1.138J/2.062J/18.376J, WAVE PROPAGATION

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CHAPTER TWO

ONE DIMENSIONAL WAVES

1 General solution to wave equation

It is easy to verify by direct substitution that the most general solution of the one dimensional wave equation:

$$\frac{\partial^2\phi}{\partial t^2} = c^2\frac{\partial^2\phi}{\partial x^2} \tag{1.1}$$

can be solved by

$$\phi(x,t) = f(x-ct) + g(x+ct) \tag{1.2}$$

where $f(\xi)$ and $g(\xi)$ are arbitrary functions of ξ. In the x,t (space,time) plane $f(x-ct)$ is constant along the straight line $x-ct =$ constant. Thus to the observer (x,t) who moves at the steady speed c along the positivwe x-axis., the function f is stationary. Thus to an observer moving from left to right at the speed c, the signal described initially by $f(x)$ at $t=0$ remains unchanged in form as t increases, i.e., f propagates to the right at the speed c. Similarly g propagates to the left at the speed c. The lines $x-ct =$constant and $x+ct =$ contant are called the *characteristic* curves (lines) along which signals propagate. Note that another way of writing (1.2) is

$$\phi(x,t) = F(t-x/c) + G(t+x/c) \tag{1.3}$$

Let us illustrates an application.

2 Branching of arteries

References: Y C Fung : *Biomechanics, Circulation.* Springer1997

M.J. Lighthill : *Waves in Fluids*, Cambridge 1978.

Recall the governing equations for pressure and velocity

$$\frac{\partial^2 p}{\partial t^2} = c^2 \frac{\partial^2 p}{\partial x^2} \tag{2.1}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2.2}$$

The two are related by the momentum equation

$$\rho \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \tag{2.3}$$

The general solutions are :

$$p = p_+(x - ct) + p_-(x + ct) \tag{2.4}$$

$$u = u_+(x - ct) + u_-(x + ct) \tag{2.5}$$

Since

$$\frac{\partial p}{\partial x} = p'_+ + p'_-,$$

and

$$\rho \frac{\partial u_-}{\partial t} = -\rho c u'_+ + \rho c u'_-$$

where primes indicated ordinary differentiation with repect to the argument. Equation (2.3) can be satisfied if

$$p_+ = \rho c u_+, \quad p_- = -\rho c u_- \tag{2.6}$$

Denote the discharge by $Q = uA$ then

$$ZQ_\pm = u_\pm A = \pm p_\pm \tag{2.7}$$

where

$$Z = \pm \frac{p_\pm}{Q_\pm} = \frac{\rho c}{A} \tag{2.8}$$

is the ratio of pressure to flux rate and is call the *impedance*. It is the property of the tube.

Now we examine the effects of branching; Referening to figure x, the parent tubes branches into two charaterized by wave speeds C_1 and C_2 and impedaces Z_1 and Z_2. An incident wave approaching the junction will cause reflection in the same tube

$$p = p_i(t - x/c) + p_r(t + x/c) \tag{2.9}$$

and transmitted waves in the branches are $p_1(t-x/c_1)$ and $p_2(t-x/c_2)$. At the junction $x = 0$ we expect the continuity of pressure and fluxes, hence

$$p_i(t) + p_r(t) = p_1(t) = p_2(t) \tag{2.10}$$

$$\frac{p_i - p_r}{Z} = \frac{p_1}{Z_1} + \frac{p_2}{Z_2} \tag{2.11}$$

Define the reflection coefficient R to be the amplitude ratio of reflected wave to incident wave, then

$$R = \frac{p_r(t)}{p_i(t)} = \frac{\frac{1}{Z} - \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)}{\frac{1}{Z} + \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)} \tag{2.12}$$

Similarly the tranmission coefficients are

$$T = \frac{p_1(t)}{p_i(t)} = \frac{p_2(t)}{p_i(t)} = \frac{\frac{2}{Z}}{\frac{1}{Z} + \left(\frac{1}{Z_1} + \frac{1}{Z_2}\right)} \tag{2.13}$$

Note that both coefficients are constants depending only on the impedances. Hence the transmitted waves are similar in form to the incident waves except smaller by the factor T. The total wave on the incidence side is however very different.

3 Waves in an infinite domain due to initial disturbances

Recall the governing equation for one-dimensional waves in a taut string

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad -\infty < x < \infty. \tag{3.1}$$

Let the initial transverse displacement and velocity be given along the entire string

$$u(x, 0) = f(x) \tag{3.2}$$

$$\frac{\partial u}{\partial t}(x, t) = g(x), \tag{3.3}$$

where $f(x)$ and $g(x)$ are non-zero only in the finite domain of x. At infinities $x \to \pm\infty$, u and $\partial u/\partial t$ are zero for any finite t. These conditions are best displayed in the space-time diagram as shown in Figure 1.

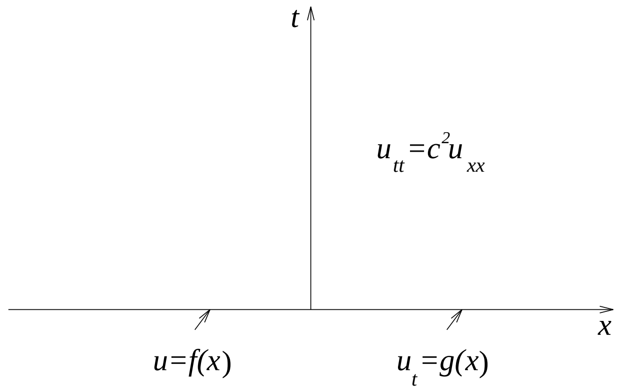

Figure 1: Summary of the initial-boundary-value problem

In (3.1.1) the highest time derivative is of the second order and initial data are prescribed for u and $\partial u/\partial t$. Initial conditions that specify all derivatives of all orders less than the highest in the differential equation are called the Cauchy initial conditions.

Recall that the general solution is

$$u = \phi(\xi) + \psi(\eta) = \phi(x + ct) + \psi(x - ct), \tag{3.4}$$

where ϕ and ψ are so far arbitrary functions of the characteristic variables $\xi = x + ct$ and $\eta = x - ct$ respectively.

From the initial conditions we get

$$\begin{aligned} u(x, 0) &= \phi(x) + \psi(x) = f(x) \\ \frac{\partial u}{\partial t}(x, 0) &= c\phi'(x) - c\psi'(x) = g(x). \end{aligned} \tag{3.5}$$

The last equation may be integrated with respect to x

$$\phi - \psi = \frac{1}{c}\int_{x_0}^{x} g(x')dx' + K, \tag{3.6}$$

where K is an arbitrary constant. Now ϕ and ψ can be solved from (3.1.6) and (3.1.7) as functions of x,

$$\phi(x) = \frac{1}{2}[f(x) + K] + \frac{1}{2c}\int_{x_0}^{x} g(x')dx'$$

$$\psi(x) = \frac{1}{2}[f(x) - K] - \frac{1}{2c}\int_{x_0}^{x} g(x')dx',$$

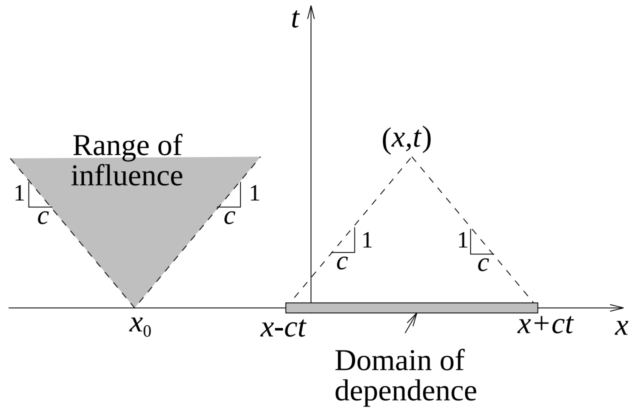

Figure 2: Domain of dependence and range of influence

where K and x_0 are some constants. Replacing the arguments of ϕ by $x+ct$ and of ψ by $x-ct$ and substituting the results in u, we get

$$\begin{aligned} u(x,t) &= \frac{1}{2}f\,(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} g\,dx' \\ &\quad + \frac{1}{2}f\,(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} g\,dx' \\ &= \frac{1}{2}\left[f(x-ct)+f(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(x')\,dx', \end{aligned} \tag{3.7}$$

which is d'Alembert's solution to the homogeneous wave equation subject to general Cauchy initial conditions.

To see the physical meaning, let us draw in the space-time diagram a triangle formed by two characteristic lines passing through the observer at x, t, as shown in Figure 2. The base of the triangle along the initial axis $t = 0$ begins at $x - ct$ and ends at $x + ct$. The solution (3.1.9) depends on the initial displacement at just the two corners $x - ct$ and $x + ct$, and on the initial velocity only along the segment from $x - ct$ to $x + ct$. Nothing outside the triangle matters. Therefore, to the observer at x, t, the *domain of dependence* is the base of the characteristic triangle formed by two characteristics passing through x, t. On the other hand, the data at any point x on the initial line $t = 0$ must influence all observers in the wedge formed by two characteristics drawn from $x, 0$ into the region of $t > 0$; this characteristic wedge is called the *range of influence*.

Let us illustrate the physical effects of initial displacement and velocity separately.

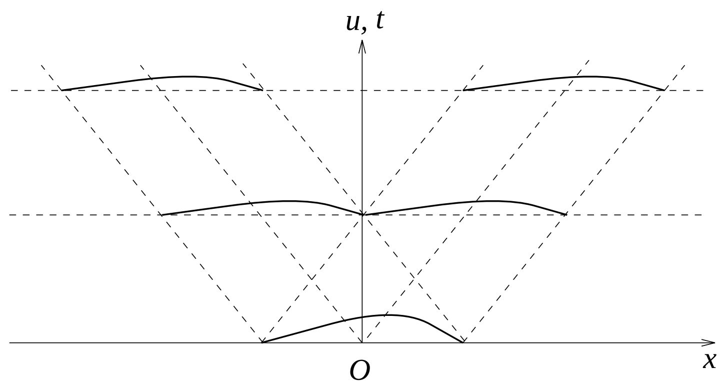

Figure 3: Waves due to initial displacement

Case (i): Initial displacement only: $f(x) \neq 0$ and $g(x) = 0$. The solution is

$$u(x,t) = \frac{1}{2}f(x - ct) + \frac{1}{2}f(x + ct)$$

and is shown for a simple $f(x)$ in Figure 3 at successive time steps. Clearly, the initial disturbance is split into two equal waves propagating in opposite directions at the speed c. The outgoing waves preserve the initial profile, although their amplitudes are reduced by half.

Case (ii): Initial velocity only: $f(x) = 0$, and $g(x) \neq 0$. Consider the simple example where

$$\begin{aligned} g(x) &= g_0 \quad \text{when} \quad |x| < b, \quad \text{and} \\ &= 0 \quad \text{when} \quad |x| > 0. \end{aligned}$$

Referring to Figure 4, we divide the $x \sim t$ diagram into six regions by the characteristics with B and C lying on the x axis at $x = -b$ and $+b$, respectively. The solution in various regions is:

$$u = 0$$

in the wedge ABE;

$$u = \frac{1}{2c}\int_{-b}^{x+ct} g_0\, dx' = \frac{g_o}{2c}(x + ct + b)$$

in the strip $EBIF$;

$$u = \frac{1}{2c}\int_{x-ct}^{x+ct} g_o dx' = g_o t$$

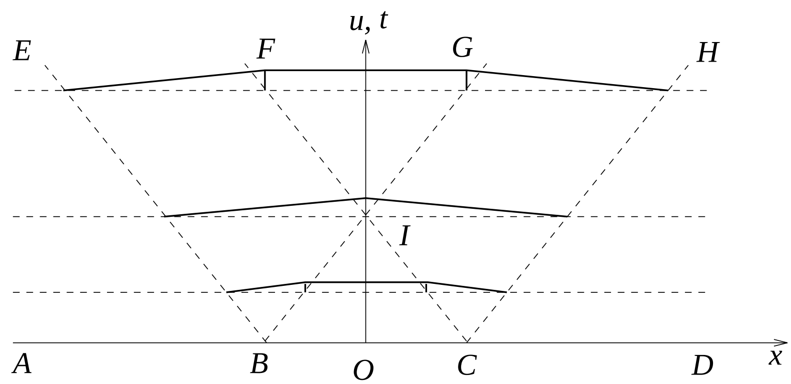

Figure 4: Waves due to initial velocity

in the triangle BCI;

$$u = \frac{1}{2c}\int_{-b}^{b} g_0\, dx' = \frac{g_o b}{c}$$

in the wedge FIG;

$$u = \frac{1}{2c}\int_{x-ct}^{b} g_0\, dx' = \frac{g_o}{2c}(b - x + ct)$$

in the strip $GICH$; and

$$u = 0$$

in the wedge HCD. The spatial variation of u is plotted for several instants in Figure 4. Note that the wave fronts in both directions advance at the speed c. In contrast to Case (i), disturbance persists for all time in the region between the two fronts.

4 Reflection from the fixed end of a string

Let us use the d'Alembert solution to a problem in a half infinite domain $x > 0$. Consider a long and taut string stretched from $x = 0$ to infinity. How do disturbances generated near the left end propagate as the result of initial displacement and velocity?

At the left boundary $x = 0$ must now add the condition

$$u = 0, \quad x = 0,\ t > 0. \tag{4.1}$$

In the space-time diagram let us draw two characteristics passing through x, t. For an observer in the region $x > ct$, the characteristic triangle does not intersect the time axis because t is still too small. The observer does not feel the presence of the fixed end at $x = 0$, hence the solution (3.1.9) for an infinitely long string applies,

$$u = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\tau)d\tau, \quad x > ct. \tag{4.2}$$

But for $x < ct$, this result is no longer valid. To ensure that the boundary condition is satisfied we employ the idea of mirror reflection. Consider a fictitious extension of the string to $-\infty < x \leq 0$. If on the side $x < 0$ the initial data are imposed such that $f(x) = -f(-x)$, $g(x) = -g(-x)$, then $u(0,t) = 0$ is assured by symmetry. We now have initial conditions stated over the entire x axis

$$u(x,0) = F(x) \quad \text{and} \quad u_t(x,0) = G(x) \quad -\infty < x < \infty,$$

where

$$F(x) = \begin{cases} f(x) & \text{if } x > 0 \\ -f(-x) & \text{if } x < 0 \end{cases}$$

$$G(x) = \begin{cases} g(x) & \text{if } x > 0 \\ -g(-x) & \text{if } x < 0. \end{cases}$$

These conditions are summarized in Figure 5. Hence the solution for $0 < x < ct$ is

$$\begin{aligned} u &= \frac{1}{2}\left[F(x+ct) + F(x-ct)\right] + \frac{1}{2c}\left(\int_{x-ct}^{0} + \int_{0}^{x+ct}\right) G(x')dx' \\ &= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\left(\int_{ct-x}^{0} + \int_{0}^{x+ct}\right) g(x')dx' \\ &= \frac{1}{2}\left[f(x+ct) - f(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} g(x')dx'. \end{aligned} \tag{4.3}$$

The domain of dependence is shown by the hatched segment on the x axis in Figure 6.

5 Forced waves in an infinite domain

Consider the inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2} + h(x,t) \qquad t > 0,\ |x| < \infty,$$

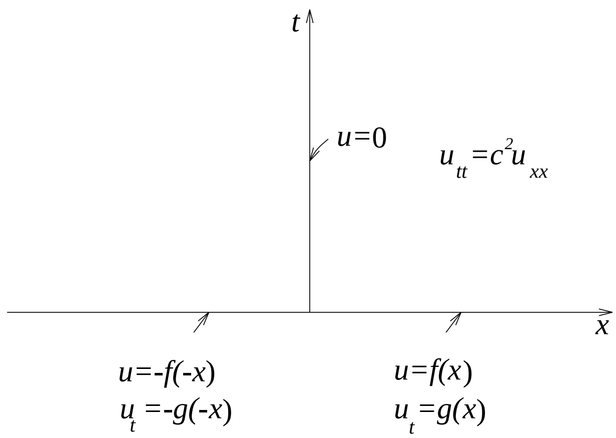

Figure 5: Initial-boundary-value problem and the mirror reflection

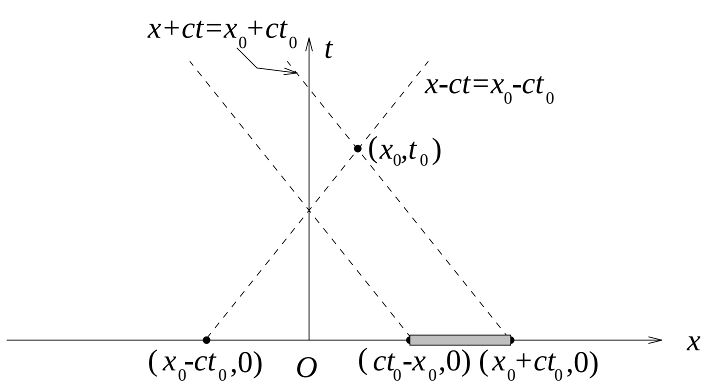

Figure 6: Reflection from a fixed end

where $h(x,t)$ represents forcing. Because of linearity, we can treat the effects of initial data separately. Let us therefore focus attention only to the effects of persistent forcing and let the initial data be zero,

$$u(x,0) = 0, \qquad \left[\frac{\partial u}{\partial t}\right]_{t=0} = 0, \tag{5.1}$$

The boundary conditions are

$$u \to 0, \quad |x| \to \infty. \tag{5.2}$$

Let the Fourier transform of any function $f(x)$ and its inverse $\bar{f}(\alpha)$ be defined by

$$\bar{f}(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{-i\alpha x}\, dx \tag{5.3}$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{f}(\alpha)\, e^{i\alpha x}\, d\alpha \tag{5.4}$$

The transformed wave equation is now an ordinary differential equation for the transform of $u(x,t)$, i.e., $\bar{u}(\alpha,t)$,

$$\frac{d^2\bar{u}}{dt^2} + c^2\alpha^2\bar{u} = \bar{h} \qquad t > 0$$

where $\bar{h}(\alpha,t)$ denotes the transform of the forcing function. The initial conditions for $\bar{u}$ are:

$$\bar{u}(\alpha,0) = \bar{f}(\alpha), \qquad \frac{d\bar{u}(\alpha,0)}{dt} = \bar{g}(\alpha).$$

Let us hide the parametric dependence on α for the time being. The general solution to the the inhomogeneous second-order ordinary differential equation is

$$\bar{u} = C_1\bar{u}_1(t) + C_2\bar{u}_2(t) + \int_0^t \frac{\bar{h}(\tau)}{W}\left[\bar{u}_1(\tau)\bar{u}_2(t) - \bar{u}_2(\tau)\bar{u}_2(t)\right] d\tau, \tag{5.5}$$

where $\bar{u}_1$ and $\bar{u}_2$ are the homogeneous solutions

$$\bar{u}_1 = e^{-i\alpha ct} \qquad \bar{u}_2 = e^{i\alpha ct}$$

and W is the Wronskian

$$W = \bar{u}_1\bar{u}_2' - \bar{u}_2\bar{u}_1' = 2i\alpha c = \text{constant}.$$

The two initial conditions require that $C_1 = C_2 = 0$, hence the Fourier transform is

$$\bar{u} = \int_0^t \frac{\bar{h}(\alpha,\tau)}{2i\alpha c}\left[e^{i\alpha c(t-\tau)} - e^{-i\alpha c(t-\tau)}\right] d\tau. \tag{5.6}$$

The inverse transform is

$$u(x,t) = \frac{1}{2\pi}\int_0^t d\tau \frac{1}{2c}\int_{-\infty}^{\infty} \frac{\bar{h}(\alpha,\tau)}{i\alpha}\left[e^{i\alpha(x+c(t-\tau))} - e^{i\alpha(x-c(t-\tau))}\right] d\alpha \tag{5.7}$$

The integrand can be written as an integral:

$$u = \frac{1}{2\pi}\int_0^t d\tau \frac{1}{2c}\int_{-\infty}^{\infty} d\alpha\, \bar{h}(\alpha,\tau)\int_{x-c(t-\tau)}^{x-c(t-\tau)} d\xi\, e^{i\alpha\xi}$$

By interchanging the order of integration

$$u = \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x-c(t-\tau)} d\xi \frac{1}{2\pi}\int_{-\infty}^{\infty} d\alpha\, e^{i\alpha\xi}\,\bar{h}(\alpha,\tau)$$

and using the definition of Fourier inversion, we get

$$u(x,t) = \frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x-c(t-\tau)} d\xi\, h(\xi,\tau) \tag{5.8}$$

The right-hand side is the integration of h over the characteristic triangle defined by the two characteristics passing through (x,t). in the $x-t$ plane. Thus the observer is affected only by the forcing inside the characteristic triangle.

For non-zero initial data $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we get by linear superposition the full solution of D'Alambert

$$\begin{aligned} u(x,t) &= \frac{1}{2}[f(x+ct)+f(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} d\xi g(\xi) \\ &+\frac{1}{2c}\int_0^t d\tau \int_{x-c(t-\tau)}^{x+c(t-\tau)} d\xi\, h(\xi,\tau), \end{aligned} \tag{5.9}$$

The domain of dependence is entirely within the characteristic triangle.

6 String embeded in an elastic surounding

Reference : Graff : *Waves in Elastic Solids*

If the lateral motion of the string is restrained by elastic springs along the entire length, the governing equation can be found from §1.1 by replacing the external pressure by the elastic restoring force $-KV$ per unit length,

$$\rho\frac{\partial^2 V}{\partial t^2} = T\frac{\partial^2 V}{\partial x^2} - KV \tag{6.1}$$

which can be written as

$$\frac{1}{c_o^2}\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial x^2} - \frac{K}{T}V \tag{6.2}$$

where

$$c_o = \sqrt{\frac{T}{\rho}} \tag{6.3}$$

6.1 Monochromatic waves

For any linearized wave problem, if the range of the spatial corrdinate x is $(-\infty, \infty)$, and all coefficients are independent of x, t, then the first task is to examine the physics of sinusoidal wave train of the form:

$$V(x,t) = |A|\cos(kx - i\omega t - \phi_A) = \Re\left(Ae^{ikx-i\omega t}\right) \tag{6.4}$$

where $A = |A|e^{i\phi_A}$ is a complex number with magnitude $|A|$ and phase angle ϕ_A. After examining the physical meaning of this special type of waves, it is possible to use the principle of superposition to construct more general solutions. It is customary to omit the symbol $\Re$= "the real part of" for the sake of brevity, i.e.,

$$V(x,t) = Ae^{ikx-i\omega t} \tag{6.5}$$

First of all a few definitions about sinusoidal waves in general. We shall call

$$\theta(x,t) = kx - \omega t \tag{6.6}$$

the *wave phase*. Clearly the trigonometric function is periodic in phase with the period 2π. In the x, t plane, V has a constant value along a line of contant phase. In particular, $\theta = 2n\pi, (n = 0, 1, 2, \cdots)$ correspond to the wave crests where $V = |A|$ is the greatest. On the other hand, $\theta = (2n + 1)\pi, (n = 0, 1, 2, \cdots)$ correspond to the wave troughs where $V = -|A|$ is the smallest. $|A|$ is half of the separation between adjacent crests and troughs and is called the *wave amplitude*; we also call A the complex amplitude. Clearly $\frac{\partial\theta}{\partial x}$ represents the number of phase lines per unit distance, i.e., the density of phase lines, at a given instant; it is called the *wavenumber*,

$$\text{wavenumber} = k = \frac{\partial\theta}{\partial x} \tag{6.7}$$

On the other hand $-\frac{\partial\theta}{\partial t}$ represensts the number of phase lines passing across a fixed x per unit time; it is called the *wave frequency.*

$$\text{wave frequency } = \omega = -\frac{\partial\theta}{\partial t} \tag{6.8}$$

To stay with a particular line of contant phase, say a crest, one must have

$$d\theta = kdx - \omega dt = 0$$

namely one must move at the *phase velocity*,

$$c = \left.\frac{dx}{dt}\right|_{\theta=\text{constant}} = \frac{\omega}{k} \tag{6.9}$$

Now back to the string. Substituting (6.10) in (6.1) we get

$$\left(-k^2 - \frac{K}{T} + \frac{\omega^2}{c_o^2}\right) V = 0 \tag{6.10}$$

or

$$\omega = c_o\sqrt{k^2 + \frac{K}{T}} \tag{6.11}$$

The phase speed is

$$c = \frac{\omega}{|k|} = c_o\left(1 + \frac{K}{Tk^2}\right)^{1/2} > c_o \tag{6.12}$$

See figure **??**. Note that, due the stiffening by the lateral support, the phase speed is always greater than c_o, and decreases monotonically with the wave number. Longer waves (small $|k|$) are faster, while shorter waves (larger $|k|$) are slower. As $|k|$ increases, c approaches the finite limit c_o for the shortest waves. In general a sinusoidal wave whose phase velocity depends on the wavelength, i.e., ω is a nonlinear function of k, is called a *dispersive wave*, and (6.11) or its equivalent (6.20) is called the *dispersion relation.*

An interesting physical feature for dispersive waves in general can be found by superposing two trains of sighltly different frequencies and wave numbers:

$$V = A\left(e^{i(k^+x-\omega^+t)} + Ae^{i(k^-x-\omega^-t)}\right) \tag{6.13}$$

where

$$k^\pm = k \pm k', \quad \omega^\pm = \omega(k^\pm), \quad k' \ll k. \tag{6.14}$$

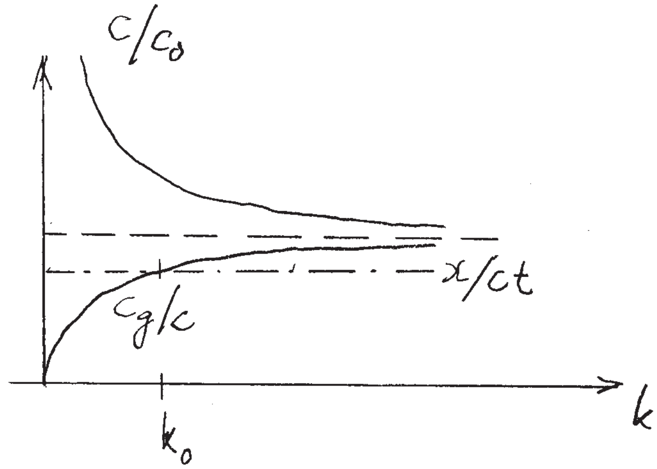

Figure 7: Phase and group velocities of a string in elastic surrounding. Point of Stationary phase k_0.

Let us approximate $\omega^{\pm}$ by

$$\omega^{\pm} = \omega(k) \pm k' \frac{d\omega}{dk} + O(k'^2)$$

then

$$V = Ae^{ikx-i\omega t}\left(e^{-ik'(x-(d\omega/dk)t)} + e^{-ik'(x-(d\omega/dk)t)}\right) = \mathcal{A}(x,t)e^{ikx-i\omega t} \tag{6.15}$$

where

$$\mathcal{A} = 2\cos k'\left(x - (d\omega/dk)t\right) \tag{6.16}$$

The factor $\exp(ikx - i\omega t)$ is called the *carrier wave* and $\mathcal{A}(x,t)$ the *envelope*. Thus the result is a sinusoidal wave train with a slowly varying envelope which has a very long wavelength $2\pi/k' \gg 2\pi/k$ and moves at the *group velocity*

$$c_g = \frac{d\omega}{dk} \tag{6.17}$$

which is in general different from the phase velocity for dispersive waves.

In our string problem, the group velocity is easily found from the dispersion relation (6.11)

$$c_g = \frac{c_o^2}{c} = \frac{Tk}{\rho\omega}, \quad \text{hence } \frac{c_g}{c} = \frac{c_o^2}{c^2} < 1 \tag{6.18}$$

Thus the group velocity is always smaller than the phase velocity, and increases with the wavenumber from 0 for the longest wave to the finite limit c_o (equal to the phase velocity) for the shortest waves.

When $\omega > \omega_c$ where

$$\omega_c = c_o\sqrt{\frac{K}{T}}. \tag{6.19}$$

is called the cutoff frequency ω_c, k is real

$$k = \pm|k|, \quad \text{with} \quad |k| = \left(\frac{\omega^2}{c_o^2} - \frac{K}{T}\right)^{1/2} \tag{6.20}$$

being the real and positive square root. $V(x,t)$ is a propagating wave, with the plus (minus) sign corresponding to right- (left-) going wave. If $\omega < \omega_c$, $k = \pm i\kappa$ is imaginary, with κ being the postive root:

$$\kappa = \left(\frac{K}{T} - \frac{\omega^2}{c_o^2}\right)^{1/2} \tag{6.21}$$

Then $e^{ikx} = e^{\mp\kappa x}$. For boundedness one must choose the minus (plus) sign for $x > 0$ ($x < 0$). Oscillations are localized or evanecent; there is no wave radiation.

As a simple application, (6.4) is the response in a semi-infinite string forced to oscillate at the left end $x = 0$,

$$y(0,t) = Ae^{-i\omega t} \tag{6.22}$$

If $\omega > \omega_c$, then

$$V(x,t) = \Re\left(Ae^{i|k|x - i\omega t}\right) \tag{6.23}$$

where $|k|$ is defined by (6.20). The requirement that waves due to a local disturbance can only radiate outwards is called the *radiation condition*. More will be said on it later. If $\omega < \omega_c$, we must require boundedness at infinity so that

$$V(x,t) = \Re Ae^{-\kappa x - i\omega t}, \quad x > 0 \tag{6.24}$$

At the cutoff frequency, $k = 0$, V is constant in x; the infinitely long string would oscillate in unison. This absurd result signfies the breakdown of the the linearized theory.

6.2 Energy transport

Along a unit length of the string the densities of kinetic energy per unit length is,

$$KE = \frac{\rho}{2}\left(\frac{\partial V}{\partial t}\right)^2 \tag{6.25}$$

The potential energy in any segment dx of the string is the work needed to deform it from static equilibrium. The part due to lengthening of the string against tension is

$$T(ds - dx) = T\sqrt{1 + \left(\frac{\partial V}{\partial x}\right)^2}dx - Tdx \approx \frac{T}{2}\left(\frac{\partial V}{\partial x}\right)^2 dx$$

Adding the part against the springs, the total potential energy per unit length is

$$PE = \frac{T}{2}\left(\frac{\partial V}{\partial x}\right)^2 + \frac{K}{2}V^2 \tag{6.26}$$

We now calculate their time averages. If two time-harmonic functions are written in the complex form:

$$a = \Re\left(Ae^{-i\omega t}\right), \quad b = \Re\left(Be^{-i\omega t}\right) \tag{6.27}$$

the time average of their product is given by

$$\overline{ab} \equiv \frac{\omega}{2\pi}\int_t^{t+\frac{2\pi}{\omega}} ab\, dt = \frac{1}{2}\Re(AB^*) = \frac{1}{2}\Re(A^*B) \tag{6.28}$$

Using this formula, the period-averaged energy densities are,

$$\overline{KE} = \frac{\rho}{2}\overline{\Re\left(-i\omega Ae^{i\theta}\right)^2} = \frac{\rho}{4}\omega^2|A|^2$$

$$\overline{PE} = \frac{T}{2}\overline{\Re\left(ikAe^{\imath\theta}\right)^2} + \frac{K}{2}\overline{(\Re Ae^{\imath\theta})^2} = \left(\frac{T}{4}k^2 + \frac{K}{4}\right)|A|^2$$

where $\theta \equiv |k|x - \omega t$ is the wave phase. Hence

$$\overline{E} = \overline{KE} + \overline{PE} = \frac{T}{4}\left(\frac{\rho\omega^2}{T} + k^2 + \frac{K}{T}\right)|A|^2 = \frac{\rho}{2}\omega^2|A|^2 \tag{6.29}$$

after using the dispersion relation.

Now the averaged rate of energy influx across any station x is

$$\overline{-T\frac{\partial V}{\partial x}\frac{\partial V}{\partial t}} = -T\,\overline{\Re(ikV)\Re(-i\omega V)} = \frac{T}{2}k\omega|A|^2 = \left(\frac{\rho}{2}\omega^2|A|^2\right)\left(\frac{Tk}{\rho\omega}\right) = \overline{E}c_g$$

Thus the speed of energy transport is the group velocty;. This result is quite general for many physical problems and is not limited to the springs.

7 Dispersion from a localized initial disturbance

The solution for monochromatic waves already shows that waves of different wavelengths move at different velocities. What then is the consequence of an initial disturbance? Since a general initial disturbance bounded in space can be represented by a Fourier integral which amounts to the sum of infinitely many sinusoids with a wide spectrum, we shall employ the tools of Fourier transform.

In addition to the governing equation:

$$\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = \frac{\partial^2 V}{\partial x^2} - \frac{KV}{T}, \qquad -\infty < x < \infty, \quad t > 0 \tag{7.1}$$

we add the initial (Cauchy) conditions

$$V(x,0) = f(x), \quad \frac{\partial V(x,0)}{\partial t} = 0 \tag{7.2}$$

Let us define the Fourier transform and its inverse by

$$\bar{f}(k) = \int_{-\infty}^{\infty} e^{-ikx} f(x)\, dx, \quad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}\bar{f}(k)\, dk \tag{7.3}$$

The transforms of (7.1) is

$$\frac{1}{c^2}\frac{d^2\bar{V}}{dt^2} = -k^2\bar{V} - \frac{K\bar{V}}{T}, \quad t > 0 \tag{7.4}$$

and of the intitial conditions are,

$$\bar{V}(k,0) = \bar{f}(k), \quad \frac{d\bar{V}(k,0)}{dt} = 0 \tag{7.5}$$

The solution for the tranform is

$$\bar{V}(k,t) = \bar{f}(k)\cos\omega t \tag{7.6}$$

where

$$\omega = c_o\sqrt{k^2 + \frac{K}{T}} \tag{7.7}$$

The Fourier inversion is

$$V(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\bar{f}(k)e^{ikx}\cos\omega t \tag{7.8}$$

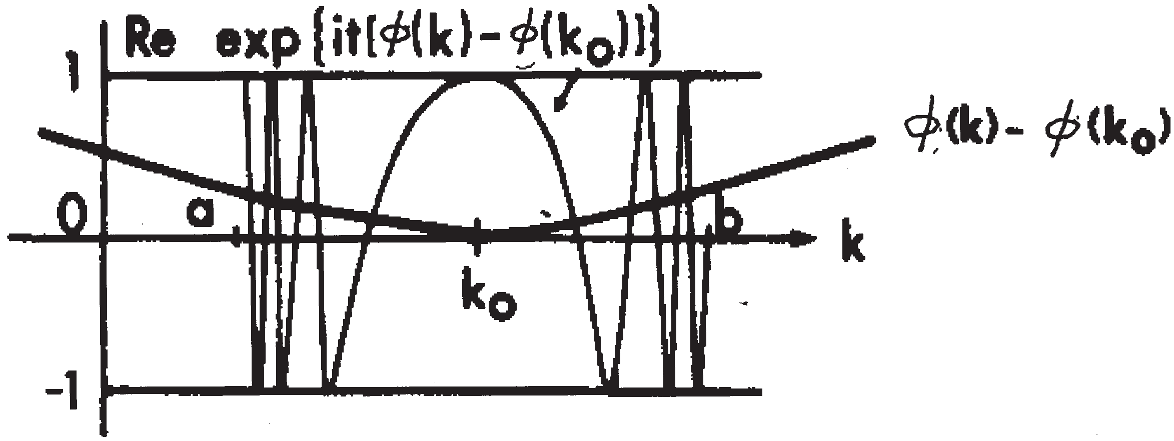

Figure 8: Neighborhood of Stationary Phase

Any real function $f(x)$ can be expressed as the sum of an even and an odd function of x. For simplicity let us assume that $f(x)$ is even in x so that $\bar{f}(k)$ is real and even in k, then

$$V(x,t) = \frac{1}{\pi}\int_0^\infty dk\bar{f}(k)\cos kx\cos\omega t$$

which can be manipulated to

$$V(x,t) = \frac{1}{2\pi}\Re\int_0^\infty dk\bar{f}(k)\left(e^{ikx-i\omega t} + e^{ikx+i\omega t}\right)dk \tag{7.9}$$

The first term in the integand represents the right-going wave while the second, left-going. Each part correponds to a superposition of sinusoidal wave trains over the entire range of wave numbers, within the small range $(k, k+dk)$ the amplitude is $\bar{f}(k)/2$. The function $\bar{f}(k)/2$ is called the Fourier amplitude spectrum. In general explicit evaluation of the Fourier integrals is not feasible. We shall therefore only seek approximate information. The method of stationary phase is particularly useful here. It aims at the asymptotic approximation of the integral

$$I(t) = \int_a^b F(k)e^{it\phi(k)}\,dk \tag{7.10}$$

for large t. Let us first give a quick derivation of the mathematical result. Assume that $F(k), \phi(k)$ are ordinary functions of k. If t is large, then as k increases along the path of integration both the real and imaginary parts of the exponential function

$$\cos(t\phi(k)) + i\sin(t\phi(k))$$

oscillates rapidly between -1 to +1, unless there is a point of stationary phase k_o within (a,b) so that

$$\frac{d\phi(k_o)}{dk} = \phi'(k_o) = 0, \quad a < k_o < b. \tag{7.11}$$

Then important contribution to the Fourier integral comes only from the neighborhood of k_o. Near the point of stationary phase, we approximate the phase by

$$\phi(k) = \phi(k_o) + \frac{1}{2}(k-k_o)^2\phi''(k_o) + \cdots$$

and the integral by

$$I(t) \approx F(k_o)e^{it\phi(k_o)}\int_a^b \exp\left(\frac{it}{2}(k-k_o)^2\phi''(k_o)\right)dk$$

With an error of $O(1/t)$, we also replace the limits of the last integral by $\pm\infty$; the justification is omitted here. Now it is known that

$$\int_{-\infty}^{\infty} e^{\pm itk^2}\,dk = \sqrt{\frac{\pi}{t}}e^{\pm i\pi/4}$$

It follows that

$$I(t) = \int_a^b F(k)e^{it\phi(k)}\,dk \approx F(k_o)e^{it\phi(k_o)\pm i\pi/4}\left[\frac{2\pi}{t|\phi''(k_o)|}\right]^{1/2} + O\left(\frac{1}{t}\right), \quad \text{if } \; k_o \in (a,b), \tag{7.12}$$

where the sign is + (or $-$) if $\phi''(k_o)$ is positive (or negative). It can be shown that if there is no stationary point in the range(a,b), then the integral $I(t)$ is small

$$I(t) = O\left(\frac{1}{t}\right), \quad \text{if } \; k_o \notin (a,b). \tag{7.13}$$

Let us apply this result to the right-going wave

$$V_+(x,t) = \frac{1}{4\pi}\Re\int_0^\infty dk\bar{f}(k)e^{it(kx/t-i\omega)}dk \tag{7.14}$$

For a fixed x/t =constant, we have

$$\phi(k) = kx/t - \omega(k), \quad \text{with} \quad x/t = \xi, \quad \omega = c_o\sqrt{k^2 + \frac{K}{T}} \tag{7.15}$$

For an observer travelling at the speed x/t, there is a stationary point k_o at the root of

$$x/t = \omega'(k_o) = \frac{Tk_o}{\rho\omega(k_o)} = \frac{c_o k_o}{\sqrt{k_o^2 + \frac{K}{T}}} \tag{7.16}$$

which increases from zero for long waves to the maximum c_o for very short waves. For any observer slower than c_o, there is a stationary point k_o. If the observer is faster than c_o, there is no stationary point.

Since

$$\phi''(k_o) = -\omega''(k_o) < 0, \tag{7.17}$$

the final result is that

$$\begin{aligned} V_+(x,t) &\sim \frac{1}{2\pi}\Re\left\{\bar{f}(k_o)\sqrt{\frac{2\pi}{t\omega''(k_o)}}e^{it\phi(k_o)-i\pi/4}\right\} \\ &= \frac{1}{2\pi}\Re\left\{\bar{f}(k_o)\sqrt{\frac{2\pi}{t\omega''(k_o)}}e^{ik_o x-\omega(k_o)t-i\pi/4}\right\} \end{aligned} \tag{7.18}$$

The transform $\bar{f}(k)$ depends on the specific profile of the initial disturbance. For the special case where

$$f(x) = \frac{Sb}{\pi(x^2 + b^2)} \tag{7.19}$$

which has an area S and characteristic width b, the Fourier transform is

$$\bar{f}(k) = Se^{-|k|b} \tag{7.20}$$

To a specific observer identified by the speed of travel x/t, the approximate result can be viewed as a simple harmonic wave train with wave number k_o, frequency $\omega(k_o)$ and amplitude

$$\mathcal{A} = \frac{\bar{f}(k_o)}{2\pi}\sqrt{\frac{2\pi}{t\omega''(k_o)}} \tag{7.21}$$

Since $\omega'(k)$ increases from 0 with increasing k to the finite maximum $T/\rho c_o = c_o$, an observer faster than c_o sees no waves. However any observer slower than c_o is accompanied by a train of progressive sinusoidal waves. The local wavelength k_o is such that the group velocty matches the observer's speed. The faster the observer, the shorter

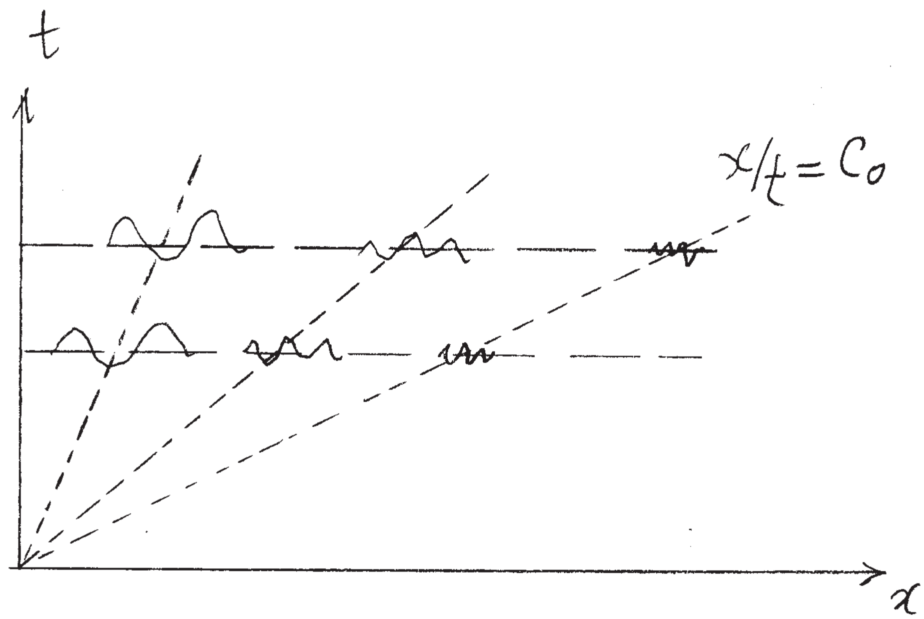

Figure 9: Snapshots of wave field on the side $x > 0$.

the waves. If a snapshot is taken, then the shortest wave, whose phase velocity is the lowest are seen at the front which moves as fast as the crests of the shortest waves. The crests of longer waves advance much faster than the local envelope. Because $\bar{f}(k)$ is the greatest at $k = 0$, the envelope of the longest waves which stays near the source, is the biggest. The envelope of the shortest waves is lower in amplitude and spreads out with the wave front. The entire disturbance attenuates in time as $t^{-1/2}$. See figure 7 for an overview.

Note also that very near the wave front, $c_g \to c_o$; the second derivative $\phi''(k_o) = -\omega''(k_o)$ vanishes. Hence the asymptotic formula breaks down. A better approximation is needed, but is omitted here. (See C. C. Mei, 1989, *Applied Dynamics of Ocean Surface Waves*). Finally we examine the propagation of wave energy in this transient problem. Using (7.21) the local energy density is:

$$E = \frac{1}{2}\rho\omega^2|\mathcal{A}|^2 = \frac{\rho\omega^2}{4\pi t}\frac{\bar{f}(k_o)^2}{\omega''(k_o)}$$

At any given t, the waves between two observers moving at slightly different speeds,

$c_g(k_1)$ and $c_g(k_2)$, i.e., between two points $x_1/t = c_g(k_1)$ and $x_2/t = c_g(k_2)$ are essentially simple harmonic so that the total energy is

$$\int_{x_1}^{x_2} dxE = \int_{x_1}^{x_2} dx \frac{\rho\omega^2}{4\pi t} \frac{(\bar{f}(k_o))^2}{\omega''(k_o)}$$

Since $x = \omega'(k_o)t$ for fixed t, we have

$$\frac{dx}{t} = \omega''(k_o)dk_o$$

Now for $x_2 > x_1$, $k_2 > k_1$, it follows that

$$\int_{x_1}^{x_2} dxE = \int_{k_1}^{k_2} dk_o \frac{\rho\omega^2}{4\pi} (\bar{f}(k_o))^2 = \text{constant}$$

Therefore the total energy between two observers moving at the local group velocity remains the same for all time. In other words, waves are transported by the local group velocity even in transient dispersion.

8 Scattering of sinusoidal waves

If along a long rod there are some localized inhomogeneities, an incoming train of sinusoidal waves will be partly reflected and partly transmitted. The scattered signals tell us something about the scatterer. To determine the scattering properties for a known scatterer is called the scattering problem. To determine the scatterer from the scattering data is called the inverse scattering problem. We shall only study the former.

Various mathematical techniques are needed for different cases: (i) Weak scatterers characterized by small amplitude relative to the wavelength, or slow variation within a wavelength, (ii) Strong scatterers if their dimensions are comparable to the wave length.

8.1 Weak scattering

Let the long rod have a slightly nonuniform cross section,

$$S(x) = S_o(1 + \epsilon a(x)), \quad \epsilon \ll 1. \tag{8.1}$$

where $a(x)$ diminishes to zero at $x \sim \pm\infty$. The wave equation reads

$$\rho S(x) \frac{\partial^2 u}{\partial t^2} = E \frac{\partial}{\partial x} \left(S(x) \frac{\partial u}{\partial x} \right) \tag{8.2}$$

For sinusoidal waves

$$u(x,t) = \Re\left[U(x)e^{-i\omega t}\right]$$

the spatial factor is governed by

$$\frac{d}{dx}\left(S\frac{dU}{dx}\right) + \rho\omega^2 SU = 0 \tag{8.3}$$

Then the solution can be sought by the perturbation method

$$U(x) = U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + \cdots \tag{8.4}$$

Substituting this into the governing equation

$$ES_o\frac{d}{dx}\left\{(1+\epsilon a(x))\frac{d}{dx}\left[U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + \cdots\right]\right\}$$
$$+\rho\omega^2 S_o(1+\epsilon a(x))\left[U_0(x) + \epsilon U_1(x) + \epsilon^2 U_2(x) + \cdots\right] = 0 \tag{8.5}$$

and equating the coefficients of like powers of ϵ to zero, we get from the terms of order $O(\epsilon^0)$:

$$ES_o\frac{d^2U_0}{dx^2} + \rho\omega^2 S_0 U_0 = 0 \tag{8.6}$$

The solution is simply the incident wave

$$U_0 = Ae^{ikx}, \quad \text{with} \quad k = \omega\sqrt{\rho/E} \tag{8.7}$$

At the order $O(\epsilon)$ we get

$$ES_o\frac{d^2U_1}{\partial x^2} + \rho\omega^2 S_0 U_1 + a(x)\left(ES_o\frac{d^2U_0}{\partial x^2} + \rho\omega^2 S_0 U_0\right) + ES_o\frac{da}{dx}\frac{dU_0}{dx} = 0$$

or,

$$\frac{d^2U_1}{dx^2} + k^2 U_1 = -\frac{da}{dx}\frac{dU_0}{dx} \tag{8.8}$$

Thus the scattered wave is at most of the order ϵ and is governed by an inhomogeneous equation. Let us define the fundamental solution (Green's function) by

$$\frac{d^2G(x,x')}{dx^2} + k^2 G(x,x') = \delta(x-x') \tag{8.9}$$

$$G \sim \text{outgoing waves at} \pm\infty \tag{8.10}$$

It will be shown in Appendix A that

$$G(x,x') = \frac{e^{ik|x-x'|}}{2ik} \tag{8.11}$$

By straightforward differentiation, it can be verified that

$$U_1(x) = -\int_{-\infty}^{\infty} \frac{da(x')}{dx'}\frac{dU_0(x')}{dx'}G(x,x')dx' = -\frac{1}{2ik}\int_{-\infty}^{\infty} \frac{da(x')}{dx'}\frac{dU_0(x')}{dx'}e^{ik|x-x'|}dx' \tag{8.12}$$

Since $U_0(x') = e^{ikx'}$, we have

$$\begin{aligned} U_1 &= -\frac{A}{2}\int_{-\infty}^{\infty} \frac{da(x')}{dx'}e^{ikx'}e^{ik|x-x'|}dx' \\ &= -\frac{A}{2}\left\{e^{ikx}\int_{-\infty}^{x}\frac{da}{dx'}dx' + e^{-ikx}\int_{x}^{\infty}\frac{da}{dx'}e^{2ikx'}dx'\right\} \qquad (8.13) \\ &= -\frac{A}{2}\left\{e^{ikx}a(x) + e^{-ikx}\int_{x}^{\infty}\frac{da}{dx'}e^{2ikx'}dx'\right\} \\ &= -\frac{A}{2}\left\{a(x)e^{ikx} + e^{-ikx}\int_{x}^{\infty}\left[\frac{d}{dx'}\left(a(x')e^{2ikx'}\right)dx' - a2ike^{2ikx'}\right]dx'\right\} \\ &= ikAe^{-ikx}\int_{x}^{\infty} a(x')e^{2ikx'}dx' \qquad (8.14) \end{aligned}$$

Far to the right we get

$$U_1 = 0, \quad x \to \infty. \tag{8.15}$$

Thus on the transmission side modification of the incident waves can at most be of the order $O(\epsilon^2)$. Far to the left $x \sim -\infty$,

$$U \cong \epsilon U_1 = Ae^{-ikx}\left(i\epsilon k\int_{-\infty}^{\infty} a(x')e^{2ikx'}dx'\right), \quad x \to -\infty. \tag{8.16}$$

Let us define the reflection coefficient by

$$R = i\epsilon k\int_{-\infty}^{\infty} a(x')e^{2ikx'}dx', \tag{8.17}$$

For instance if a is a hump of unit total dimensionless area and length b,

$$a(x) = \frac{b}{\pi(x^2+b^2)} \tag{8.18}$$

then

$$R = i\epsilon kbe^{-2kb} \tag{8.19}$$

Hence reflection is small if kb is large, i.e., long and gentle obstacles are not effective in reflecting short waves . For very long waves or a short obstacle, reflection is also

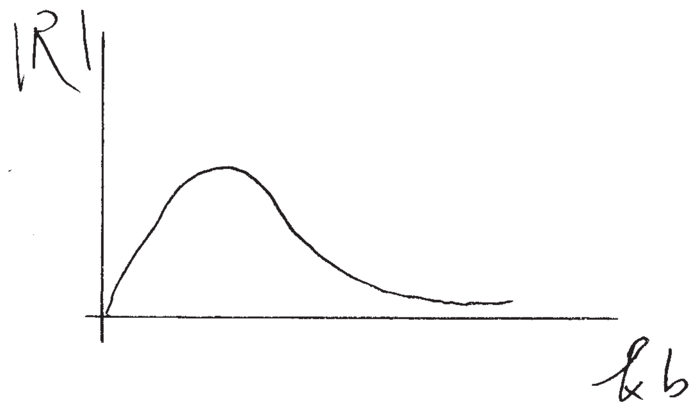

Figure 10: Reflection coefficient given by (8.19).

vanishingly small. The maximum reflection coefficient is $R = i\epsilon a_o/2$ when $kb = 1/2$. Note that far to the right, $U_1(x) \to 0$ as $x \to \infty$. The transmission vwave is

$$U \cong U_0 + \epsilon U_1 = Ae^{ik_0 x}(1 + O(\epsilon^2)). \tag{8.20}$$

The tranmission coefficient is $T = 1 + O(\epsilon^2)$. This is consistent with the general law of energy conservation $|R|^2 + |T|^2 = 1$ to be proven later.

Homework: Check if the governing equation at $O(\epsilon^2)$ for U_2 is:

$$ES_o \frac{d^2 U_2}{\partial x^2} + \rho\omega^2 S_0 U_2 = ES_o \left[a \frac{da}{dx}\frac{dU_0}{dx} - \frac{da}{dx}\frac{dU_1}{dx} \right] \tag{8.21}$$

Carry out the solution to find the transmission coefficient T from the amplitude of $U_0 + \epsilon U_1 + \epsilon^2 U_2$ at $x \sim \infty$, and show that energy is conserved to the order $O(\epsilon^2)$.

8.2 Strong scattering of long water-waves by a shelf

Consider an ocean bottom with a step-wise variation of depth.

$$h = \begin{cases} h_1, & x < -a; \\ h_2, & -a < x < a; \\ h_3 = h_1, & x > a \end{cases} \tag{8.22}$$

How does the step interupt the propagation of an incident wave of unit amplitude arriving from $x \sim -\infty$?

In each zone of contant depth ($i = 1, 2, 3$), the shallow water equations read:

$$\frac{\partial \zeta_i}{\partial t} + h_i \frac{\partial u_i}{\partial x} = 0 \tag{8.23}$$

$$\frac{\partial u_i}{\partial t} + g \frac{\partial \zeta_i}{\partial x} = 0 \tag{8.24}$$

For sinusoidal waves

$$\zeta_i = \eta_i e^{-i\omega t}, \quad u_i = U_i e^{-i\omega t} \tag{8.25}$$

we get

$$-i\omega\eta_i + h_i \frac{\partial U_i}{\partial x} = 0 \tag{8.26}$$

$$-i\omega U_i + g \frac{\partial \eta_i}{\partial x} = 0 \tag{8.27}$$

or

$$\frac{d^2\eta_i}{dx^2} + k_i^2 \eta_i = 0, \quad \text{where} \quad k_i = \frac{\omega}{\sqrt{g h_i}} \tag{8.28}$$

At a junction, the pressure and the flux must be equal, hence

$$\eta_1 = \eta_2, \quad \text{and} \quad h_1 \frac{d\eta_1}{dx} = h_2 \frac{d\eta_2}{dx}, \quad x = -a; \tag{8.29}$$

$$\eta_2 = \eta_3, \quad \text{and} \quad h_2 \frac{d\eta_2}{dx} = h_1 \frac{d\eta_3}{dx}, \quad x = a. \tag{8.30}$$

The forms of the solutions in each zone of constant depth are:

$$\eta_1 = e^{ik_1(x+a)} + R e^{-ik_1(x+a)}, \quad x < -a; \tag{8.31}$$

$$\eta_2 = A e^{ik_2 x} + B e^{-ik_2 x}, \quad -a < x < a \tag{8.32}$$

$$\eta_3 = T e^{ik_1(x-a)}, \quad x > a \tag{8.33}$$

The reflection and transmission coefficients R and T as well as A and B are yet unknown.

Applying the matching conditions at the left junction, we get two relations

$$1 + R = A e^{ik_2 a} + B e^{ik_2 a} \tag{8.34}$$

$$k_1 h_1 (1 - R) = k_2 h_2 (A e^{-ik_2 a} - B e^{ik_2 a}). \tag{8.35}$$

Similarly the matching conditions at $x = a$ gives

$$Ae^{-ik_2 a} + Be^{-ik_2 a} = T \tag{8.36}$$

$$k_2 h_2 (Ae^{-ik_2 a} - Be^{ik_2 a}) = k_1 h_1 T. \tag{8.37}$$

These four equations can be solved to give

$$T = \frac{4s}{(1+s)^2 e^{2ik_2 a} - (1-s)^2 e^{-2ik_2 a}} \tag{8.38}$$

$$R = \frac{-(1-s)^2 (e^{-2ik_2 a} - e^{2ik_2 a})}{(1+s)^2 e^{2ik_2 a} - (1-s)^2 e^{-2ik_2 a}} \tag{8.39}$$

$$A = \frac{T}{2} e^{-ik_2 a} (1+s) \tag{8.40}$$

$$B = \frac{T}{2} e^{ik_2 a} (1-s) \tag{8.41}$$

where

$$s = \frac{k_1 h_1}{k_2 h_2} = \sqrt{\frac{h_1}{h_1}} = \frac{c_1}{c_2} \tag{8.42}$$

The energy associated with the tranmsitted and reflected waves are :

$$|T|^2 = \frac{4s^2}{4s^2 + (1-s^2)^2 \sin^2 2k_2 a} \tag{8.43}$$

$$|R|^2 = \frac{(1-s^2) \sin^2 2k_2 a}{4s^2 + (1-s^2)^2 \sin^2 2k_2 a} \tag{8.44}$$

It is evident that $|R|^2 + |T|^2 = 1$.

Over the shelf the free surface is given by

$$\eta_2 = \frac{2s \left[(1+s) e^{ik_2 (x-a)} + (1-s) e^{-ik_2 (x-a)} \right]}{(1+s)^2 e^{-ik_2 a} - (1-s)^2 e^{ik_2 a}} \tag{8.45}$$

Recalling the time factor $e^{-i\omega t}$, we see that the free surface over the shelf consists of two wave trains advancing in oppopsite directions. Therefore along the shelf the two waves can interfere each other constructively, with the crests of one coinciding with the crests of the other at the same moment. At other places the interference is destructive, with the crests of one wave train coinciding with the troughs of the other. The envelope of energy on the shelf is given by

$$|\eta|^2 = \frac{4s^2 \left[\cos^2 k_2 (x-a) + s^2 \sin^2 k_2 (x-a) \right]}{4s^2 + (1-s)^2 \sin^2 2k_2 a} \tag{8.46}$$

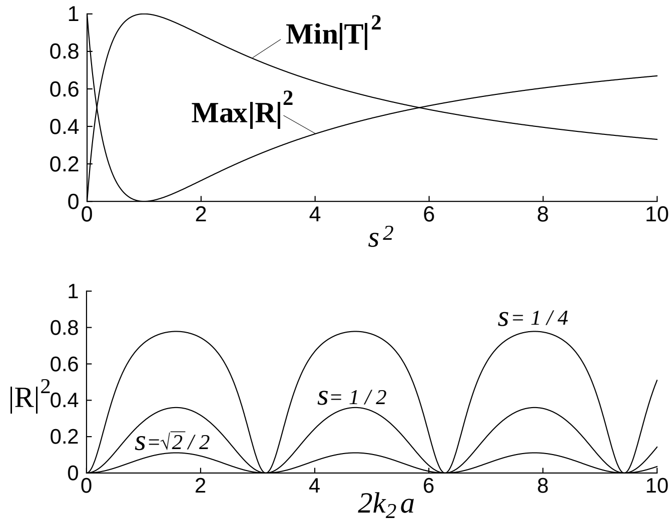

Figure 11: Transmission and reflection coefficients for a rectangular shelf. From Mei, C.C., *Applied Dynamics of Ocean Surface Waves.*

At the downwave edge of the shelf, $x = a$, the envelope is

$$|\eta|^2 = \frac{4s^2}{4s^2 + (1-s)^2 \sin^2 2k_2 a} \tag{8.47}$$

Note that the reflection and transmission coefficients are oscillatory in $k_2 a$. In particulatr for $2k_2 a = n\pi, n = 1, 2, 3...$, that is $4a/\lambda = n$, $|R| = 0$ and $|T| = 1$. The shelf is transparent to the incident waves, correspoinidng to the most constructive interference and the strongest transmission Mininum transmision and maximum reflection occur when $2k_2 a = (n - 1/2)\pi$, or $4a/\lambda = n - 1/2$, when the interference is the most destructive. The corresponding transimssion and reflection coefficients are

$$\min|T|^2 = \frac{4s^2}{(1+s^2)^2}, \quad \max|R|^2 = \frac{(1-s^2)^2}{(1+s^2)^2}. \tag{8.48}$$

See Figure 11

The features of interference can be explained physically. The incident wave train consists of periodic crests and troughs. When one of the crests first strikes the left edge at $x = a$, part of the it is transmitted onto the shelf and part is reflected towards $x \sim -\infty$. After reaching the right edge at $x = a$, the tranmitted crest has a part reflected to the left and re-reflected by the edge $x = -a$ to the right again. When the re-reflected crest arrives at the right edge the second time, its total travel distance is $4a$. If $4a$ is an integral multple of the wave length λ_2, the crest is in phase with all the other crests entering the shelf either before (the third, fourth, fifth, ... time) or after. Thus all the crests reinforce one another at the right edge. This is constructive interference, leading to the strongest tranmission to the right $x \sim \infty$. On the other hand if $2k_2a = (n - 1/2)\pi$ or $4a/\lambda = n - 1/2$, some crests will be in opposite phase to some other crests, leading to the most destructive interference at the right edge, and smallest transmission.

9 General identities in scattering by arbitrary inhomogeneities

Scattering due to inhomogeneities, caused by nonuniformities either in geometry or in material properties, requires the solution of ordinary differential equations with variable coefficients. In general one must resort to numerical means. For the sake of checking numerical accuracy and for gaining physical insight, identities which must be true are useful. They are often deduced by arguments typical in the derivation or the use of Green's theorem.

We illustrate these identities by the example of an infinitely long rod with variable cross section $S(x)$,

$$\frac{\partial}{\partial x}\left(S\frac{\partial u}{\partial x}\right) = \frac{\rho S}{E}\frac{\partial^2 u}{\partial t^2} \tag{9.1}$$

E and ρ are taken to be constants. $S(x)$ is nonuniform only in a finite neighborhood around $x = 0$. Elsewhere $S = S_o$ =constant. Consider monochromatic waves

$$u(x,t) = U(x)e^{-i\omega t} \tag{9.2}$$

so that $U(x)$ is governed by the ordinary differential equation

$$(SU')' + b^2SU = 0, \quad b^2 = \frac{\omega^2\rho}{E} \tag{9.3}$$

The boundary conditions depend on the source of the incident waves.

If the incident waves arrive from $x \sim -\infty$, the asymptotic form of the solution must be

$$U_1(x) \sim A_- e^{ikx} + B_- e^{-ikx} = A_- \left(e^{ikx} + R_1 e^{-ikx} \right), \quad kx \sim -\infty \tag{9.4}$$

and

$$U_1(x) \sim A_+ e^{ikx} = A_- T_1 e^{ikx} \quad kx \sim \infty \tag{9.5}$$

where T_1 and R_1 are the transmission and reflection coefficients which ar a part of the unknown solution of the left-incidence problem. Let us define the Jost function by

$$f_1(x) = \frac{U_1}{A_- T_1} \tag{9.6}$$

then

$$f_1(x) \sim \frac{1}{T_1} e^{ikx} + \frac{R_1}{T_1} e^{-ikx} \quad kx \ll -1 \tag{9.7}$$

and

$$f_1(x) \sim e^{ikx} \quad kx \gg 1 \tag{9.8}$$

On the other hand, if the incident waves arrive from $x \sim \infty$, the asymptotic form of the solution must be

$$U_2(x) \sim b_- e^{-ikx} = b_+ T_2 e^{-ikx}, \quad kx \sim -\infty \tag{9.9}$$

and

$$U_2(x) \sim a_+ e^{ikx} + b_+ e^{-ikx} = b_+ \left(R_2 e^{ikx} + e^{-ikx} \right), \quad kx \sim \infty \tag{9.10}$$

where T_2 and R_2 are the unknown transmission and reflection coefficients of the right-incidence problem. Similarly we define the Jost function by the following asymptotic solution

$$f_2(x) \sim e^{-ikx}, \quad kx \sim -\infty \tag{9.11}$$

and

$$f_2(x) \sim \frac{1}{T_2} e^{-ikx} + \frac{R_2}{T_2} e^{ikx}, \quad kx \sim \infty \tag{9.12}$$

Since both f_1 and f_2 satisfy (9.3), we have

$$(S f_1')' + b^2 S f_1 = 0, \qquad \text{and} \quad (S f_2')' + b^2 S f_2 = 0, \tag{9.13}$$

Multiplying the first by f_2 and the second by f_1, then taking the difference, we get after partial integration,

$$\frac{d}{dx}\left(S(f_1 f_2' - f_2 f_1')\right) = 0,$$

or

$$W(f_1, f_2) = (f_1 f_2' - f_2 f_1') = \frac{C}{S(x)} \tag{9.14}$$

The constant C can be related to the asymptotic limits at $k|x| \gg 1$ where $S \to S_o$. Far on the incident side, the Wronskian is

$$\begin{aligned}
&(f_1 f_2' - f_2 f_1')_{x\sim-\infty} \\
&\quad = \left(\frac{1}{T_1}e^{ikx} + \frac{R_1}{T_1}e^{-ikx}\right)(-ik)e^{-ikx} - e^{-ikx}\left(\frac{ik}{T_1}e^{ikx} - \frac{ikR_1}{T_1}e^{-ikx}\right) \\
&\quad = -\frac{2ik}{T_1}
\end{aligned}$$

Far on the transmission side, the Wronskian is

$$\begin{aligned}
&(f_1 f_2' - f_2 f_1')_{x\sim\infty} \\
&\quad = e^{ikx}\left(\frac{-ik}{T_2}e^{-ikx} + \frac{ikR_2}{T_2}e^{ikx}\right) - \left(\frac{1}{T_2}e^{-ikx} + \frac{R_2}{T_2}e^{ikx}\right)(ik)e^{ikx} \\
&\quad = -\frac{2ik}{T_2}
\end{aligned}$$

Since the two Wronskians are equal, we conclude that

$$T_1 = T_2 \tag{9.15}$$

Thus, the complex transmission coefficients are equal for incident waves coming from either side, no matter how asymmetric the geometry may be in the near field!

Next we consider the left-incidence problem. Multiplying the first of (9.3) by the complex conjugate of U_1, i.e., U_1^*, and taking the difference of the result with its complex conjugate, we get

$$U_1 S(U_1^{*\prime})' - U_1^*(SU_1')' = \frac{d}{dx}\left(S(U_1 U_1^{*\prime} - U_1{}^* U_1')\right) = 0 \tag{9.16}$$

hence

$$S(U_1 U_1^{*\prime} - U_1^* U_1') = \text{constant} \tag{9.17}$$

By using the asymptotic expression (9.4) on the incidence side, we get

$$\begin{aligned}
&\left[S(U_1 U_1^{*\prime} - S U_1^* U_1^\prime)\right]_{x\sim-\infty} \\
&= S_o\left(A_- e^{ikx} + B_- e^{-ikx}\right)\left(-ikA_-^* e^{-ikx} + ikB_-^* e^{ikx}\right) \\
&- S_o\left(A_-^* e^{-ikx} + B_-^* e^{ikx}\right)\left(ikA_- e^{ikx} - ikB_- e^{-ikx}\right) \\
&= -2ik|A_-|^2 + 2ik|B_-|^2
\end{aligned}$$

Similarly by using the asymptotic expression (9.5) on the transmission side, we get

$$\begin{aligned}
&\left[S(U_1 U_1^{*\prime} - U_1^* U_1^\prime)\right]_{x\sim\infty} \\
&= S_o\left(A_+ e^{ikx}\right)\left((-ik)A_+^* e^{-ikx}\right) - S_o\left(A_+^* e^{-ikx}\right)\left((ik)A_+ e^{ikx}\right) \\
&= -2ikS_o|A_+|^2
\end{aligned}$$

Equating the two limits we conclude that

$$|A_-|^2 = |B_-|^2 + |A_+|^2 \tag{9.18}$$

or

$$|R_1|^2 + |T_1|^2 = 1. \tag{9.19}$$

This is merely a statement of conservation flux : the flux rate of total scattered energy (reflected and transmitted) is equal to the flux rate of the incident wave energy; the speed of energy tranport being the same on both sides of the scatterer. Clearly the same energy conservation must hold for the right-incidence problem.

$$|R_2|^2 + |T_2|^2 = 1 \tag{9.20}$$

If the scattering problem is solved numerically by, say, finite elements, it is necesssary that the computed scattering coefficients R and T satisfy the identities (9.15) (9.19) and (9.20).

10 Refraction in a slowly varying medium

For time-harmonic cases, we again use shallow water waves for demonstration. The governing equation is

$$\frac{d}{dx}\left(h\frac{d\eta}{dx}\right) + \frac{\omega^2}{g}\eta = 0 \tag{10.1}$$

Consider a sea depth which varies slowly within a wavelength, i.e.,

$$\frac{1}{kh}\frac{dh}{dx} = O(\mu) \ll 1 \tag{10.2}$$

Earlier analysis suggests that reflection is negligibly small. Thus the solution is expected to be a locally progressive wave with both the wavenunmber and amplitude varying much more slowly than the wave phase in x . Hence we try the solution

$$\eta = A(x)e^{i\theta(x)} \tag{10.3}$$

where $\theta(x) - \omega t$ is the phase function and

$$k(x) = \frac{d\theta}{dx} \tag{10.4}$$

is the local wave number and A is the complex amplitude. Note the spatial rate of variation of the phase function, i.e., the wave number is in gneral not small, therefore θ itself is not a slowly varying function of x. Let us calculate the first derivative:

$$\frac{d\eta}{dx} = \left(ikA + \frac{dA}{dx}\right)e^{i\theta}$$

and assume

$$\frac{\frac{dA}{dx}}{kA} = O(kL)^{-1} \ll 1$$

In fact we shall assume each derivative of h, A or k is μ times smaller than kh, kA or k^2. Futhurmore,

$$\frac{d}{dx}\left(h\frac{d\eta}{dx}\right) + \frac{\omega^2}{g}\eta = \left[ik\left(ikh + h\frac{dA}{dx}\right) + \frac{d}{dx}\left(h\frac{dA}{dx}\right) + i\frac{d(khA)}{dx} + \frac{\omega^2 A}{g}\right]e^{i\theta} = 0$$

Note that the complex amplitude A can be written as

$$A = |A(x)|e^{i\theta_A(x)} \tag{10.5}$$

The phase of A: θ_A, is a slowly varying function of x. It can be clonsidered as a part of the wave phase, although its spatial gradient is much smaller than k.

Now let's expand

$$A = A_0 + A_1 + A_2 + \cdots \tag{10.6}$$

with $A_1/A_0 = O(\mu), A_2/A_0 = O(\mu^2), \cdots$. From $O(\mu^0)$ the dispersion relation follows:

$$\omega^2 = ghk^2, \quad \text{or} \quad k = \frac{\omega}{\sqrt{gh}} \tag{10.7}$$

Thus the local wave number and the local depth are related to frequency according to the well known dispersion relation for constant depth. As the depth decreases, the wavenumber increases. Hence the local phase velocity

$$c = \frac{\omega}{k} = \sqrt{gh} \tag{10.8}$$

also decreases.

From $O(\mu)$ we get,

$$ikh\frac{dA_0}{dx} + i\frac{d(khA_0)}{dx} = 0$$

or, after multiplication by A_0^*,

$$\frac{d}{dx}(kh|A_0|^2) = 0 \tag{10.9}$$

which means

$$kh|A_0|^2 = C^2 = \text{constant}$$

or,

$$\sqrt{gh}|A_0|^2 = \text{constant} = \sqrt{gh_\infty}|A_\infty|^2 \tag{10.10}$$

Since in shallow water the group velocity equals the phase velocity, the above result means that the rate of energy flux is the same for all x and is consistent with the original assumption of unidirectional propagation. Furthermore, the local amplitude increases with depth as

$$\frac{A_0(x)}{A_\infty} = \left(\frac{h_\infty}{h}\right)^{1/4} \tag{10.11}$$

This result is called Green's law.

In summary, the leading order solution is

$$\zeta = A_\infty \left(\frac{h_\infty}{h}\right)^{1/4} e^{i\theta - i\omega t} = A_\infty \left(\frac{h_\infty}{h}\right)^{1/4} \exp\left(i\int^x k(x')dx' - i\omega\right) \tag{10.12}$$

A Solution for the Green function

Let us divide the domain into two parts, corresponding to two sides of the concentrated forcing. On the right side $x > x_0$, we define $G(x, x_0) \equiv G_+(x, x_0)$ which must be governed by :

$$\frac{d^2G}{dx^2} + k_0^2 G_+ = 0, \quad x > x_0 \tag{A.1}$$

To satisfy the radiation condition we take the sol,ution to beTo satisfy the radiation condition we take

$$G_+ = Ce^{ik_0x} \tag{A.2}$$

On the left side we define $G(x, x_0) = G_-(x, x_0)$, which must be governed by

$$\frac{d^2G_-}{dx^2} + k_0^2G_- = 0, \quad x < x_0 \tag{A.3}$$

To satisfy the radiation condition we take

$$G_+ = De^{-ik_0x} \tag{A.4}$$

At $x = x_0$ we require continuity:

$$G_+(x_0, x_0) = G_-(x_), x_)) \tag{A.5}$$

Hence

$$Ce^{ik_ox_0} = De^{-ik_0x_0} \tag{A.6}$$

In addition we get another matching condition at $x = x_0$ by integrating eq(8.8) across the concentrated forcing

$$\int_{x_0-\delta}^{x_0+\delta} \left\{ \frac{d^2G}{dx^2} + k_0^2G \right\} dx = \int_{x_0-\delta}^{x_0+\delta} \delta(x - x_0)dx = 1.$$

The left-hand side can be integrated to give the second matching codition,

$$\left[\frac{dG_+}{dx}\right]_{x_0} - \left[\frac{dG_-}{dx}\right]_{x_0} = 1 \tag{A.7}$$

implying that

$$ik_0Ce^{ik_0x_0} + ik_0De^{-ik_0x_o} = 1 \tag{A.8}$$

Thus

$$C_0 = \frac{e^{-ik_0x_0}}{2ik_0}, \quad D_0 = \frac{e^{ik_0x_0}}{2ik_0}, \tag{A.9}$$

The final solution is

$$G_+ = \frac{e^{ik_0(x-x_0)}}{2ik_0}, \quad x - x_0 > 0; \quad G_- = \frac{e^{-ik_0(x-x_0)}}{2ik_0}, \quad x - x_0 < 0.$$

or more compactly,

$$G(x, x_0) = \frac{e^{ik_0|x-x_0|}}{2ik_0} \tag{A.10}$$