3.5 Karman’s momentum integral approach

Ref: Schlichting: Boundary layer theory

With a general pressure gradient the boundary layer equations can be solved by a variety of modern numerical means. An alternative which can still be employed to simplify calculations is the momentum integral method of Karman. We explain this method for a transient boundary layer along the $x$-axis forced by an unsteady pressure gradient outside. This pressure gradient can be due to some unsteady and nonuniform flow such as waves or gust.

3.5.1 General ideas

Let us limit our discussion to a two dimensional flow. Recall the continuity

$$u_x + w_z = 0$$

and boundary layer equation:

$$\rho(u_t + uu_x + wu_z) = -p_x + \mu u_{zz}$$

$$0 = -p_z$$

The boundary conditions are

$$u = w = 0, \quad z = 0; \quad u \to U, \quad w \to 0, \quad z \to \infty.$$  

For general $U(x,t)$ the longitudinal pressure gradient is

$$-p_x = \rho(U_t + UU_x)$$

so that

$$\rho(u_t + uu_x + wu_z) = \rho(U_t + UU_x) + \mu u_{zz}$$

Instead of solving the initial-boundary-value problem accurately we introduce a moment method by integrating the momentum equation across the boundary layer, then make a reasonable assumption on the velocity profile to get the longitudinal variation of the boundary layer thickness. This is called the Karman’s momentum integral approximation in boundary layer theory.
Let us add the following two equations

\[-\mu u_{zz} = \rho \{ U_t + UU_x - (u_t + uu_x + wu_z) \} \]

and

\[0 = (U - u)u_x + (U - u)w_z\]

to get,

\[-\mu u_{zz} = \rho \left\{ \frac{\partial}{\partial t} (U - u) + \frac{\partial}{\partial x} [u(U - u)] + (U - u) \frac{\partial U}{\partial x} + \frac{\partial}{\partial z} [w(U - u)] \right\} \]

Now let us integrate across the boundary layer,

\[\mu u_z = \rho \left\{ \frac{\partial}{\partial t} \int_0^\infty (U - u) dz + \frac{\partial}{\partial x} \int_0^\infty u(U - u) dz + \frac{\partial U}{\partial x} \int_0^\infty (U - u) dz + \rho [w(U - u)] \right\}_0^\infty \]

The boundary terms again vanish. Finally, we have,

\[
\frac{\partial}{\partial t} \int_0^\infty \rho (U - u) dz + \frac{\partial}{\partial x} \int_0^\infty \rho (u - u^2) dz + \frac{\partial U}{\partial x} \int_0^\infty (U - u) dz = \mu \frac{\partial u}{\partial z} \bigg|_0 \tag{3.5.7}
\]

This is the Kármán momentum integral equation, representing the momentum balance across the thickness of the boundary layer.

Figure 3.5.1: Displacement thickness

Karman also introduces the displacement thickness as a measure of the lost volume

\[U \delta_1 = \int_0^\infty (U - u) dz, \quad \text{or} \quad \delta_1 = \int_0^\infty \left( 1 - \frac{u}{U} \right) dz, \tag{3.5.8}\]
and the momentum thickness as a measure of the lost momentum

\[ U^2 \delta_2 = \int_0^\infty u(U - u)dz, \quad \text{or} \quad \delta_2 = \int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dz, \tag{3.5.9} \]

both due to the slowing down of the fluid in the boundary layer. See Figure 3.5.1.

In certain cases when the velocity profile in the boundary layer can be reasonably guessed in advance, the Karman momentum integral equation can be the basis of obtaining an approximate solution. The procedure is to assume a reasonable profile:

\[ u = U f \left( \frac{y}{\delta} \right) \tag{3.5.10} \]

and substitute (3.5.10) into (3.5.7) equation. After evaluating the integrals a differential equation is obtained for the boundary layer thickness \( \delta(x, t) \), which can be more easily solved for certain initial and boundary conditions.

### 3.5.2 Application to a transient boundary layer along a flat plate

Consider the two dimensional boundary layer near edge of half infinite plane along the \( x \) axis due to the impulsive start along its own plane. There is no motion anywhere for \( t < 0 \). At \( t = 0 \) the plane suddenly advances from right to left perpendicular to its leading edge. What is the boundary layer flow? Referring to Figure 3.5.2 where the coordinate system is fixed on the plane. For all \( t > 0 \),

\[ \frac{\partial U}{\partial t} = \frac{\partial U}{\partial x} = 0 \]

The boundary layer equation reads:

\[ \rho(u_t + uu_x + uu_z) = \mu u_{zz} \tag{3.5.11} \]

and the Karman momentum integral equation reads

\[ \frac{\partial}{\partial t} \int_0^\infty \rho(U - u)dz + \frac{\partial}{\partial x} \int_0^\infty \rho \left(Uu - u^2\right) dz = \mu \frac{\partial u}{\partial z} \bigg|_0 \tag{3.5.12} \]
Let us assume
\[ \frac{u}{U} = f(\eta), \quad \eta \equiv \frac{z}{\delta(x, t)} \quad \text{(3.5.13)} \]
then
\[ \delta_1 = \delta \int_0^\infty (1 - f(\eta)) d\eta = \alpha_1 \delta \quad \text{(3.5.14)} \]
\[ \delta_2 = \delta \int_0^\infty f(\eta)(1 - f(\eta)) d\eta = \alpha_2 \delta \quad \text{(3.5.15)} \]
and
\[ u_z = \frac{U}{\delta} f'(0). \quad \text{(3.5.16)} \]
Substituting into Eq. (3.5.12)
\[ \alpha_1 U \frac{\partial \delta}{\partial t} + \alpha_2 U^2 \frac{\partial \delta}{\partial x} = \nu \frac{U}{\delta} f'(0) \]
Therefore,
\[ (\alpha_1) \frac{\partial (\delta^2/2)}{\partial t} + (\alpha_2 U) \frac{\partial (\delta^2/2)}{\partial x} = \nu f'(0) \quad \text{(3.5.17)} \]
which is a first order wave (hyperbolic) equation for \( \delta^2/2 \). We must add the boundary and initial conditions:
\[ \delta(0, t) = 0, \quad \forall t > 0 \quad \text{(3.5.18)} \]
\[ \delta(x, 0) = 0, \quad \forall x > 0 \quad \text{(3.5.19)} \]

**Heuristic solution**

The fact that the boundary value in (3.5.18) is independent of \( t \) for all \( t \) suggests that the solution is independent \( t \) for sufficiently large \( x \), i.e.,
\[ \frac{\partial}{\partial x} \left( \frac{\delta^2}{2} \right) = \frac{2\nu f'(0)}{\alpha_2 U} \quad \text{(3.5.20)} \]
subject to the boundary condition (3.5.18). The solution is simply
\[ \delta = \sqrt{\frac{2\nu f'(0)x}{\alpha_2 U}} \quad \text{(3.5.21)} \]
This is the approximate version of the solution by Blasius who solved the steady boundary layer equation
\[ uu_x + vu_y = \nu u_{yy} \quad \text{for } x > 0, \quad y > 0 \quad \text{(3.5.22)} \]
by the method of similarity.

On the other hand, the fact that the initial value in (3.5.18) is independent of \( x \) for all \( x \) suggests that the solution is independent of \( x \) for sufficiently large \( t \), i.e.,
\[ \frac{\partial}{\partial t} \left( \frac{\delta^2}{2} \right) = \frac{2\nu f'(0)}{\alpha_1} \quad \text{(3.5.23)} \]
subject to the initial condition (3.5.19). The solution is simply

\[ \delta = \sqrt{\frac{2\nu f'(0)t}{\alpha_1}} \]  

(3.5.24)

This is the approximate version of the solution by Rayleigh’s problem which is known to be governed by

\[ u_t = \nu u_{yy} \quad \text{for} \quad t > 0, \ y > 0 \]  

(3.5.25)

To find the range of each solution in the \( x \sim t \) plane, we equate the two \( \delta \)'s and get

\[ x = \frac{\alpha_2 Ut}{\alpha_1} \]  

(3.5.26)

Thus (3.5.21) holds in the wedge \( x > \frac{\alpha_2 Ut}{\alpha_1} > 0 \) and (3.5.24) holds in the wedge \( 0 < x < \frac{\alpha_2 Ut}{\alpha_1} \).

See Figure 3.5.3.

Physically: if \( t > \alpha_1 x/\alpha_2 U \), (3.5.21) applies and one has the steady Blasius flow past a semi-infinite plate; the boundary layer is already in the steady state. See Figure 3.5.4. If \( t < \alpha_1 x/\alpha_2 U \), (3.5.24) applies and one has Rayleigh’s problem of impulsively started plane infinite in both \( x < 0 \) and \( x > 0 \). The boundary layer is still in the initial stage and the effect of the leading edge is not felt elsewhere. The result is summarized in Figure 3.5.5.

To calculate \( \alpha_1 \) and \( \alpha_2 \) let us make a special choice of the velocity profile so that \( f(0) = 0, \ f'(1) = f''(1) = 0 \) (due to Karman and Polhausen)

\[ f(\eta) = 2\eta - 2\eta^3 + \eta^4, \ 0 < \eta < 1 \]  

(3.5.27)

Note that

\[ f(0) = 0, \ f'(\eta) = 2 - 6\eta^2 + 4\eta^3, \ f''(\eta) = -12\eta + 12\eta \]  

(3.5.28)

then

\[ \alpha_1 = \int_0^1 (1 - 2\eta + 2\eta^3 - \eta^4) \, d\eta = 3/10 = 0.3 \]

\[ \alpha_2 = \int_0^1 (2\eta - 2\eta^3 + \eta^4) \left(1 - 2\eta + 2\eta^3 - \eta^4\right) \, d\eta = 37/315 \cong 0.117 \]

and

\[ f'(0) = 2, \quad \frac{\partial u}{\partial z} \bigg|_{z=0} = \frac{2U}{\delta} \]

Hence the displacement thickness of the Blasius boundary layer is

\[ \delta_1 = \alpha_1 \delta = \alpha_1 \sqrt{\frac{2\nu f'(0)x}{\alpha_2 U}} = 1.754 \sqrt{\frac{\nu x}{U}} \]  

(3.5.29)

The momentum integral method is the special case of the moment method, since the Karman equation is the zeroth moment of the boundary layer equation. For the classical steady boundary layer problem solved exactly by Blasius using the similarity method, the momentum integral approximation gives fairly good results, even with various crude profiles, see Table 3.1.
Figure 3.5.3: Evolution of a transient boundary layer. (a) Blasius region and Rayleigh region

Figure 3.5.4: Evolution of a transient boundary layer. (b) Blasius region

**Formal solution**

A more formal solution can be obtained as follows: Let

\[ a = \alpha_1 \quad b = \alpha_2 U \quad c = \nu f'(0) \]

then

\[ a \frac{\partial}{\partial t} \left( \frac{\delta^2}{2} \right) + b \frac{\partial}{\partial x} \left( \frac{\delta^2}{2} \right) = c \]  

(3.5.30)

The solution can be facilitated by a change of coordinates from \((x, t)\) to \((\xi, \zeta)\) where

\[ \xi = ax + bt \quad \zeta = ax - bt \]  

(3.5.31)

then

\[
\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi} \frac{\partial}{\partial t} + \frac{\partial}{\partial \zeta} \frac{\partial}{\partial t}; \quad \frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial}{\partial \zeta} \frac{\partial}{\partial x} \\
\]

\[
\left( \frac{\delta^2}{2} \right)_t = \left( \frac{\delta^2}{2} \right)_\xi b + \left( \frac{\delta^2}{2} \right)_\zeta (-b) \\
\left( \frac{\delta^2}{2} \right)_x = \left( \frac{\delta^2}{2} \right)_\xi a + \left( \frac{\delta^2}{2} \right)_\zeta a \\
\]

Therefore, from (3.5.17),

\[
\frac{1}{2} \left[ ab \left( \frac{\delta^2}{2} \right)_\xi - ab \left( \frac{\delta^2}{2} \right)_\zeta + ab \left( \frac{\delta^2}{2} \right)_\xi + ab \left( \frac{\delta^2}{2} \right)_\zeta \right] = c
\]
Figure 3.5.5: Evolution of a transient boundary layer. (c) Rayleigh region

<table>
<thead>
<tr>
<th>Velocity profile</th>
<th>Displacement boundary layer thickness</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{u}{U} = f(\eta), \quad \eta = \frac{u}{\delta(x)}$</td>
<td>$\delta_1 \sqrt{\frac{U}{\nu}}$</td>
</tr>
<tr>
<td>$\eta$</td>
<td>1.732</td>
</tr>
<tr>
<td>$\sin \frac{2\eta}{3}$</td>
<td>1.741</td>
</tr>
<tr>
<td>$2\eta - 2 \eta^3 + \eta^4$</td>
<td>1.754</td>
</tr>
<tr>
<td>Blasius</td>
<td>1.721</td>
</tr>
</tbody>
</table>

Table 3.1: Comparison of approximate solution by momentum integral method with the exact solution of Blasius

\[
\frac{1}{2} \left( \delta^2 \right)_t = \frac{c}{2ab} = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}
\]

Integrate once

\[
\frac{\delta^2}{2} = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U} \xi + G(\zeta) \tag{3.5.32}
\]

where $G$ is an arbitrary function of $\zeta$.

To determine $G(\zeta)$ we first use the initial condition that $\delta = 0$ at $t = 0$ for all $x > 0$,

\[
0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(\xi)_{t=0} + G(\zeta)_{t=0}
\]

or

\[
0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(ax) + G(ax) \quad \text{for} \quad x > 0.
\]

Therefore

\[
G(\zeta) = -\frac{\nu f'(0)}{2\alpha_1\alpha_2 U} \zeta, \quad \zeta > 0 \tag{3.5.33}
\]

What is $G(\zeta)$ for $\zeta < 0$? Let us use the boundary condition at $x = 0$ that $\delta = 0$ for all $t > 0$.

\[
0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(\xi)_{x=0} + G(\zeta)_{x=0}
\]

or

\[
0 = \frac{\nu f'(0)}{2\alpha_1\alpha_2 U}(-bt) + G(-bt), \quad t > 0
\]
Note that for \( t > 0, \ -bt < 0 \). Therefore,

\[
G(\zeta) = \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} \zeta \quad \text{for all} \quad \zeta < 0. \tag{3.5.34}
\]

Eqs. (3.5.33) and (3.5.34) complete the solution for all \( \zeta > 0 \) and \( \zeta < 0 \).

Let us return to \( x \) and \( t \)

\[
\frac{\delta^2}{2} = \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} \xi - \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} \xi = \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} 2bt \quad \text{for} \quad ax - bt > 0
\]

\[
= \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} \xi + \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} \xi = \frac{\nu f'(0)}{2 \alpha_1 \alpha_2 U} 2ax \quad \text{for} \quad ax - bt < 0
\]

Therefore,

\[
\delta = \sqrt{\frac{2\nu f'(0)}{\alpha_1} t} \quad \text{for} \quad x > \frac{\alpha_2 U}{\alpha_1} t \quad \tag{3.5.35}
\]

and

\[
\delta = \sqrt{\frac{2\nu f'(0)}{\alpha_2 U} x} \quad \text{for} \quad x < \frac{\alpha_2 U}{\alpha_1} t \quad \tag{3.5.36}
\]

These are the same as (3.5.24) and (3.5.21).