Lecture Notes on Fluid Dynmics
(1.63J/2.21J)
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3-2Hi-Re-bl.tex

3.2 Viscous Flow at High Reynolds Numbers

Let us first give a heuristic estimates of boundary layer in steady flows.

Consider a particle near the wall to be influenced by viscosity. After traveling a distance x from the edge, it has been under viscous influence for a time of $t = x/U$. Let U be large. For finite x, t is small so that vorticity is spread sideways to the width $(\nu t)^{1/2} \sim (\nu x/U)^{1/2}$. Let us define this width to be the boundary layer, which has thickness $\delta = O\,(\nu x/U)^{1/2}$.

Alternatively we start from Navier-Stokes equations :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{3.2.1}$$

$$u\,\frac{\partial u}{\partial x} + v\,\frac{\partial u}{\partial y} = -\frac{1}{\rho}\,\frac{\partial p}{\partial x} + \nu\,\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.2.2}$$

$$u\,\frac{\partial v}{\partial x} + v\,\frac{\partial v}{\partial y} = -\frac{1}{\rho}\,\frac{\partial p}{\partial y} + \nu\,\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{3.2.3}$$

When viscosity is important $y = O(\delta), x = O(L)$, convective inertia is comparable to viscous stresses.

From continuity

$$\frac{u}{L} \sim \frac{v}{\delta}$$

From $x-$momentum

$$\begin{aligned} u\,u_x &\sim \nu\,u_{yy} \\ \frac{U^2}{L} &\sim \nu\,\frac{U}{\delta^2} \end{aligned}$$

Therefore,

$$\delta \sim (\nu L/U)^{1/2} \tag{3.2.4}$$

and

$$\frac{\delta}{L} \sim \left(\frac{\nu}{UL}\right)^{1/2} = \frac{1}{Re^{1/2}}. \tag{3.2.5}$$

Shear stress on the awall :

$$\frac{\tau_0}{\rho} = \nu\,\left.\frac{\partial u}{\partial y}\right|_0 = \nu\,\frac{U}{\delta} \sim \nu U\,\sqrt{\frac{U}{\nu L}}$$

Hence the drag coefficient is,

$$C_D = \frac{\tau_0}{\frac{1}{2}\rho U^2} = 2\sqrt{\frac{\nu}{Ux}} = \frac{2}{\text{Re}}.$$

For water $\nu = 10^{-5}\,\text{ft}^2/\text{sec}$. Let $U = 1$ ft/sec $\quad L = 1$ ft, then $\text{Re} = 10^5$. Hence,

$$O\left(\frac{\delta}{L}\right) \propto \frac{1}{\sqrt{\text{Re}}} \sim \frac{1}{3}\,10^{-2} \qquad (\delta \sim 0.003\text{ ft})$$

and

$$C_D \sim 0.003.$$

Experiments for flat plates (Schlichting, p. 133) show that: $C_D \sim 0.002$, but experiments for a circular cylinder show that $C_D \approx 0(1)$ because flow is separated for most Re .

3.2.1 Systematic Boundary-layer Approximation

Let $u = O(U),\ x = O(L),\ y = 0(\delta)$. From continuity, $v = O(U\delta/L)$. Let $u \to Uu,\ v \to \frac{U\delta}{L}\,v,\ x \to Lx,\ y \to \delta y$

$$\frac{U}{L}\,(u_x + v_y) = 0. \tag{3.2.6}$$

$$\frac{U^2}{L}\,(uu_x + vu_y) = -\frac{P}{\rho L}\frac{\partial p}{\partial x} + \frac{\nu U}{L^2}\,u_{xx} + \frac{\nu U}{\delta^2}\,u_{yy}. \tag{3.2.7}$$

$$\frac{\delta}{L}\frac{U^2}{L}\,(uv_x + vv_y) = -\frac{P}{\rho\delta}\,p_y + \frac{\nu U}{L^2}\frac{\delta}{L}\,v_{xx} + \frac{\nu U}{\delta^2}\frac{\delta}{L}\,v_{yy}. \tag{3.2.8}$$

From Eqn. (3.2.6)

$$u_x + v_y = 0. \tag{3.2.9}$$

From Eqn. (3.2.7)

$$uu_x + vu_y = -\frac{P}{\rho U^2}\,p_x + \frac{1}{\text{Re}}\left(u_{xx} + \frac{L^2}{\delta^2}\,u_{yy}\right). \tag{3.2.10}$$

From Eqn. (3.2.8)

$$uv_x + vv_y = -\frac{PL^2}{\rho\delta^2U^2}\,p_y + \frac{1}{\text{Re}}\left(v_{xx} + \frac{L^2}{\delta^2}\,v_{yy}\right). \tag{3.2.11}$$

To keep the dominant viscous stress term in Eqn. (3.2.10), we must have

$$\left(\frac{\delta}{L}\right)^2 = \frac{1}{\text{Re}} \qquad \text{or} \qquad \frac{\delta}{L} = \text{Re}^{-1/2}. \tag{3.2.12}$$

From Eqn. (3.2.11)

$$p_y = O\left(\frac{\delta^2}{L^2}\right) \tag{3.2.13}$$

and from Eqn. (3.2.10)

$$uu_x + vu_y = -\frac{P}{\rho U^2}\, p_x + u_{yy}. \tag{3.2.14}$$

In physical variables, we have to leading order

$$u_x + v_y = 0 \tag{3.2.15}$$

$$uu_x + vu_y = -\frac{1}{\rho}\, p_x + \nu u_{yy} \tag{3.2.16}$$

The pressure is constant across the boundary layer and must be the same as the pressure just outside. In the inviscid outer flow

$$UU_x + VU_y = -\frac{1}{\rho}\, p_x. \tag{3.2.17}$$

Since $V = 0$ on the wall, $p_x = -\rho UU_x$. Hence, inside the boundary layer:

$$uu_x + vu_y = UU_x + \nu u_{yy}. \tag{3.2.18}$$

This is the classical boundary layer approximation for high Re flows, due to Prandtl (1905).