

## Lecture Notes on Fluid Dynamics

(1.63J/2.21J)

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6-3SaTay.tex

May 11, 2003

### 6.3 Saffman-Taylor instability in porous layer- Viscous fingering

Refs:

P. G. Saffman & G. I. Taylor, 1958, The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more visous fluid. *Proc. Royal Society*, 245, 312-329.

G. Homsy; 1987. Viscous fingering in porous media. *Annual Rev of Fluid Mech.* 19, 271 - 314.

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In petroleum recovery water is often used to drive oil from the reservoir. An oil reservoir can also be covered by a layer of water from above. Phenomenon of fingering often occurs when oil is extracted frome beneath the water layer. Although known to mining engineers, Saffman & Taylor (1958) gave the first theory and performed simulated experiments in a Hele-Shaw cell.

Consider a moving interface in a stationary coordinate system. Let the initial seepage velocity  $V$  be vertical and the interface be a plane, then

$$y = \frac{Vt}{n} \quad (6.3.1)$$

where  $n$  is the porosity. If the interface is disturbed then its position is at

$$y = \frac{Vt}{n} + \eta(x, t) \quad (6.3.2)$$

At any interior point,  $\phi$  is the velocity potential,

$$\phi = -\frac{k}{\mu} (p + \rho gy) \quad (6.3.3)$$

where  $k$  is the permeability related to conductivity  $K$  by

$$K = \frac{\rho g k}{\mu} \quad (6.3.4)$$

The pressure is

$$p = -\frac{\mu}{k} \phi - \rho gy \quad (6.3.5)$$

Thus in fluid 1(upper fluid)

$$p_1 = -\frac{\mu_1}{k_1}g\phi_1 - \rho_1gy \quad (6.3.6)$$

By continuity,

$$\nabla^2\phi_1 = 0, \quad y > \frac{Vt}{n} + \eta(x, t), \quad (6.3.7)$$

In the lower fluid (2),

$$p_2 = -\frac{\mu_2}{k_2}g\phi_2 - \rho_2gy \quad (6.3.8)$$

and

$$\nabla^2\phi_2 = 0, \quad y < \frac{Vt}{n} + \eta(x, t) \quad (6.3.9)$$

Let us first examine the basic uniform flow where the interface is plane ( $\eta = 0$ ). The potentials are

$$\phi_1^o = Vy + f_1(t) = -\frac{k_1}{\mu_1}(p_1^o + \rho_1gy) \quad (6.3.10)$$

$$\phi_2 = Vy + f_2(t) = -\frac{k_2}{\mu_2}(p_2^o + \rho_2gy) \quad (6.3.11)$$

Note that an arbitrary function of  $f(t)$  is added to the potential without affecting the velocity field. The pressures are

$$p_1^o = -\left(\frac{\mu_1V}{k_1} + \rho_1g\right)y - \frac{\mu_1f_1(t)}{k_1}, \quad y > \frac{Vt}{n} \quad (6.3.12)$$

and in the lower fluid (2),

$$p_2^o = -\left(\frac{\mu_2V}{k_2} + \rho_2g\right)y - \frac{\mu_2f_2(t)}{k_2}, \quad y < \frac{Vt}{n} \quad (6.3.13)$$

In order that pressure is continuous at  $y = Vt/n$  for all  $t$ , we must have

$$f_1(t) = F_1t, \quad f_2(t) = F_2t \quad (6.3.14)$$

where  $F_1, F_2$  are constants and

$$-\left(\frac{\mu_1V}{k_1} + \rho_1g\right)\frac{V}{n} - \frac{\mu_1F_1}{k_1} = -\left(\frac{\mu_2V}{k_2} + \rho_2g\right)\frac{V}{n} + \frac{\mu_2F_2}{k_2}$$

Thus

$$\frac{\mu_2F_2}{k_2} - \frac{\mu_1F_1}{k_1} = -\frac{V}{n} \left\{ \left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)V + (\rho_2 - \rho_1g) \right\} \quad (6.3.15)$$

Note that it is only the difference that matters.

We now consider a small disturbance on the interface .

$$y = \frac{Vt}{n} + \eta(x, t) \quad (6.3.16)$$

where

$$\eta = ae^{i\alpha x - i\omega t} \quad (6.3.17)$$

is small. The total solution is

$$\phi_1 = Vy + F_1t + B_1e^{i\alpha x - \alpha(y - Vt/n) - i\omega t}, \quad y > \frac{Vt}{n} + \eta(x, t) \quad (6.3.18)$$

$$\phi_2 = Vy + F_2t + B_2e^{i\alpha x + \alpha(y - Vt/n) - i\omega t}, \quad y < \frac{Vt}{n} + \eta(x, t) \quad (6.3.19)$$

The linearized kinematic boundary condition is that velocities must be continuous.

$$n \frac{\partial \eta}{\partial t} = \frac{\partial \phi_1}{\partial y} \Big|_{y=Vt/n} = \frac{\partial \phi_2}{\partial y} \Big|_{y=Vt/n} \quad (6.3.20)$$

Thus

$$-i\omega n a e^{i\alpha x - i\omega t} = -\alpha a B_1 e^{i\alpha x - i\omega t} = \alpha a B_2 e^{i\alpha x - i\omega t} \quad (6.3.21)$$

hence,

$$B_1 = -B_2 = \frac{i\omega n a}{\alpha} \quad (6.3.22)$$

Now we require continuity of pressure on  $y = Vt/n + \eta$ ,

$$-\frac{\mu_1}{k_1} \left( V\eta + \frac{i\omega n \eta}{\alpha} \right) - \rho_1 g \eta = -\frac{\mu_2}{k_2} \left( V\eta - \frac{i\omega n \eta}{\alpha} \right) - \rho_2 g \eta \quad (6.3.23)$$

Eliminating  $\eta$  we get

$$i\omega = \frac{\alpha (\rho_2 - \rho_1) g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right)}{n \left( \frac{\mu_2}{k_2} + \frac{\mu_1}{k_1} \right)} \quad (6.3.24)$$

Clearly  $i\omega$  is real. If  $i\omega > 0$ , or

$$(\rho_2 - \rho_1)g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) > 0 \quad (6.3.25)$$

the flow is stable. If  $i\omega < 0$ , or

$$(\rho_2 - \rho_1)g + V \left( \frac{\mu_2}{k_2} - \frac{\mu_1}{k_1} \right) < 0, \quad (6.3.26)$$

the flow is unstable.

From the simple model of a tubular porous medium,

$$K = \frac{n\rho g R^2}{8\mu} = \frac{\rho g k}{\mu} \quad (6.3.27)$$

hence

$$k = \frac{nR^2}{8} \quad (6.3.28)$$

is independent of viscosity and depends only on  $n$  and the pore size. Assume therefore  $k_1 = k_2$  and that oil (lighter more viscous) lies above water  $\rho_1 < \rho_2$  and  $\mu_1 > \mu_2$ . If  $V < 0$  (water pushed downward by oil ) then the flow is always stable. Consider  $V > 0$ . The flow is unstable if

$$V > V_c = \frac{(\rho_2 - \rho_1)g}{\left(\frac{\mu_1}{k_1} - \frac{\mu_2}{k_2}\right)} \quad (6.3.29)$$

Too high an extraction rate causes instability which marks the onset of fingers.

If the water layer is on top of the oil layer, then  $\rho_2 - \rho_1 < 0$ ; the flow is unstable even if  $V = 0$ . Since  $\mu_2/k_2 - \mu_1/k_1 > 0$  a downward flow (water toward oil) is always unstable. A upward flow can be unstable if

$$0 < V < V_c = \frac{(\rho_1 - \rho_2)g}{\left(\frac{\mu_2}{k_2} - \frac{\mu_1}{k_1}\right)} \quad (6.3.30)$$

Note also that the growth(decay) rate is higher for shorter waves.

A gallery of beautiful photographs of fingering taken fromn Hele-Shaw erperiments can be found in the survey by Homsy. Here are two samples.

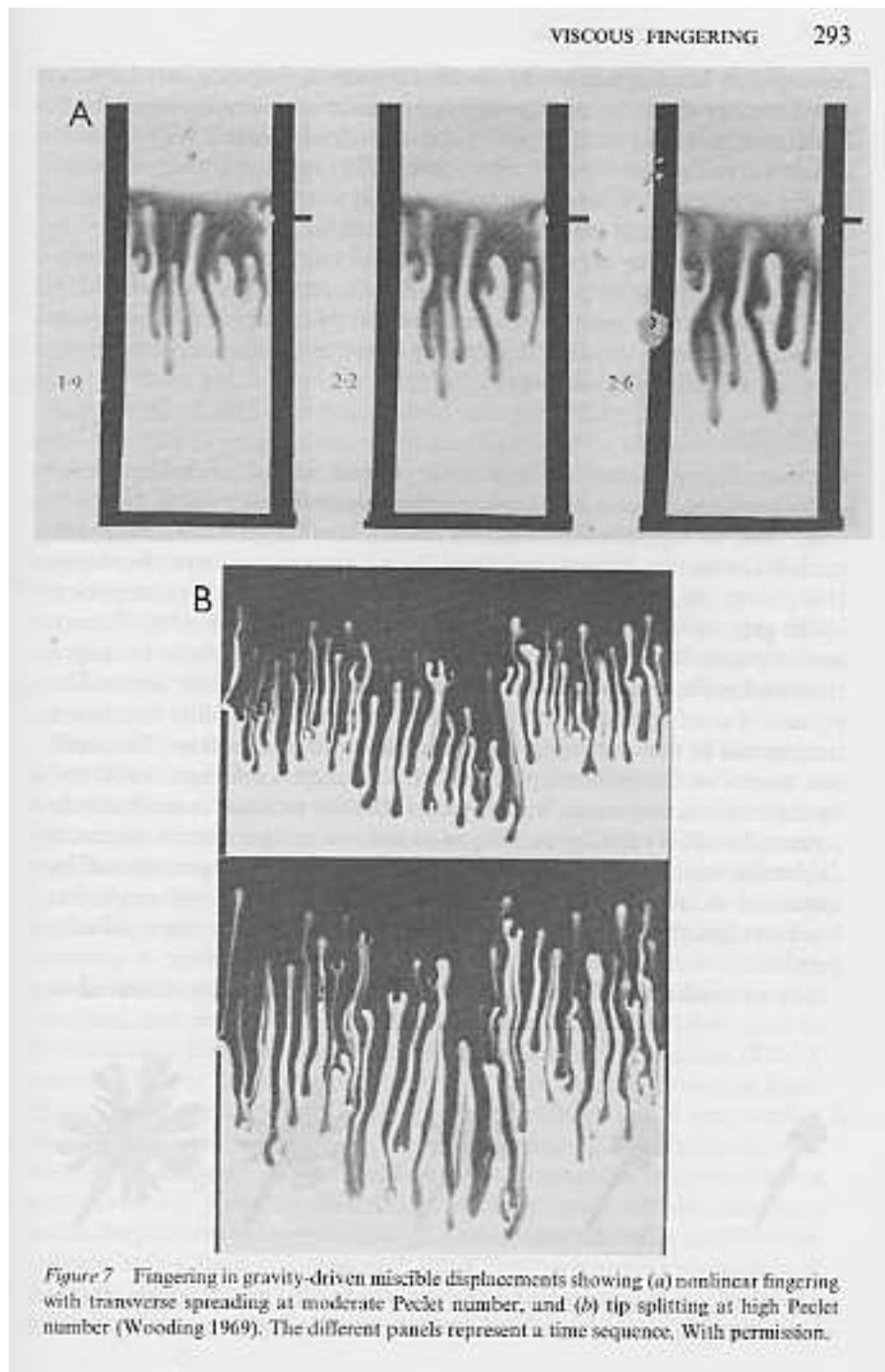


Figure 6.3.1: Fingering in a Hele-Shaw model. (Wooding) From Homsy.

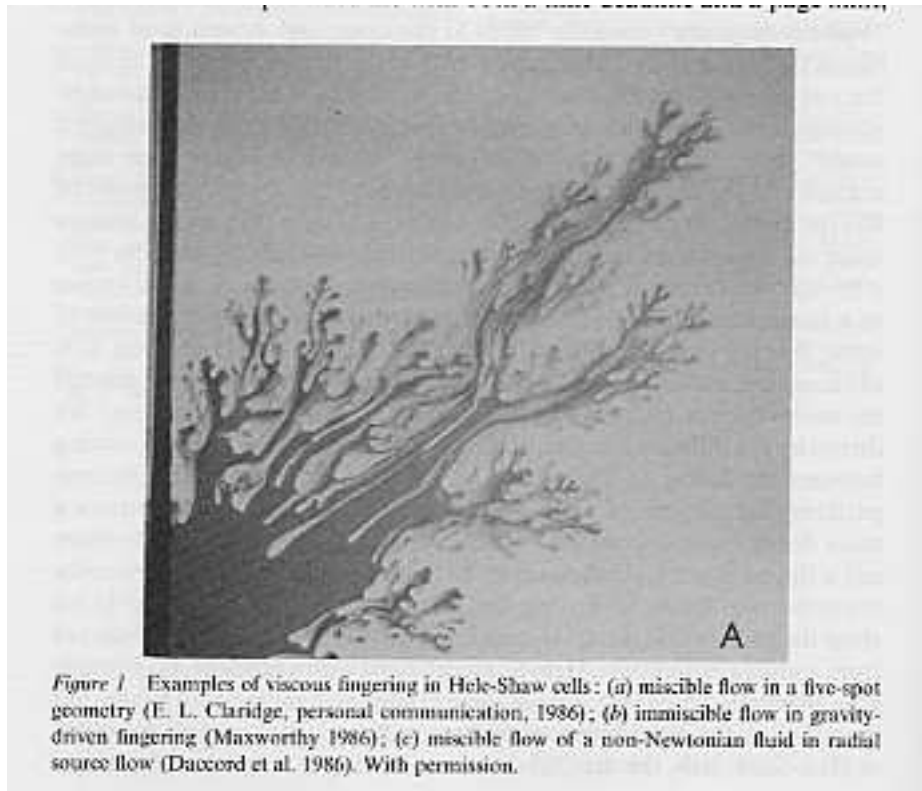


Figure 6.3.2: Fingering in a Hele shaw model. From Homsy.