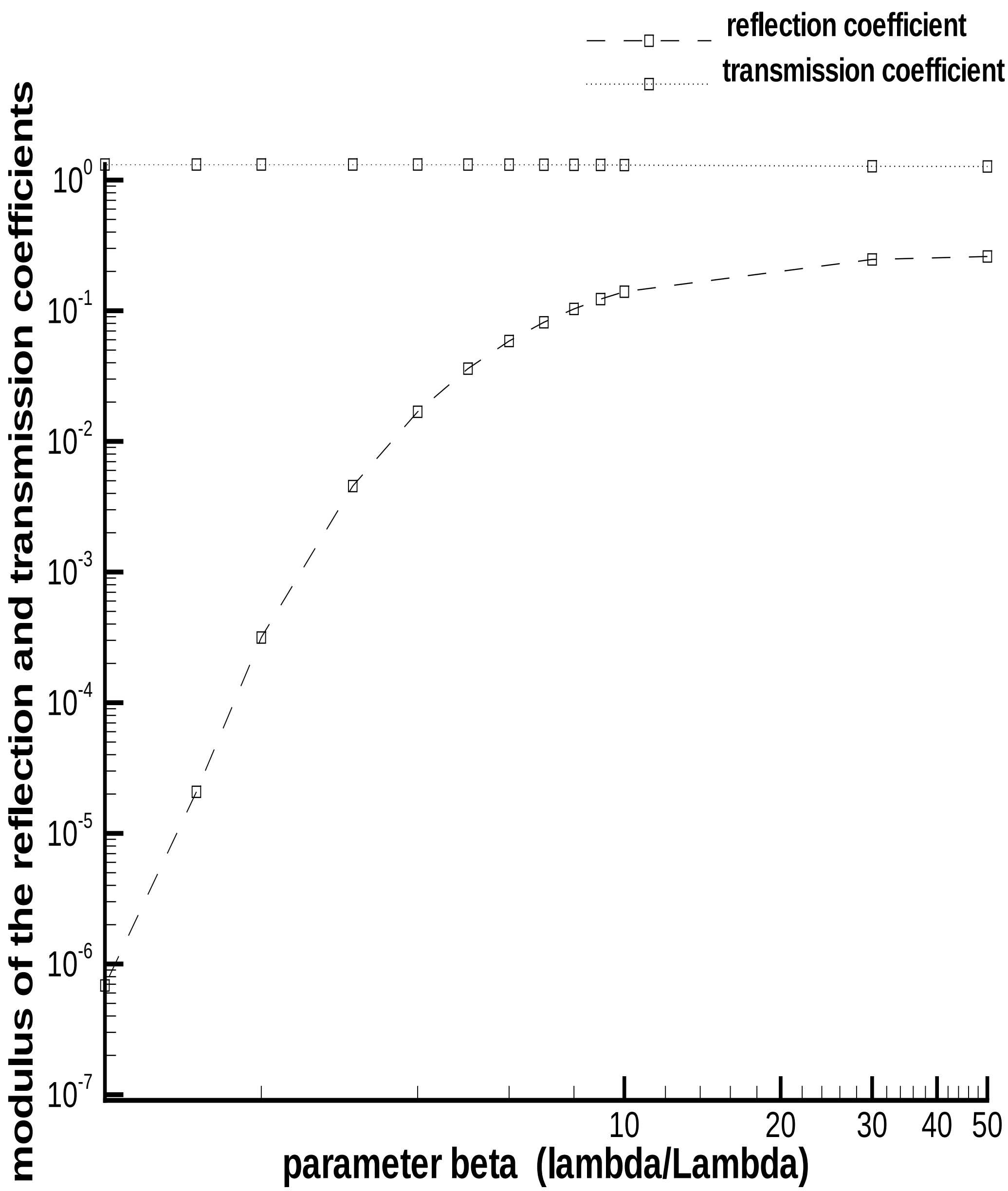

Figure 10: Modulus of the reflection and transmission coefficients as a function of the parameter $\beta = \frac{\lambda}{\Lambda}$ (λ - wavelength) for wave incidence from the deep side of the bottom (wave incidence from left). $A_b = 0.5$ in equation (7.12) for the depth function $h(x)$.

According to both figures 9 and 10, the reflection coefficient is of the order of $O(10^{-6})$ for β around 1. We check the validity of Green's formula given by equation (7.10). In figure 11 we display the ratio $A(x)/A_\infty$ computed numerically and by using Green's law for two values of β. $A(x)$ is the wave amplitude along the topography and A_∞ is the wave amplitude at $x \to +(-)\infty$ for the case of wave incidence from the shallow (deep) side of the bottom. In figure 12, we display the free-surface displacement computed numerically and given by Green's law for two values of the parameter β. In part (A) of figures 11 and 12, we display results for $\beta = 5$, and in part (B) of figures 11 and 12, we display results for $\beta = 1$. In the case where $\beta = 1$, the reflected wave is exponentially small, so we should observe basically wave refraction and the Green's law is verified, as is confirmed in figures 11 and 12 below.

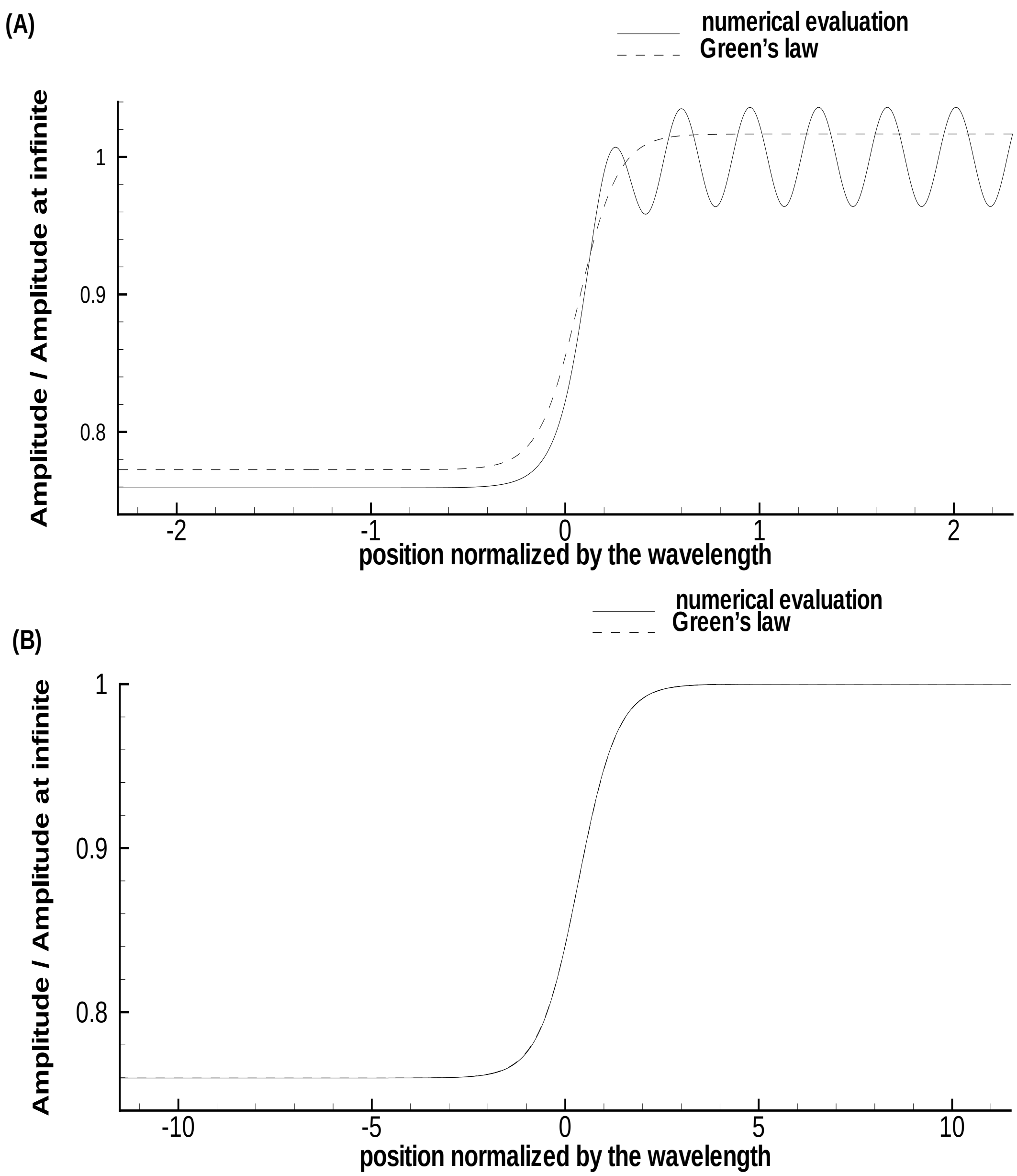

Figure 11: Wave amplitude along the bottom topography computed through the finite difference scheme and through the Green's law given by equation (7.10) for: (A) - $\beta = 5$, (B) - $\beta = 1$. Depth function $h(x)$ given by equation (7.12).

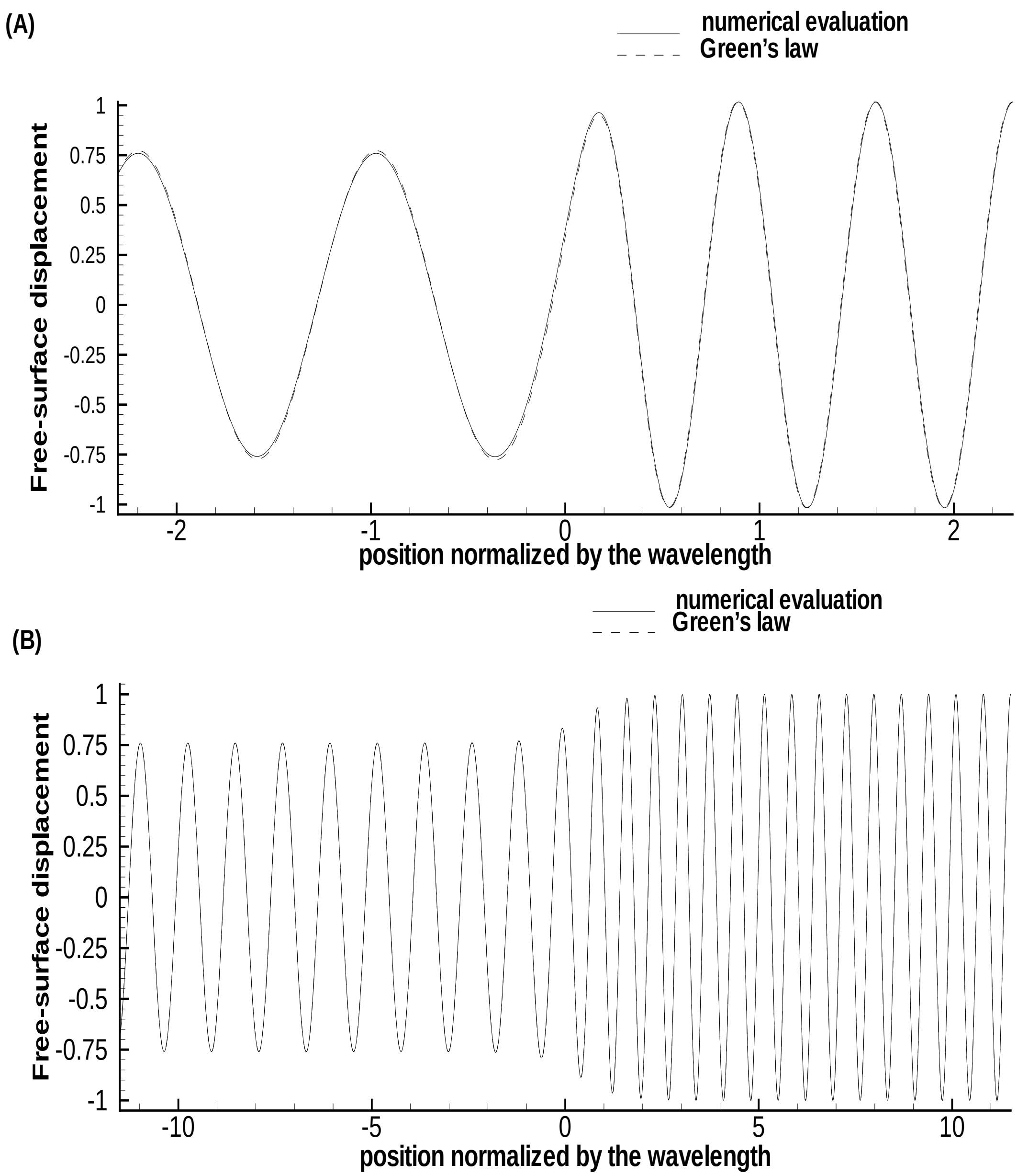

Figure 12: Surface wave displacement along the bottom topography computed through the finite difference scheme and through the equation (7.11) for: (A) - $\beta = 5$, (B) - $\beta = 1$. Depth function $h(x)$ given by equation (7.12).