Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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5.4 Viscous effects on the instability of parallel flow

The instability of parallel flows (in pipes, channels, boundary layers, jet wakes, plumes) is important to understand the transition to turbulence and has been studied by numerous theoreticians and experimentalists. The inviscid theory does not tell us why some flows without the point of inflection are unstable, as observed in pipes by Reynolds.

It turns out that viscosity can be destabilizing. This topic is very fully described in books by Lin, Drazin and Reid, and Drazin. We only give a brief sketch for the steady two-dimensional boundary layer on an semi infinite flat plate (Blasius' problem) here.

5.4.1 Role of viscosity

By considering a control volume one wavelength long, it is possible to show that the balance of mechanical energy requires that

$$\frac{\partial E}{\partial t} = \rho M - \mu N \tag{5.4.1}$$

where

$$E = \frac{\rho}{2}\iint \overline{(u'^2 + v'^2)}dxdy \tag{5.4.2}$$

is the kinetic energyof the disturbance in the volume,

$$\rho M = -\rho\lambda \iint \overline{u'v'}\frac{dU}{dy}dxdy \tag{5.4.3}$$

is the rate of working by Reynolds stress against the shear in the basic flow, and

$$\mu N = \mu \iint (\overline{\zeta'^2})dxdy \tag{5.4.4}$$

is the rate of dissippation, with

$$\zeta' = \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y}$$

being the vorticity. Derivation of this formula is left as a homework exercise.

For large μ, $\frac{\partial E}{\partial t} < 0$, hence viscosity is stabilizing. However for small μ the first term which is also affected by viscosity can overwhelm the second term near the wall, hence viscosity can also be destabilizing.

An argument due to C. C. Lin (MIT) is sketched below.

Near the wall of a parallel shear flow such as Blasius boundary layer of thickness δ (or a channel flow of depth δ), a wave-like disturbance introduces an oscillatory Stokes boundary

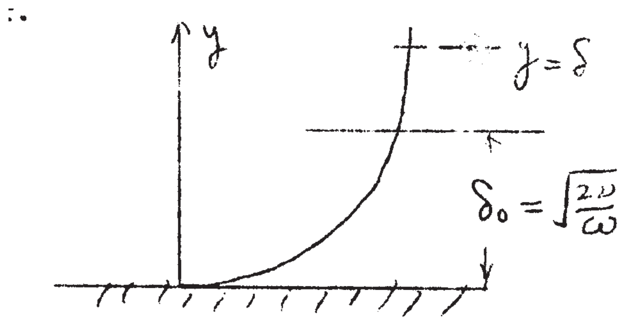

Figure 5.4.1: An oscillatory Stokes boundary layer

layer. Refering to Figure 5.4.1, the oscillatory boundary layer has the thickness of the order $\delta_o = \sqrt{2\nu/\omega}$. Let us calculate the Reynolds stress due to wave disturbances in this boundary layer.

For this purpose let us consider first the vorticity disturbance in the Stokes boundary layer. After linearization, the vorticity satisfies

$$\zeta'_t + U\zeta'_x + v'\bar{\zeta}_y = \nu(\zeta'_{xx} + \zeta'_{yy}) \tag{5.4.5}$$

In the case of traveling wave disturbance

$$(u', v', \zeta') = ((\hat{u}'(y), \hat{v}'(y), \hat{\zeta}'(y))e^{i(kx-\omega t)}$$

Inside the Stokes layer, $\zeta' \cong -u'_y$ so that $u' \sim \delta_o \zeta'$. From continuity we have $v' \sim k\delta_o u'$.Now consider the two convection terms:

$$\frac{U\zeta'_x}{\zeta'_t} \sim \frac{Uk\zeta'}{\omega\zeta'} \sim \frac{U}{C}$$

where $C = \omega/k$, and

$$\frac{v'\bar{\zeta}_y}{\zeta'_t} \sim \frac{k\delta_o^2 U\zeta'}{\omega\zeta'\delta^2} \sim \frac{U}{C}\frac{\delta_o^2}{\delta^2}$$

Close enough to the wall $U/C \ll 1$; the convective terms are negligible.

On the other hand,

$$\zeta'_{xx}/\zeta'_{yy} \sim (k\delta)^2$$

Hence in the wall boundary layer, we have approximately

$$\zeta'_t = \nu\zeta'_{yy} \tag{5.4.6}$$

The boundary conditions are :

$$\zeta' = Ae^{i(kx-\omega t)}, \quad y = 0 \tag{5.4.7}$$

and

$$\zeta' \to 0, \quad y \to \infty. \tag{5.4.8}$$

This boundary value problem is just the Stokes problem of a plate oscillating in its own plane. The solution is, in complex form,

$$\zeta' = \left(Ae^{i\theta}e^{\alpha y}\right), \quad \text{where } \theta = kx - \omega t, \quad \alpha = \frac{-1+i}{\delta_o} \tag{5.4.9}$$

The real part is to be taken at the end.

Let us calculate u' from the vorticity,

$$u' = -\int_0^y \zeta'\,dy = -Ae^{i\theta}\int_0^y e^{\alpha y}dy = \frac{A}{\alpha}e^{i\theta}(1 - e^{\alpha y}) \tag{5.4.10}$$

and v' from continuity,

$$v' = -\int_0^y u'_x dy = -\frac{ikA}{\alpha}e^{i\theta}\int_0^y (1 - e^{\alpha y})dy = -\frac{ikA}{\alpha}e^{i\theta}\left[y - \frac{1}{\alpha}(e^{\alpha y} - 1)\right] \tag{5.4.11}$$

For $y/\delta_o \ll 1$ or $\alpha y \ll 1$ we can approximate u', v' as power series

$$u' \cong -\frac{A}{\alpha}e^{i\theta}\left(\alpha y + \frac{\alpha^2 y^2}{2} + \cdots\right) \tag{5.4.12}$$

and

$$v' \cong \frac{ikA}{\alpha}e^{i\theta}\left(\frac{\alpha y^2}{2} + \frac{\alpha^2 y^3}{6} + \cdots\right) \tag{5.4.13}$$

Now we shall take the time average to get the Reynolds Stress $\overline{u'v'}$. For two sinusoidal functions $f = \Re F e^{-i\omega t}$ and $g = \Re G e^{i\omega t}$, it can be shown that the time average over a period is

$$\overline{fg} = \frac{\omega}{2\pi}\int_t^{t+2\pi/\omega} fg\,dt = \frac{1}{2}\Re(FG^*) = \frac{1}{2}\Re(F^*G) \tag{5.4.14}$$

Here the time average is equal to the space average over a wavelength and to the phase average.

Substitituting (5.4.12) and (5.4.13) into (5.4.14), we get

$$\begin{aligned}
-\overline{u'v'} &= -\frac{1}{2}\Re\left\{\left[-\frac{A^*}{\alpha^*}\left(\alpha^* y + \frac{\alpha^{*2}y^2}{2} + \cdots\right)\right]\left[\frac{ikA}{\alpha}\left(\frac{\alpha y^2}{2} + \frac{\alpha^2 y^3}{6} + \cdots\right)\right]\right\} \\
&= -\frac{1}{2}\Re\left\{\frac{ik|A|^2}{2}y^3 - \frac{ik|A|^2}{|\alpha|^2}\left(\frac{\alpha(\alpha^*)^2}{4} + \frac{\alpha^*\alpha^2}{6}\right)y^4 + \cdots\right\} \\
&= -\frac{1}{2}\Re\left\{ik|A|^2\left[\frac{1}{4}\frac{-1-i}{\delta_o} + \frac{1}{6}\frac{-1+i}{\delta_o}y^4\cdots\right]\right\} \\
&= \frac{k|A|^2}{24\delta_o}y^4 + \cdots
\end{aligned}$$

The Reynold stress is therefore positive

$$-\rho\overline{u'v'} = \frac{\rho|A|^2}{24} y^4 \frac{k}{\delta} > 0 \tag{5.4.15}$$

to the leading order and is inversely proportional to $\sqrt{\mu}$. On the other hand, the rate of dissipation is proportional to the integral of

$$\mu\overline{\zeta'}^2 \propto \mu|A|^2 \tag{5.4.16}$$

For sufficiently small viscosity work done by the Reynolds stress to the mean shear can therefore overwhelm dissipation, leading to instability. This estimate has also been cofirmed by fuller anslysis (see Lin, p 62ff.) .

Homework Use the solution (5.4.9), (5.4.10) and (5.4.11) without assuming small y/δ_0 and compute ρM and μN and discuss the result.

5.4.2 Orr-Sommerfeld theory

The complete solution of the viscous instability problem begins with the linearized Navier-Stokes equations. Starting from a basic shear flow $\vec{Q} = (U(y), 0)$, we consider an infinitesimal disturbance with the velocity field $\vec{q} = (u(x, y, t), v(x, y, t))$ which is much smaller than U. Introduce the stream fundtion

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

and assume a normal mode disturbance with real k and complex C,

$$\psi = f(y)e^{ik(x-Ct)}. \text{ where } C = \omega/k$$

subject to the boundary conditions that

$$\psi = 0, \psi_y = 0, \quad y = 0, \text{ on the wall} \tag{5.4.17}$$

$$\psi = 0, \psi_y = 0, \quad y = d(\text{or } \infty) \tag{5.4.18}$$

After nondimensionalization and lineaization of the Navier-Stokes equations we get the Orr-Sommerfeld equation for $f(y)$

$$f'''' - 2k^2 f'' + k^4 f = -ikR\left[(U(y) - C)(f'' - k^2 f) - U'' f\right] \tag{5.4.19}$$

where R is the Reynolds number. The boundary conditions are

$$f = f' = 0, \quad y = 0, 1 \tag{5.4.20}$$

the eignevalue problem can be solved numerically.

For the boundary layer, computations (Tolmien and Schlichting) show that the threshold of instability $C_i = 0$ is a hairpin-like curve in the plane of $k\delta_1(x)$ vs. $R_1(x) = U\delta_1/\nu$ where

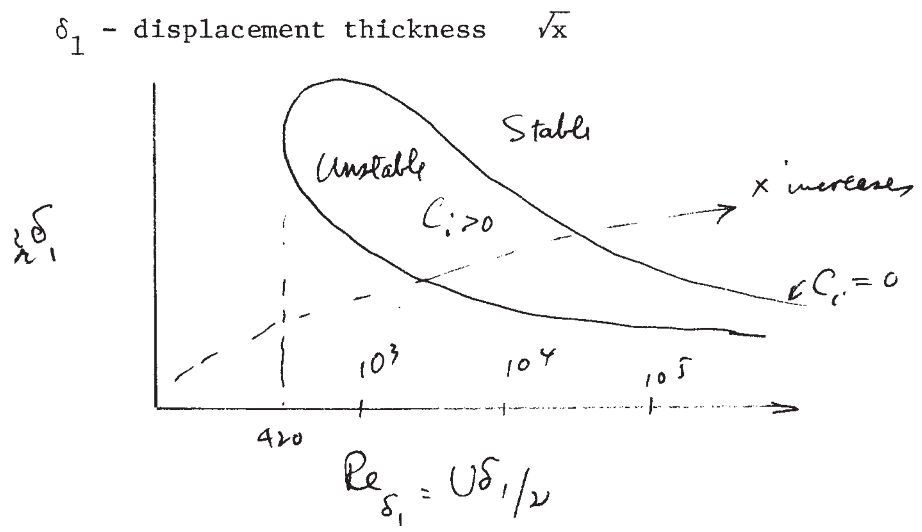

Figure 5.4.2: Instability region of a Blasius boundary layer. $k_r\delta_1$ (ordinate) vs. Re_{δ_1} (abscissa) where δ_1 is the displacement thickness which increases with $\sqrt{x}$.

$\delta_1(x) \propto \sqrt{x}$ is the displacement thickness, as shown in Figure 5.4.2. At a fixed x, a sinudoidal disturbance of wave number k is stable if the Reynolds number is within certain hairpin-like band. The is also a critical Reynolds number below which all disturbaqnces are stable, and above which some disturbances are unstable.

An improved solution of Orr-Sommerfeld equation was later obtained by Shen (1954) using a method of Lin (1944). Shen's result on the neutral stability curve is shown in Figure 5.4.3.

Verifications were first made in a landmark experiment by Schubauer, Skramstad and Dryden (1945). With controled disturbances where the amplitude and wavelength of the sinusodial distrubance were introduced by a loudspeaker, the Tolmien-Schlichting threshold was confirmed. Under the test condition

$$U = 53ft/s, \quad \frac{\sqrt{u'^2}}{U} = 3 \times 10^{-4}$$

they found that natural disturbances were still wave-like at $x = 7ft$, but waves are amplified at 9 ft. Occasional bursts appeared at 10.5 feet and signals become random at 11 ft. Later experiments (See Figure 5.4.4) and nonlinear theories showed the following development as x increases :

1. For small x a laminar boundary layer is stable.

2. For sufficiently long waves, linear instability leads to Tolmien-Schlichting waves.

3. Tolmien-Schlichting waves grow large. Nonlinearity sets in. Disturbances become 3-dimensional. Spanwise periodicity appears. Local instablitiy occurs.

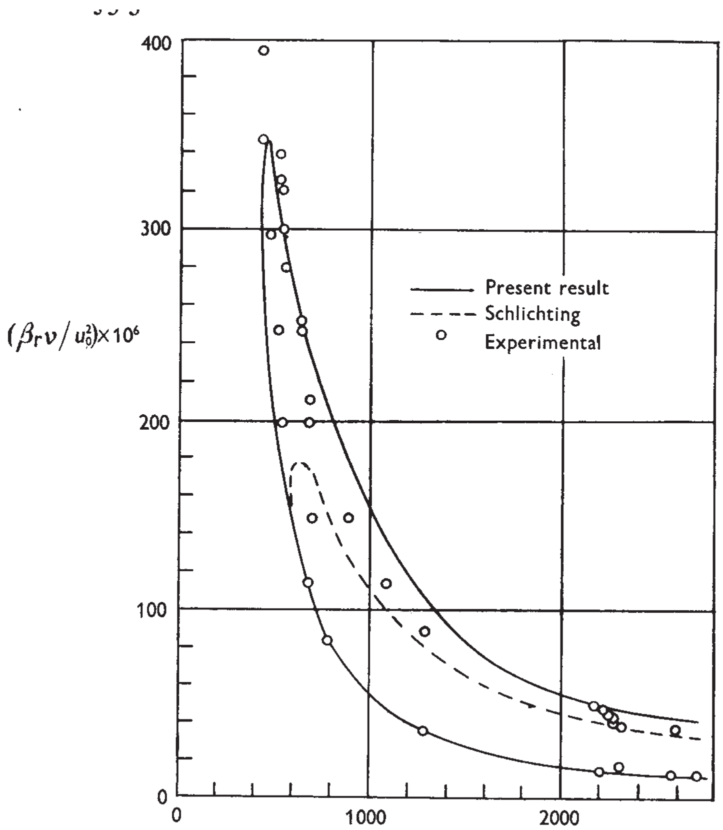

Fig. 5.1. Curve of neutral stability for Blasius flow (after Shen, 1954).

Figure 5.4.3: Comparison of theories by Schlichting and S. F. Shen (1954), and experiments on boundary layer instability.

4. Reynolds-stress generates higher harmonics. Turbulent spots appear.

5. Turbulent spots grow and overlap.

6. Boundary layer becomes turbulent.

This evolution is summarized in Figure 5.4.5

Fig. 16.17. Oscillogram of the u'-component of fluctuations caused by random ("natural") disturbances in the laminar boundary layer on a flat plate in a stream of air. Measurements on transition from laminar to turbulent flow due to Schubauer and Skramstad [57]

Distance from wall: 0·57 mm; free-stream velocity $U_\infty = 24$ m/sec, interval between time marks: $^1/_{30}$ sec

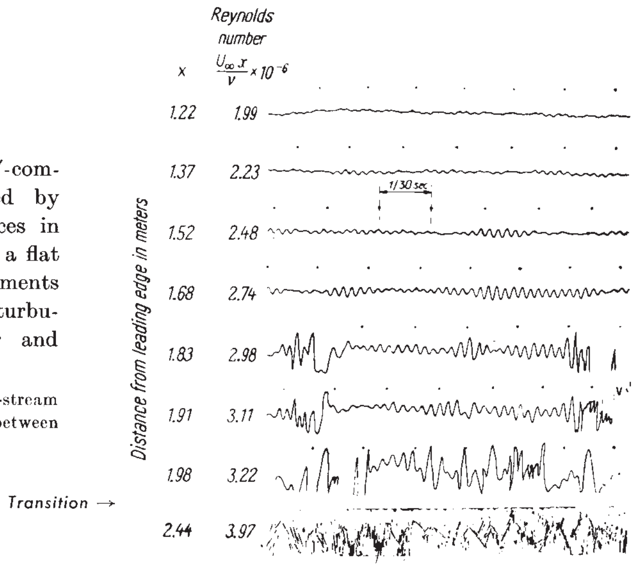

Figure 5.4.4: Experiments on boundary layer instability. From Schlichting.

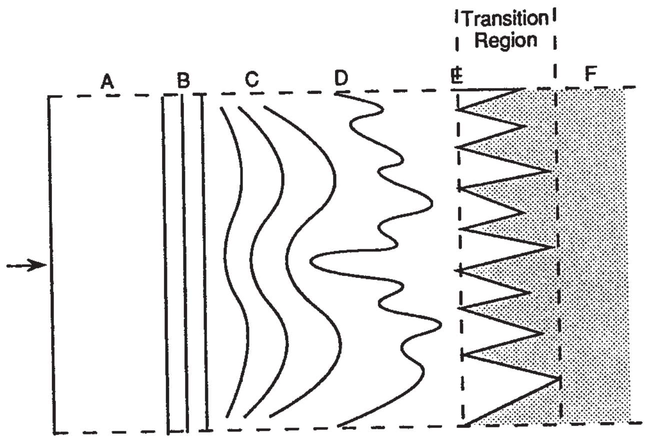

Figure 8.11 A symbolic sketch indicating roughly the regions of development of instability in Blasius's boundary layer on a plate at zero incidence in a low-turbulence stream: plan view. A, laminar flow; B, Tollmien–Schlichting (two-dimensional small-amplitude) waves; C, three-dimensional wave amplification; D, nonlinear peak-valley development with streamwise vortices; E, breakdown with formation and growth of turbulent spots; F, fully developed turbulence. (After Young, 1989, Fig. 5.13; reproduced by permission of Blackwell Science Ltd.)

Figure 5.4.5: Spatial evolution of Blasius' boundary layer from instability to turbulence instability. From Drazin, 2002.