Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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2-8-aCSYih.tex

2.8.a. Suppression of vertical motion in a stratified fluid

Note: This section should precede §2.8

ref: C. S. Yih.*Dynamics of Inhomogeneous Fluids* , MacMillan

There is a useful theorem, due to C. S. Yih, that the vertical motion of a stratified fluid is supressed when the velocity is low.

Consider the three dimensional flow of an inviscid incompressible fluid which is nondiffusive :

$$u_x + v_y + w_z = 0 \tag{2.8.1}$$

$$u\rho_x + v\rho_y + w\rho_z = 0 \tag{2.8.2}$$

$$\rho(uu_x + vu_y + wu_z) = -p_x \tag{2.8.3}$$

$$\rho(uv_x + vv_y + wv_z) = -p_y \tag{2.8.4}$$

$$\rho(uw_x + vw_y + ww_z) = -p_z - g\rho \tag{2.8.5}$$

Let us choose the following normalization:

$$\rho = \rho_o\rho', \quad (u, v, w) = U(u', v', w'),$$

$$(x, y, z) = L(x', y,' z'), \quad p = p'\rho_o U^2$$

where

$$L = \left(-\frac{1}{\rho_o}\frac{d\rho_o}{dz}\right)^{-1}, \quad U = \sqrt{gL}.$$

Note that the velocity scale is chosen on dimensional basis alone and may not represent the actual magnitude. Therefore the dimensionless velocity components need not be all of $O(1)$. After omitting primes for brevity, we have,

$$u_x + v_y + w_z = 0 \tag{2.8.6}$$

$$u\rho_x + v\rho_y + w\rho_z = 0 \tag{2.8.7}$$

$$\rho(uu_x + vu_y + wu_z) = -p_x \tag{2.8.8}$$

$$\rho uv_x + vv_y + wv_z) = -p_y \tag{2.8.9}$$

$$\rho(uw_x + vw_y + ww_z) = p_z - \rho \tag{2.8.10}$$

Since L represents the scale of stratification,

$$\rho_z = O(1) \tag{2.8.11}$$

must be true. In lakes or coastal seas, the typical stratification depth is $L = O(5m)$ then $\sqrt{gL} = O(7)$ m/s, which can be much greater than the flow velocity. Hence we assume

$$(u, v, w) \leq O(\epsilon) \ll 1 \tag{2.8.12}$$

In physical dimensions, $(u, v, w) = O(\epsilon)\sqrt{gL}$. Eliminating p from eqs (2.8.8) and (2.8.10) we get

$$\rho_x = \frac{\partial}{\partial z}[\rho(uu_x + vu_y + wu_z)] - \frac{\partial}{\partial x}[\rho(uw_x + vuw_y + ww_z)]$$

which implies that

$$\rho_x = O(\epsilon^2) \tag{2.8.13}$$

Similarly we eliminate p from (2.8.9) and (2.8.10) and get

$$\rho_y = O(\epsilon^2) \tag{2.8.14}$$

It follows from (2.8.7) that

$$w = O(\epsilon^3) \tag{2.8.15}$$

giving the first indication that the flow is primarily horizontal. Eq (2.8.6), can then be approximated as

$$u_x + v_y = O(\epsilon^3) \tag{2.8.16}$$

From (2.8.8)

$$\rho(uu_x + vu_y) + O(\epsilon^4) = -p_x \tag{2.8.17}$$

which implies that

$$p_x = O(\epsilon^2) \tag{2.8.18}$$

Similarly, from (2.8.9) we get

$$\rho(uv_x + vv_y) + O(\epsilon^4) = -p_y$$

$$p_y = O(\epsilon^2) \tag{2.8.19}$$

From (2.8.10) we get instead

$$O(\epsilon^4) = -p_z - \rho \tag{2.8.20}$$

so that the pressure is hydrostatic.

In summary, at the leading order, the flow of an inviscid stratified fluid is two dimensional in the horizontal plane. Returning to physical variables, the approximate equations are:

$$u_x + v_y = 0 \tag{2.8.21}$$

$$\rho(uu_x + vu_y) = -p_x \tag{2.8.22}$$

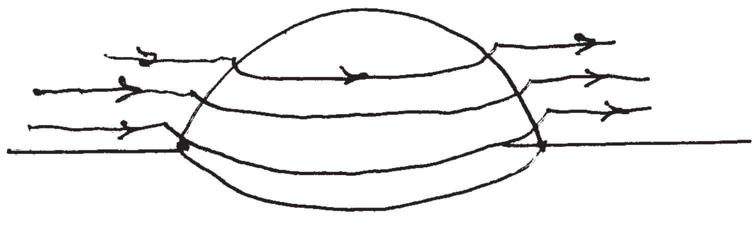

Figure 2.8.1: Suppression of vertical motion in a stratified fluid

$$\rho(uv_x + vv_y) = -p_y. \tag{2.8.23}$$

Comparing (2.8.18, 2.8.19) with (2.8.20), we see that the horizontal pressure gradient is much smaller than the vertical gradient. To understand this let us split p into the static and dynamic parts,

$$p = p^{(s)} + p^{(d)} \tag{2.8.24}$$

where the static part $p^{(s)}(z)$ is related to $\rho^{(s)}(z)$ by

$$-p_z^{(s)} - \rho^{(s)} = 0. \tag{2.8.25}$$

Therefore

$$\rho(uu_x + vu_y) = -p_x^{(d)} \tag{2.8.26}$$

$$\rho(uv_x + vv_y) = -p_y^{(d)}. \tag{2.8.27}$$

implying that $p^{(d)} = O(\epsilon^2)$. Eq (2.8.20) also implies that

$$-p_z^{(d)} - \rho^{(d)} = O(\epsilon^2) \tag{2.8.28}$$

Thus the dynamic parts $p^{(d)}$ and $\rho^{(d)}$ are of the order (ϵ^2), consistent also with (2.8.13), and (2.8.14).

Similar to the Taylor-Proudman theorem in rotating fluids, Yih's theorm has useful physical implications in stratified fluids. For example a horizontal current encounters a three dimensional obstacle can only pass the body on the sides in a horizontal plane, but not above or below. A two dimensional flow incident on a horizontal cylinder of infinite length will be blocked, leaving a horizontal layer of stagnant fluid in the wake of the same height as the cylinder. If fluid is withdrawn into a sink, only a thin layer of fluid on the level of the sink is moved. The last case is discussed in detail next.