3.9 Oscillatory Boundary Layers

3.9.1 Stokes problem

Near the solid bottom under a wave there is a boundary layer. Let the outside flow have the tangential velocity \( u = \Re U(x)e^{-i\omega t} \). Consider the ratio

\[
\frac{uu_x, vu_y}{u_t} = O\left(\frac{U_o}{\omega L}\right)
\]

If

\[
1 \gg \frac{U_o}{\omega L} \gg \frac{\nu}{\omega L^2}
\]

then, to a high degree of accuracy in \( \delta^2/L^2 \), we have, in physical variables:

\[
\begin{align*}
    u_x + v_y &= 0 \quad (3.9.1) \\
    u_t + (uu_x + vu_y) &= U_t + UU_x + \nu u_{yy} \quad (3.9.2)
\end{align*}
\]

Keep in mind that the quadratic terms are one order in \( U_o/\omega L \) smaller than the linear terms.

Let us introduce, artificially for the sake of keeping track of the small terms the ordering parameter

\[
\epsilon = \frac{U_o}{\omega L} \ll 1
\]

and insert it in front of the nonlinear terms so that (3.9.2) becomes

\[
u_t + \epsilon(uu_x + vu_y) = U_t + \epsilon UU_x + \nu u_{yy} \quad (3.9.3)
\]

Let us apply the perturbation method by assuming an approximate solution in the form of a series,

\[
u = u_1 + \epsilon u_2 + O(\epsilon^2), \quad (3.9.4)
\]

Substituting (3.9.4) into the boundary layer equations and separate terms of different orders, we get perturbation equations of different orders: \( O(\epsilon)^0 \) and \( O(\epsilon) \), etc. After this mission is accomplished, the order parameter will be discarded.
From the leading order $O(U_o/\omega L)^0$,

$$\frac{\partial u_1}{\partial t} = \frac{\partial U}{\partial t} + \nu \frac{\partial^2 u_1}{\partial y^2} \quad y > 0 \tag{3.9.5}$$

subject to the boundary conditions that

$$u_1 \to U(x, t) \quad y \to \infty \tag{3.9.6}$$

and

$$u_1 = 0 \quad y = 0 \tag{3.9.7}$$

From here on we only consider simple harmonic time dependence, i.e., the tangential inviscid velocity at the outer edge of the boundary layer is

$$U(x, t) = \Re \left[ U(x) e^{-\omega t} \right].$$

Let

$$u_1 = \Re \left[ \hat{u}_1(x, y) e^{-i\omega t} + U e^{-i\omega t} \right] \tag{3.9.8}$$

then

$$-i\omega U - i\omega \hat{u}_1 = -i\omega U + \nu \frac{d^2 \hat{u}_1}{dy^2}$$

Therefore,

$$\frac{d^2 \hat{u}_1}{dy^2} + \frac{i\omega}{\nu} \hat{u}_1 = 0 \tag{3.9.9}$$

$$\hat{u}_1 \to 0, \quad y \to \infty \tag{3.9.10}$$

$$\hat{u}_1 = -U(x), \quad y = 0 \tag{3.9.11}$$

The solution (due to Stokes) is

$$\hat{u}_1 = -U(x) \exp \left[-(1 - i) y \sqrt{\frac{\omega}{2\nu}} \right] \tag{3.9.12}$$

or,

$$u_1 = \Re \left\{ U(x) \left[ 1 - \exp \left(- (1 - i) y \sqrt{\frac{\omega}{2\nu}} \right) \right] e^{-i\omega t} \right\} \tag{3.9.13}$$

The sign of $\sqrt{-i}$ is chosen so that (3.9.10) is satisfied. The boundary layer thickness is

$$\delta = \sqrt{\frac{2\nu}{\omega}} \tag{3.9.14}$$

To complete the leading order solution we calculate the transverse velocity component $v_1$ in the boundary layer. By continuity:

$$v_1 = -\int_0^y \frac{\partial u_1}{\partial x} dy = ie^{-i\omega t} \frac{dU}{dx} \int_0^y \left[ 1 - e^{-(1-i) y/\delta} \right] dy \tag{3.9.15}$$

$$= -e^{-i\omega t} \frac{dU}{dx} \left\{ y - \frac{\delta}{1 - i} \left[ 1 - e^{-(1-i) y/\delta} \right] \right\}$$

which is valid in $y \leq O(\delta)$ only. Thus if the inviscid outer flow has tangential variation $\frac{dU}{dx} \neq 0$, then there is transverse flow $v_1$ in the boundary layer.
3.9.2 Induced Streaming

The second-order perturbation equation is:

\[
\frac{\partial u_2}{\partial t} - \nu \frac{\partial^2 u_2}{\partial y^2} = U \frac{\partial u_1}{\partial x} - \left( u_1 \frac{\partial u_1}{\partial x} + v_1 \frac{\partial u_1}{\partial y} \right)
\]

\[
= U \frac{\partial u_1}{\partial x} - \left[ \frac{\partial (u_1 u_1)}{\partial x} + \frac{\partial (u_1 v_1)}{\partial y} \right]
\]

Since \(U, u_1\) and \(v_1\) are simple harmonic in time, the solution for \(u_2\) must have zeroth and second harmonics. Focussing on the zeroth harmonic by taking the average over a period

\[
-\nu \frac{\partial^2 \bar{u}_2}{\partial y^2} = \bar{U} \frac{\partial U}{\partial x} - \left( \bar{u}_1 \frac{\partial u_1}{\partial x} + \bar{v}_1 \frac{\partial u_1}{\partial y} \right)
\]

On the right-hand-side the last two terms \(u_1 u_1, u_1 v_1\) are wave-induced Reynolds stresses. In particular \(\rho u_1 u_1\) is the rate of transporting \(x\)-momentum in the \(x\)-direction, and \(\rho u_1 v_1\) is the rate of transporting \(x\)-momentum in \(y\)-direction.

Alternatively:

\[
-\nu \frac{\partial^2 \bar{u}_2}{\partial y^2} = \frac{1}{2} \frac{\partial}{\partial x} \bar{U}^2 - \frac{1}{2} \frac{\partial}{\partial x} \bar{u}_1^2 - \bar{v}_1 \frac{\partial u_1}{\partial y}
\]

Let

\[
\alpha = (1 - i)/\delta
\]

Since

\[
v_1 = i e^{-i\omega_1} \frac{1}{\alpha} \frac{dU}{dx} \left( \alpha y - 1 + e^{-\alpha y} \right)
\]

\[
\frac{\partial u_1}{\partial y} = \alpha U(x)e^{-i\omega t}e^{-\alpha y}
\]

Exercise: Show that if

\[
a(t) = \Re(Ae^{-i\omega t}), \quad \text{and} \quad b(t) = \Re(Be^{-i\omega t})
\]

then the time average of their product is

\[
\bar{ab} = \frac{1}{2} \Re(AB^*) = \frac{1}{2} \Re(A^*B).
\]

Using this formula it can be shown that

\[
-\bar{v}_1 \frac{\partial u_1}{\partial y} = \frac{1}{2} \Re \left[ U^* \frac{dU}{dx} \alpha^* \alpha e^{-\alpha* y} \left( \alpha y - 1 + e^{-\alpha y} \right) \right]
\]
Thus

\[-\nu \frac{\partial^2 \bar{u}_2}{\partial y^2} = G(y) \equiv \frac{1}{2} \frac{d|U|^2}{dx} \left[ 1 - \left( 1 - e^{-\alpha y} \right) \left( 1 - e^{-\alpha^* y} \right) \right] + ReU \frac{dU}{dx} \frac{\alpha^*}{\alpha} e^{-\alpha^* y} \left( \alpha y - 1 + e^{-\alpha y} \right)\]

\[\nu \frac{\partial \bar{u}_2}{\partial y} = \int_{y}^{\infty} G(y') dy'\]

\[\nu \bar{u}_2 = \int_{0}^{y} dy'' \int_{y''}^{\infty} G(y') dy'\]

\[= -y \int_{y}^{\infty} G(y') dy' + \int_{0}^{y} y'' G(y'') dy''\]

One more integration gives

\[-\omega \bar{u}_2 = Re F \frac{dU^*}{dx}\]

where \( F = -\frac{1}{2} (1 - 3i) e^{-(1-i)\eta} - \frac{i}{2} e^{-(1+i)\eta} - \frac{1}{4} (1 + i) e^{-\eta} + \frac{1}{2} (1 + i) \eta e^{-(1-i)\eta} + \frac{3}{4} (1 - i) \]

Note that as \( y \to \infty \), just outside the boundary layer,

\[\bar{u}_2 = -\frac{3}{4\omega} Re \left[ (1 - i) \frac{dU^*}{dx} \right]\] (3.9.18)

By Taylor expansion we can show that for \( \eta \ll 1 \),

\[\bar{u}_2 \approx Re \left[ \frac{(1 + i) \eta}{2\omega} \frac{dU^*}{dx} \right]\] (3.9.19)

**Example : Surface gravity waves**

On the free surface of water of constant depth \( h \), let the vertical displacement be

\[\zeta = \Re \left[ A \left( e^{ikx} + Re^{-ikx} \right) e^{-i\omega t} \right]\] (3.9.20)

where \( R \) denotes the reflection coefficient. The frequency \( \omega \) frequency and and the wavenumber \( k \) are related by

\[\omega^2 = gk \tanh kh\] (3.9.21)

The corresponding velocity potential is

\[\Phi = \Re \left[ -\frac{igA \cosh k(z + h)}{\omega \cosh kh} \left( e^{ikx} + Re^{-ikx} \right) e^{-i\omega t} \right]\] (3.9.22)
The inviscid horizontal velocity just above the bed boundary layer is
\[
\frac{\partial}{\partial x} \Phi(x, -h, t) = \frac{g k A}{\omega \cosh k h} \Im \left[ \left( e^{i k x} - Re^{-i k x} \right) e^{-i \omega t} \right]
\] (3.9.23)

We can then identify
\[
U = \frac{g k A}{\omega \cosh k h} \left( e^{i k x} - Re^{-i k x} \right)
\] (3.9.24)

For purely progressive waves, \( R = 0 \)
\[
U = \frac{g k A}{\omega \cosh k h} e^{i k x}
\] (3.9.25)

hence
\[
\frac{dU^*}{dx} = -ik \frac{g k A}{\omega \cosh k h} e^{-i k x}
\] (3.9.26)

The induced streaming velocity is,
\[
\bar{u}_2(\infty) = \frac{3}{4 \omega} k \left( \frac{g k A}{\omega \cosh k h} \right)^2
\] (3.9.27)
at the upper edge of the boundary layer, and
\[
\bar{u}_2(\eta) \approx \frac{\eta}{2 \omega} k \left( \frac{g k A}{\omega \cosh k h} \right)^2, \quad \eta \ll 1.
\] (3.9.28)

near the bottom of the boundary layer. The velocity profile is monotonic in height.

For purely standing waves \( R = 1 \), we have
\[
U = \frac{g k A}{\omega \cosh k h} 2i \sin k x
\] (3.9.29)

and
\[
\frac{dU^*}{dx} = -2ik \frac{g k A}{\omega \cosh k h} \cos k x
\] (3.9.30)

Hence
\[
U \frac{dU^*}{dx} = \left( \frac{g k A}{\omega \cosh k h} \right)^2 2k \sin 2k x
\] (3.9.31)

It follows that
\[
\bar{u}_2(\infty) = -\frac{3}{4 \omega} k \left( \frac{g k A}{\omega \cosh k h} \right)^2 \sin 2k x
\] (3.9.32)

and
\[
\bar{u}_2(\eta) \approx \frac{\eta}{2 \omega} k \left( \frac{g k A}{\omega \cosh k h} \right)^2 \sin 2k x, \quad \eta \ll 1.
\] (3.9.33)

Thus near the bottom of the boundary layer, the streaming velocity converges toward points beneath the amplitude minima. Near the top, the opposite is true. See Figure (3.9.2).
Figure 3.2  Schematic variation of mass transport velocity beneath a standing wave.
3.9.3 Physics of the Induced Streaming

Take progressive water waves as an example: We have outside the boundary layer,

\[ u_\infty = U_o \cos(\omega t - kx) \]  

and inside the boundary layer,

\[ u = U_o \left[ \cos(\omega t - kx) - e^{-y/\delta} \cos(\omega t - kx - y/\delta) \right] \]

where the velocity amplitude \( U_o \) is related to the surface amplitude \( A \) by

\[ U_o = \frac{gkA}{\omega \cosh kh} = \frac{A\omega}{\sinh kh} \]

Let us find the induced transverse velocity \( v \)

\[ \frac{\partial u}{\partial x} = U_o \sin(\omega t - kx) - U_o e^{-y/\delta} \sin(\omega t - kx - y/\delta) \]

\[ v_\infty = - \int_0^{y>\delta} \frac{\partial u}{\partial x} \, dy = -y U_o \sin(\omega t - kx) - \frac{1}{2} U_o k\delta \cos(\omega t - kx) + \frac{1}{2} U_o k\delta \sin(\omega t - kx) \]

Now

\[ u_\infty v_\infty = -\frac{1}{4} U_o^2 k\delta < 0 \]

where the \( \sin(\omega t - kx) \) terms in \( v_\infty \) are out of phase with \( u_\infty \) by \( \pi/2 \), hence does not contribute to the mean.

Now consider a slice of boundary layer one wavelength long. Because of periodicity, there is no net transfer of momentum or forces at two ends \( x_0 \) and \( x_0 + 2\pi/k \). But the momentum transfer downwards is \( \frac{U_o^2}{4} k\delta \), causing a positive shear stress. To balance it there must be a non-zero \( \mu \frac{\partial \bar{u}}{\partial y} \) at all levels \( y \) below the top. Hence, the induced streaming velocity is created and \( \bar{u} \neq 0 \).

The induced streaming in a wave boundary layer is a mechanism for the generation of sand bars, as shown in the following photographs.
Figure 3.9.1: Reynolds stress and Induced streaming in Stokes layer

Figure 3.9.2: Accumulation of heavy sand on the metallic bottom of a wave tank. Reflection coefficients vary from low to high. Note that for high reflection, heavy sand pile up beneath the wave nodes.