**Lecture Notes on Fluid Dynamics**
(1.63J/2.21J)
by Chiang C. Mei, February 6, 2007

# A.1 Appendix on Cartesian tensors

[Ref 1] : H Jeffreys, **Cartesian Tensors**;
[Ref 2] : Y. C. Fung, **Foundations of Solid Mechanics**.

A true law of physics must satisfy the following basic requirements on dimensions : (a) two physical quantities are not equal unless they are of the same dimension, and (b) a physical equation must be invariant under a change of fundamental units. Equally fundamental, a true physical law must be free of the frame of reference, and should remain valid in any coordinate system.The mathematical tool that can express physical laws in invariant forms is Tensor Analysis.

Let us first introduce the index notation. A group of three variables $x, y, z$ can be denoted alterntively by $x_i, i = 1, 2, 3$ where $x_1 = x, x_2 = y, x_3 = z$. Thus the equation of a plane in $x, y, z$ space can be written as

$$a_1x + a_2y + a_3z = p$$

or,

$$a_1x_1 + a_2x_2 + a_3x_3 = p$$

or

$$\sum_{i=1}^{3} a_i x_i = p \tag{A.1.1}$$

As a simplification, we add a rule due to A. Einstein: If an index is repeated once (and only once) in a term, summation over the full range of that index is implied but the summation sign $\sum_i$ is omitted. For example the previous equation for a plane is now written as $a_ix_i = p$.

Since tensor analysis is motivated by coordinate transformation, let us look at a transformation law for rotation.

## A.1.1 Transformation of a vector

Refering to Figure A.1.1, let $\{x', y'\}$ axes be different from $\{x, y\}$ by a simple rotation. Then the compoents of a vector $\vec{A}$ in the two coordinate systems are related by

$$\begin{aligned} A'_x &= A_x \cos(x', x) + A_y \cos(x', y) \\ A'_y &= A_x \cos(y', x) + A_y \cos(y', y) \end{aligned} \tag{A.1.1}$$

where $(x', y)$ denotes the angle between the $x'$ and $y$ axies. Using the index notation, it can also be written

$$\begin{aligned} A'_1 &= A_1 \cos(x'_1, x_1) + A_2 \cos(x'_1, x_2) \\ A'_2 &= A_1 \cos(x'_2, x_1) + A_2 \cos(x'_2, x_2) \end{aligned} \tag{A.1.2}$$

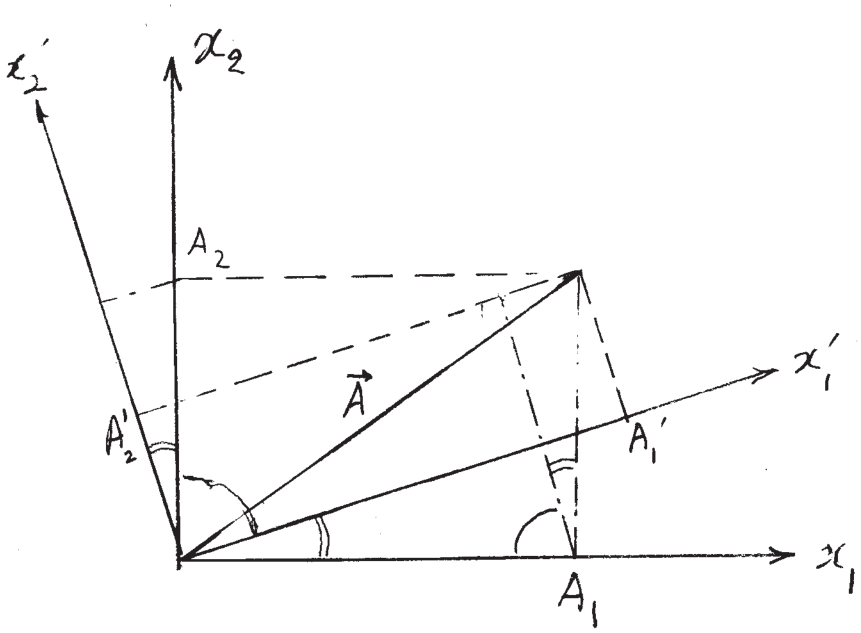


Figure A.1.1: Two coordinate systems related by rotation

In three-dimensions, the components of a vector are transformed under rotation as follows

$$\begin{aligned} A_1' &= A_1\cos(x_1', x_1) + A_2\cos(x_1', x_2) + A_3\cos(x_1', x_3) \\ A_2' &= A_1\cos(x_2', x_1) + A_2\cos(x_2', x_2) + A_3\cos(x_2', x_3) \\ A_3' &= A_1\cos(x_3', x_1) + A_2\cos(x_3', x_2) + A_3\cos(x_3', x_3) \end{aligned} \tag{A.1.3}$$

Let us introduce the shorthand notation

$$C_{ik} \equiv \cos(x_i', x_k) \tag{A.1.4}$$

then , in the summation convention, (A.1.3) can be written in matrix form

$$A_i' = \sum_k C_{ik}A_k = C_{ik}A_k \tag{A.1.5}$$

where

$$[C_{ik}] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \equiv C. \tag{A.1.6}$$

is the transformation matrix. Some properties of the matrix $C_{ik}$ are derived below:

(a) **Orthogonality of** $C_{ik}$. Since the length of vector $A$ must be invariant, i.e., the same in both coordinate systems,

$$A_i'A_i' = A_iA_i.$$

In view of (A.1.5)

$$A_i' A_i' = C_{ik} A_k C_{ij} A_j = C_{ik} C_{ij} A_k A_j = A_j A_j.$$

hence,

$$C_{ik} C_{ij} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases} \tag{A.1.7}$$

Introducing the Kronecker delta

$$\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

we have,

$$C_{ik} C_{ij} = \delta_{k_j} \tag{A.1.8}$$

Notice that the summation is performed over the first indices. This property is called the orthogonality of the transformation matrix $C$, which is a generalization of a similar concept in vectors. Equation (A.1.8) for $k, j, = 1, 2, 3$ represents six constraints among nine components of $C_{ij}$, hence only three components of $C_{ij}$ are independent.

(b) **The inverse transformation**: Consider the vector $\vec{A}$ in both systems $S'$ and system $S$ which are related by rotation. We may write

$$A_i = C_{i\ell}' A_\ell', \qquad C_{i\ell}' = \cos\left(x_i, x_\ell'\right).$$

Note that by definition

$$C_{i\ell}' = \cos\left(x_i, x_\ell'\right), \quad C_{\ell i} = \cos\left(x_\ell', x_i\right)$$

Since the angles are the same in magnitude, we have

$$C_{i\ell}' = C_{\ell i} \tag{A.1.9}$$

Clearly,

$$C_{ki}' C_{kj}' = \delta_{ij},$$

hence, it follows from (A.1.9) that

$$C_{ik} C_{jk} = \delta_{ij} \tag{A.1.10}$$

Comparing (A.1.8) and (A.1.10), we note that summation in the latter is performed over the second indices.

Remark: A general transformation is equal to a translation plus a rotation, but a vector is not affected by translation at all since only the end points matter.

### A.1.2 Definition of a cartesian tensor

A tensor $\mathsf{T}$ of rank $r$ is an array of components denoted by $T_{ijk\ldots m}$ with $r$ indices $ijk\ldots m$. In three dimensional space $\mathsf{T}$ has $3^r$ components. The defining property of a cartesian tensor is the following law : From coordinate system $S$ to $S'$ by a rotation, the components of a tensor transform according to

$$T'_{ijk\ldots m} = C_{is}C_{jt}C_{ku}\cdots C_{mv}T_{stu\ldots v}. \tag{A.1.11}$$

As special cases, a scalar is a zero-th rank tensor $T' = T$. A vector is a first rank tensor which is transformed according to $T'_i = C_{ij}T_j$ . A second rank tensor is transformed according to $T'_{ij} = C_{is}C_{jt}T_{st}$.

**Problem:** Show that $\delta_{ij}$ is a second-rank tensor.

Hint: Consider $C_{is}C_{jt}\delta_{st}\cdots$ and use the definition of Knonecker delta.

Remark: General non-cartesian tensors are defined by more general laws of transformation.

### A.1.3 The Quotient Law

A set of $3^r$ numbers form the components of a tensor of rank $r$, if and only if its scalar product with another **arbitrary** tensor is again a tensor. This is called the quotient law and can be used as a litmus test whether a set of numbers form a tensor.

We only give a one-way proof for a third rank tensor. Consider a set of numbers $A_{ijk}$. Let $\xi_\alpha$ be the components of an **arbitrary** vector. Then, if

$$A_{\alpha jk}\xi_\alpha = B_{jk}$$

is a tensor component, it must obey the transformation law of a second-order tensor, i.e.,

$$B'_{ik}(= A'_{\alpha ik}\xi'_\alpha) = C_{i\ell}C_{km}B_{\ell m}(= C_{i\ell}C_{km}A_{\beta\ell m}\xi_\beta).$$

But,

$$\xi_\beta = C_{\alpha\beta}\xi'_\alpha = C'_{\beta\alpha}\xi'_\alpha.$$

hence,

$$A'_{\alpha ik}\xi'_\alpha = \left(C_{i\ell}C_{km}C_{\alpha\beta}A_{\beta\ell m}\right)\xi'_\alpha.$$

Since $\xi'_\alpha$ is arbitrary

$$A'_{\alpha ik} = C_{i\ell}C_{km}C_{\alpha\beta}A_{\beta\ell m}.$$

it follows that $A_{\beta\ell m}$ is a third rank tensor.

### A.1.4 Tensor Algebra

(a) **Addition**. The sum of two tensors of equal rank is another tensor of the same rank. Let us give the proof for second-rank tensors only.

Given two tensors $A_{ij}$ and $B_{ij}$, we define the sum in $S$ by

$$E_{ij} = A_{ij} + B_{ij}.$$

In $S'$ system we have, by definition,

$$\begin{aligned} E'_{ij} &= A'_{ij} + B'_{ij} = C_{i\ell}C_{jm}A_{\ell m} + C_{i\ell}C_{jm}B_{\ell m} \\ &= C_{i\ell}C_{jn}\left(A_{\ell m} + B_{\ell m}\right) = C_{i\ell}C_{jn}E_{\ell m}, \end{aligned}$$

hence, $E_{ij}$ is a tensor of second rank after using linearlity.

(b) **Multiplication**. (A tensor of rank $b$) times (a tensor of rank $c$) = a tensor of rank $b + c$ with $3^{b+c}$ components

$$E_{ij\ldots krs\ldots t} = A_{ij\ldots k}B_{rs\ldots t}.$$

We only give the proof for the special case $E_{ijrs} = A_{ij}B_{rs}$. Define

$$\begin{aligned} E'_{ijrs} &= A'_{ij}B'_{rs} \,\text{in}\, S' \\ &= \left(C_{ik}C_{j\ell}A_{k\ell}\right)\left(C_{rm}C_{sn}B_{mn}\right) \\ &= C_{ik}C_{j\ell}C_{rm}C_{sn}A_{k\ell}B_{mn} = C_{ik}C_{j\ell}C_{rm}C_{sn}E_{k\ell mn}, \end{aligned}$$

hence, $E$ is a tensor of fourth rank.

(c) **Contraction** : If any pair of indices of of an r-th rank tensor with are set equal and summed over the range 12,3, the result is a tensor of rank $r - 2$.

Consider $A_{ijk\ell}$, which obeys $A'_{rstu} = C_{ri}C_{sj}C_{tk}C_{u\ell}A_{ijk\ell}$. Let $s = t$ and sum over the index $s$,

$$\begin{aligned} A'_{rssu} &= C_{ri}\left(C_{sj}C_{sk}\right)C_{u\ell}A_{ijk\ell} \\ &= C_{ri}\delta_{jk}C_{u\ell}A_{ijk\ell} \\ &= C_{ri}C_{u\ell}A_{ijj\ell}. \end{aligned}$$

This is precisely the transformation law of a second rank tensor. Hence, $A'_{rssu}$ is a second rank tensor.

Note: The scalar product $A_iB_i = D$ is a special case of contraction. $\vec{A}\vec{B}$, which is known as a "dyad," is a seocnd-order tensor $= (AB)_{ij} = A_iB_j$. Contraction makes it a zero-th order tensor, i.e., a scalar. Thus a scalar product is the result of multiplication and contraction.

### A.1.5 Tensor Calculus

(a) **Differentiation** A tensor whose components are functions of spatial coordinates $x_1, x_2, x_3$ in a region $R$ forms a **tensor field**.

a.1) Gradient: Taking the gradient of a tensor of rank $r$ gives rise to a tensor of rank $r+1$.

Let $\sigma\left(x_1, x_2, x_3\right)$ be a scalar (tensor of rank 0). Is $V_i = \partial\sigma/\partial x_i$ a tensor of rank 1 (a vector)?

Since $\vec{x} = (x_i)$ is a vector, it transforms like one: $x'_i = C_{ik}x_k$. Now,

$$\begin{aligned}
V_i &= \frac{\partial\sigma}{\partial x_i} = \frac{\partial\sigma}{\partial x'_j}\frac{\partial x'_j}{\partial x_i} = C_{ji}\frac{\partial\sigma}{\partial x'_j} \\
&= \frac{\partial\sigma}{\partial x'_j}C'_{ij} \qquad \text{by}(A.1.8) \\
&= C'_{ij}V'_j \qquad \text{by definition.}
\end{aligned}$$

Hence, $V_i$ is a vector component, and the gradient of a scalar is a vector. In other words, the gradient of a tensor of rank zero is a tensor of rank 1.

In general

$$T_{ij\ldots k\ell} = \frac{\partial R_{ij\ldots k}}{\partial x_\ell}. \tag{A.1.12}$$

(a.2) Divergence is contraction in differentiation.

Taking the divergence of a tensor of rank $r$ gives rise to a tensor of rank $r-1$.

Consider $\sigma = \partial v_i/\partial x_i$. Is $\sigma$ a scalar? Let's check the transformation law.

$$\begin{aligned}
\sigma &= \frac{\partial V_i}{\partial x_i} = \frac{\partial V_i}{\partial x'_k}\frac{\partial x'_k}{\partial x_i} = \frac{\partial\left(C'_{ij}V'_j\right)}{\partial x'_k}\frac{\partial x'_k}{\partial x_i} \\
&= C'_{ij}\frac{\partial V'_j}{\partial x'_k}C_{ki} = C'_{ij}C'_{ik}\frac{\partial V'_j}{\partial x'_k} \\
&= \delta_{jk}\frac{\partial V'_j}{\partial x'_k} = \frac{\partial V'_k}{\partial x'_k} = \sigma'.
\end{aligned}$$

Hence, $\sigma$ is a scalar and $\sigma = \sigma'$.

<u>Problem</u>: Prove that the strain components

$$e_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

and the quantities

$$\Omega_{ij} = \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j}$$

are components of tensors.