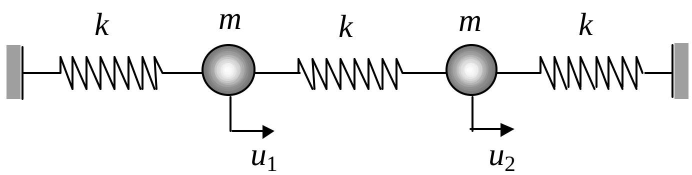
HOMEWORK #2 OF 1.581/13.801
(due September 26 2001)

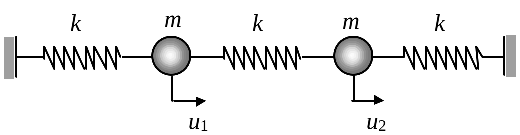
PROBLEM 1 (**P.5** on page 368 in the lecture note)

Find the two vibration frequencies and modal shapes (i.e. the relative values of u_1 and u_2) for the system shown. Apply symmetry – antisymmetry concepts.



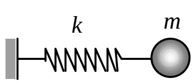
SOLUTION

- Antisymmetric Mode



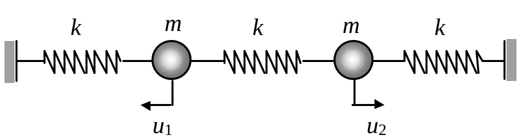
The deformation of the two masses is: $u_1 = u_2 = u$

Therefore the middle spring does not participate, and the system is equivalent to:



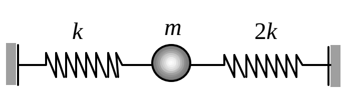
$$w_n = \sqrt{\frac{k}{m}}$$

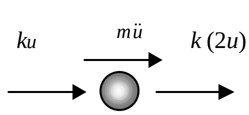
- Symmetric Mode



The deformation of the two masses is: $u_1 = u$, $u_2 = -u$

The middle spring deforms twice as much, therefore the system is equivalent to:





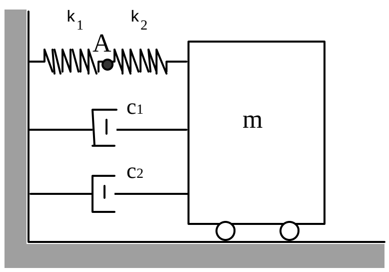
$$m\ddot{u} + 3k\,u = 0$$

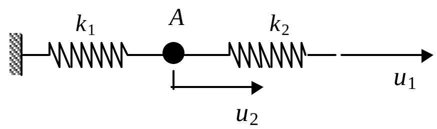
$$w_n = \sqrt{\frac{3k}{m}}$$

<u>PROBLEM 2</u> (**<u>P.6</u>** on page 368 in the lecture note)

Write the equations of motion for the following dynamic systems

SOLUTION



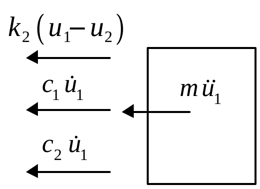


Equilibrium of point A:

$$k_1 u_2 = k_2 (u_1 - u_2)$$

$$u_2 = \frac{k_2}{k_1 + k_2} u_1 \qquad \{1\}$$

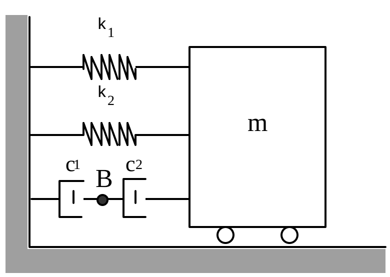
We draw the free body diagram of the system:

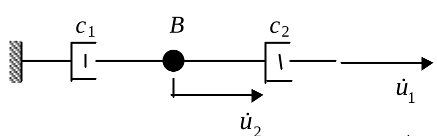


The equation of motion of the system is:

$$m\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 + k_2(u_1 - u_2) = 0$$

$$m\ddot{u}_1 + (c_1 + c_2)\dot{u}_1 + \frac{k_1 k_2}{k_1 + k_2} u_1 = 0$$



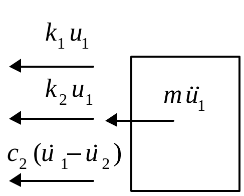


Equilibrium of point B:

$$c_1 \dot{u}_2 = c_2 (\dot{u}_1 - \dot{u}_2)$$

$$\dot{u}_2 = \frac{c_2}{c_1 + c_2} \dot{u}_1 \qquad \{2\}$$

We draw the free body diagram of the system:



The equation of motion of the system is:

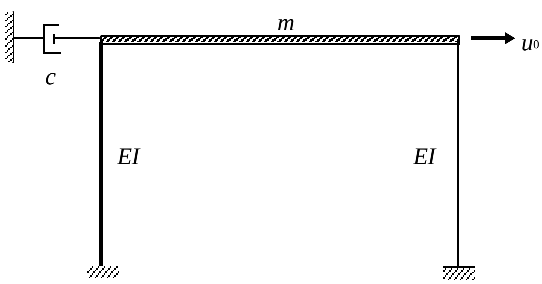
$$m\ddot{u}_1 + c_2(\dot{u}_1 - \dot{u}_2) + (k_1 + k_2)u_1 = 0$$

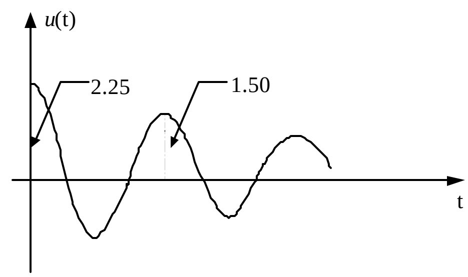
$$m\ddot{u}_1 + \frac{c_1 c_2}{c_1 + c_2}\dot{u}_1 + (k_1 + k_2)u_1 = 0$$

<u>PROBLEM 3</u> (**<u>P.7</u>** on page 368 in the lecture note)

To determine the dynamic properties of a one-story structure 3 m in height that can be idealized as a simple frame with rigid girder of mass *m* and equal massless columns with bending stiffness *EI*, a jack was used to displace the building laterally by 2.25 cm. The force required in the jack to accomplish this displacement was 625.000 N (about 62.5 metric tons). Upon instantaneous release of the structure, the maximum displacement on the return swing was 1.50 cm, and the measured period of the motion was *T* = 1 sec.

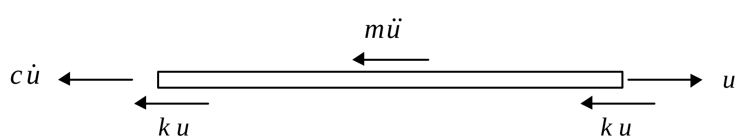
- What is the mass *m* of the girder and *EI* of the columns?
- What is the fraction of damping? The damping constant *c*?
- How many cycles are required for the free vibration to decay to 1 cm?
- What is the maximum acceleration felt by the girder, and when does it happen?





SOLUTION

The free body diagram of the girder is:



$$k = \frac{12\,EI}{H^3}$$

Therefore, the equation of motion for the free vibration of the structure is: $m\ddot{u} + c\dot{u} + 2k\,u = 0$

The *statically* applied jack force (F = 625 kN) caused a lateral deflection u_0 = 2.25 cm. Therefore, the stiffness of the structure is calculated as follows:

$$u_0 = \frac{F}{2k} = 2.25\cdot 10^{-2}\ m$$

$$k = \frac{F}{2u_0} = \frac{625\ kN}{2\cdot 2.25\cdot 10^{-2}\ m} = 13888.89\ kN/m = \frac{12\,EI}{H^3}$$

$$\therefore\quad EI = 31250\ kNm^2$$

To a first approximation, for a lightly damped system (we later can check for the value of damping, x), we can assume that the damped period of the structure is equal to the undamped period, i.e. $T_n = T_d = 1$ sec.

Therefore:

$$w_n = 2p\ f_n = \frac{2p}{T_n} \simeq \frac{2p}{T_d} = 2p\ \ rad/\sec = \sqrt{\frac{2k}{m}}$$

$$m = (2k)\,w_n^{\ 2} = 703.62\cdot 10^3\ kg$$

We will estimate the fraction of damping using the logarithmic decrement.

$$\ln\left(\frac{u(t)}{u(t+N\,T_d)}\right) = 2p\ N x \qquad \text{where } N \text{ :integernumberofcycles}$$

We therefore have: $x = \ln\left(\frac{2.25}{1.50}\right)\frac{1}{2p \cdot 1} = 0.0645$ [1]

The damping coefficient is: $c = 2\,m\,x\,w_n = 570.306 \cdot 10^3\ kg/\text{sec}$

We now apply the logarithmic decrement, to estimate the number of cycles required for the free vibration to decay to 1.0 cm.

$$N = \ln\left(\frac{u(t)}{u(t+N\,T_d)}\right)\frac{1}{2p\ x} = \ln\left(\frac{2.25}{1.00}\right)\frac{1}{2p\ \ 0.0645} \simeq 2 \text{ cycles}$$

The free vibration of the damped system is:

$$u(t) = e^{-x\,w_n\,t}\left[u(0)\cos(w_d\ t) + \frac{\dot{u}(0) + x\,w_n\,u(0)}{w_n}\sin(w_d\ t)\right]$$

$$u(0) = 2.25 \cdot 10^{-2}\ m \qquad \dot{u}(0) = 0$$

The acceleration of the system is therefore:

$$\ddot{u}(t) = \frac{d^2}{dt^2}u(t) = u(0)\,w_n^2\ e^{-x\,w_n\,t}\left[-\cos(w_d\ t) + \frac{x}{\sqrt{1-x^2}}\sin(w_d\ t)\right]$$

The maximum acceleration occurs at t = 0: $\ddot{u}_{\max} = \ddot{u}(0) = -u(0)\,w_n^2 = 0.89\ m/\text{sec}^2$

The variation of the acceleration with time is shown in the Figure below.

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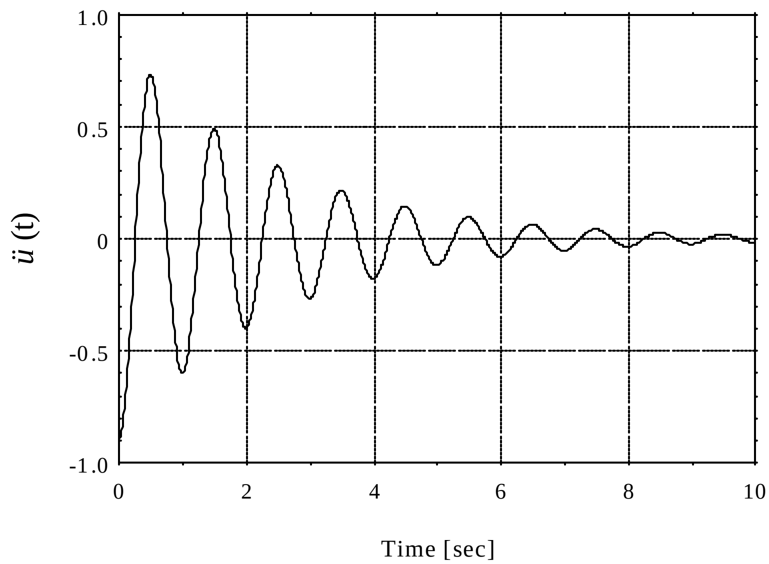
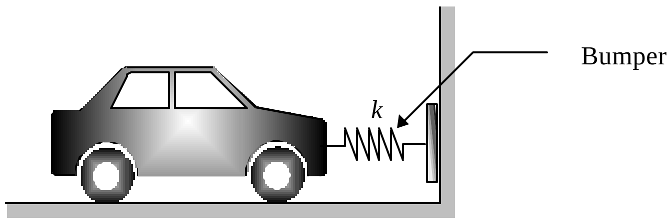
[1] Note.- For $x = 0.0645$, we have: $T_d = \frac{T_n}{\sqrt{1-x^2}} = 1.002\ T_n \simeq T_n$


1.0
0.5
0
-0.5
-1.0
ü (t)
0
2
4
6
8
10
Time [sec]

<u>PROBLEM 4</u> (**<u>P.9</u>** on page 369 in the lecture note)

A car with mass 1000 kg is moving slowly at a constant speed v = 7.2 km/h = 2 m/sec on a flat road. The car can be modeled as a rigid block with a spring–mounted front bumper. The stiffness of the bumper is such that a horizontal force equal to the weight of the car would deform the spring by 5cm (=0.05m). At t = 0, the car collides head on against a flat, rigid wall.

- What is the maximum deformation of the bumper?
- What is the maximum force exerted on the bumper?
- What is the maximum acceleration felt by the passengers?
- How long does the car remain in contact with the wall?



SOLUTION

From the given information, we know that the initial velocity, the static stiffness of the bumper, and the natural frequency as

$$\dot{u}_0 = 2 \text{ (m/sec)} \tag{1}$$

$$k = \frac{mg}{\Delta} = \frac{1000*9.8}{0.05} = 196{,}000(N/m) \tag{2}$$

$$w_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg}{m\Delta}} = \sqrt{\frac{g}{\Delta}} = \sqrt{\frac{9.8}{0.05}} = 14(rad\ /\text{sec}) \tag{3}$$

Then, we can express the motion u for the system of interest as

$$u(t) = \frac{\dot{u}_0}{w_n}\sin w_n t \tag{4}$$

What is the maximum deformation of the bumper, $u_{\max}$? $u_{\max}$ is obtained from equation (4) as

$$u_{\max} = \frac{\dot{u}_0}{w_n} = \frac{2}{14} = \frac{1}{7}(m) \tag{5}$$

What is the maximum force exerted on the bumper, $F_{\max}$? This is simply

$$F_{\max} = ku_{\max} = 196000\frac{1}{7} = 28000(N) \tag{6}$$

What is the maximum acceleration felt by the passengers, $\ddot{u}_{\max}$? The acceleration is given as

$$\ddot{u} = -w_n \dot{u}_0 \sin w_n t \tag{7}$$

Therefore, $\ddot{u}_{\max}$ is $w_n \dot{u}_0$ = 14*2=28(m/sec^2).

How long does the car remain in contact with the wall, t_d? This t_d is simply the half of the period T of the system, i.e.

$$t_d = \frac{1}{2}T_n = \frac{1}{2}\frac{1}{f_n} = \frac{1}{2}\frac{2p}{w_n} = \frac{1}{2}\frac{2p}{14} = 0.2244(\text{sec})$$