1.138J/2.062J, WAVE PROPAGATION

Fall, 2000 MIT

Notes by C. C. Mei

Homework set No. 3. Due Oct 12. 2000.

(1). Read §2.8, Notes. For a slightly nouniform rod, check if the governing equation at $O(\epsilon^2)$ for U_2 is:

$$ES_o \frac{d^2U_2}{\partial x^2} + \rho\omega^2 S_0 U_2 = ES_o \left[a\frac{da}{dx}\frac{dU_0}{dx} - \frac{da}{dx}\frac{dU_1}{dx} \right] \tag{H.3.1}$$

Carry out the solution to find the transmission coefficient T from the amplitude of $U_0 + \epsilon U_1 + \epsilon^2 U_2$ at $x \sim \infty$, and show that energy is conserved to the order $O(\epsilon^2)$.

The solution given here is based on the wisdom of Diane Jarrah, Francois Blanchette, Lixian Liu and Guangyu Wu

Let

$$U_0 + \epsilon U_i + \epsilon^2 U_2 + \cdots \tag{H.3.2}$$

then

$$U_0 = Ae^{ikx}, \tag{H.3.3}$$

$$U_1 = \frac{A}{2}\left[e^{-ikx}\int_x^\infty e^{-2ik\xi} a_\xi d\xi + a(x)e^{ikx} \right] \tag{H.3.4}$$

Let

$$R = \epsilon R_1 + \epsilon R_2 + \cdots, \quad T = 1 + \epsilon T_1 + \epsilon^2 T_2 + \cdots \tag{H.3.5}$$

Clearly

$$U_1(\infty) = T_1 = 0, \tag{H.3.6}$$

$$R_1 = -\frac{1}{2}\int_{-\infty}^\infty e^{-2ik\xi} a_\xi d\xi = ik\int_x^\infty e^{-2ik\xi} a(\xi) d\xi \tag{H.3.7}$$

For energy flux conservation $RR^* + TT^* = 1$, we need

$$1 = \epsilon^2 R_1 R_1^* + \cdots + (1 + \epsilon^2 T_1 + \cdots)(1 + \epsilon^2 T_2^* + \cdots) = \epsilon^2 R_1 R_1^* + \cdots 1 + \epsilon^2 (T_2 + T_2^*) + \cdots \tag{H.3.8}$$

Hence we need to prove that

$$R_1 R_1^* + T_2 + T_2^* = 0 \tag{H.3.9}$$

– – – – – – – – – – – – – –

Using Green's function

$$\begin{aligned}
U_2(x) &= \int_{-\infty}^{\infty} \frac{d\xi}{2ik} e^{ik|x-\xi|} a_\xi \left(a\frac{dU_0}{d\xi} - \frac{dU_1}{d\xi} \right) \\
&= \int_{-\infty}^{\infty} \frac{d\xi}{2ik} e^{ik|x-\xi|} a_\xi \left(aikAe^{ik\xi} - \frac{A}{2}\left\{ ike^{-ik\xi} \int_\xi^\infty a_\eta e^{2ik\eta} d\eta + ikae^{ik\xi} \right\} \right] \\
&= \frac{A}{2} \int_{-\infty}^{\infty} d\xi e^{ik|x-\xi|} a_\xi \left(\frac{a}{2} e^{ik\xi} - \frac{e^{-ik\xi}}{2} \int_\xi^\infty a_\eta e^{2ik\eta} d\eta \right) \\
&= \frac{A}{2} \int_{-\infty}^{x} d\xi a_\xi \left(\frac{a}{2} - \frac{e^{-2ik\xi}}{2} \int_\xi^\infty a_\eta e^{2ik\eta} d\eta \right) \\
&+ \frac{A}{2} \int_0^\infty a_\xi \left(\frac{a}{2} e^{2ik\xi} - \frac{1}{2} \int_\xi^\infty a_\eta e^{2ik\eta} d\eta \right) \qquad \text{(H.3.10)}
\end{aligned}$$

Integrationg by parts,

$$\int_\xi^\infty d\xi e^{2ik\xi} a_\xi = ae^{2ik\xi}|_\xi^\infty - 2ik \int_\xi^\infty e^{2ik\eta a} d\eta$$