Lecture Notes on Fluid Dynamics
(1.63J/2.21J)
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7-8-2layer.tex
Refs:
Csandy: *Circulation in the Coastal Ocean*
Cushman-Rosin, *Intro to Geophysical Fluid Dynamics*

7.8 Transient motion in a two-layered sea

7.8.1 Shallow water approximation

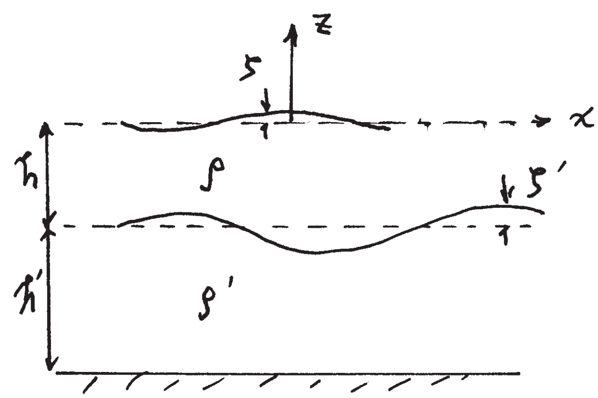

Figure 7.8.1: A two-layered sea of contant depth.

Let h be the mean depth and U, V be the depth-integrated fluxes in the upper layer, and h', U', V' be the corresponding quantities in the lower layer. Let ζ and ζ' be the vertical displacements of the free surface and interface respectively. Then

$$(U,V) = \int_{-h}^{0} (u,v)dz, \quad (U',V') = \int_{-h-h'}^{-h} (u',v')dz, \tag{7.8.1}$$

For shallow layers, we have hydrostatic pressure in the upper layer:

$$p = \rho g(\zeta - z) \tag{7.8.2}$$

On the interface, $z = -h + \zeta'$,

$$p = \rho g(\zeta + h - \zeta') \tag{7.8.3}$$

In order that pressure is continuous on the interface we must have in the lower layer

$$p' = \rho g(\zeta + h - \zeta') + \rho' g(-h + \zeta' - z) \tag{7.8.4}$$

Thus

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \zeta}{\partial x} \tag{7.8.5}$$

$$\frac{\partial p'}{\partial x} = \rho g \left(\frac{\partial \zeta}{\partial x} - \frac{\partial \zeta'}{\partial x} \right) + \rho' g \frac{\partial \zeta'}{\partial x} = \rho g \frac{\partial \zeta}{\partial x} + (\rho' - \rho) g \frac{\partial \zeta'}{\partial x} \tag{7.8.6}$$

The depth-integrated momentum equations in the upper layer are

$$\frac{\partial U}{\partial t} - fV = -\frac{1}{\rho}\frac{\partial p}{\partial x} = -gh\frac{\partial \zeta}{\partial x} + \frac{\tau_x^S}{\rho} \tag{7.8.7}$$

$$\frac{\partial V}{\partial t} + fU = -\frac{1}{\rho}\frac{\partial p}{\partial y} = -gh\frac{\partial \zeta}{\partial y} + \frac{\tau_y^S}{\rho} \tag{7.8.8}$$

$$\frac{\partial (\zeta - \zeta')}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \tag{7.8.9}$$

after using (7.8.2). Ignoring shear stresses on the interface and at the bottom, we have, for the lower layer,

$$\frac{\partial U'}{\partial t} - fV' = -\frac{1}{\rho'}\frac{\partial p'}{\partial x} = -gh'\frac{\rho}{\rho'}\frac{\partial \zeta}{\partial x} - \epsilon g h' \frac{\partial \zeta'}{\partial x} \tag{7.8.10}$$

$$\frac{\partial V'}{\partial t} + fU' = -\frac{1}{\rho'}\frac{\partial p'}{\partial y} = -gh'\frac{\rho}{\rho'}\frac{\partial \zeta}{\partial y} - \epsilon g h' \frac{\partial \zeta'}{\partial y} \tag{7.8.11}$$

$$\frac{\partial \zeta'}{\partial t} + \frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = 0 \tag{7.8.12}$$

where (7.8.2) has been used. The density contrast

$$\epsilon = \frac{\rho' - \rho}{\rho'} \tag{7.8.13}$$

is usually small in oceans.

There six equations for six unknowns : ζ, U, V, ζ', U', and V'. As in the last section we can eliminate some unknowns and reduce the number of equations. For example, by eliminating U, V in the upper layer, we can get

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} + f^2\right)(\zeta - \zeta') - gh\nabla^2\frac{\partial \zeta}{\partial t} = -\frac{1}{\rho}\frac{\partial}{\partial t}\left(\frac{\partial \tau_x^S}{\partial x} + \frac{\partial \tau_y^S}{\partial y}\right) - \frac{f}{\rho}\left(\frac{\partial \tau_y^S}{\partial x} - \frac{\partial \tau_x^S}{\partial y}\right) \tag{7.8.14}$$

Similarly by eliminating U', V' in the lower layer we get,

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial t^2} + f^2\right)\zeta' - gh'\frac{\rho}{\rho'}\nabla^2\frac{\partial \zeta}{\partial t} - \epsilon g h' \nabla^2 \frac{\partial \zeta'}{\partial t} = 0 \tag{7.8.15}$$

These two govern the coupled behavior of ζ and ζ', in terms of which the no-flux boundary conditons on the velocity can also be expressed.

We shall however introduce an alternate approach which has some advantages.

7.8.2 Transformation into normal form

To avoid solving six simultaneous equations at once, we seek a linear combination so that the governing equations look like those for a one-layer fluid with just three unknowns. This is called the *normal form*. Take $a(7.8.7) + (7.8.10)$, we get,

$$\frac{\partial}{\partial t}(aU + U') - f(aV + V') = -gh\frac{\partial}{\partial x}\left(a\zeta + \frac{h'\rho}{h\rho'}\zeta + \frac{h'}{h}\epsilon\zeta'\right) + \frac{a\tau_x^S}{\rho} \tag{7.8.16}$$

Similarly $a(7.8.8) + (7.8.11)$ gives

$$\frac{\partial}{\partial t}(aV + V') + f(aU + U') = -gh\frac{\partial}{\partial y}\left(a\zeta + \frac{h'\rho}{h\rho'}\zeta + \frac{h'}{h}\epsilon\zeta'\right) + \frac{a\tau_y^S}{\rho} \tag{7.8.17}$$

Finally $a(7.8.9) + (7.8.12)$ gives,

$$\frac{\partial}{\partial t}(a\zeta - a\zeta' + \zeta') + \frac{\partial}{\partial x}(aU + U') + \frac{\partial}{\partial y}(aV + V') = 0 \tag{7.8.18}$$

from the continuity equations.

Now if we take

$$a\zeta + \frac{h'\rho}{h\rho'}\zeta + \frac{h'}{h}\epsilon\zeta' = \beta\bar{\zeta} \tag{7.8.19}$$

$$a\zeta - a\zeta' + \zeta' = a\zeta + (1-a)\zeta' = \bar{\zeta} \tag{7.8.20}$$

and

$$\bar{U} = aU + U, \quad \bar{V} = V + V' \tag{7.8.21}$$

then (7.8.16)-(7.8.18) become

$$\frac{\partial \bar{U}}{\partial t} - f\bar{V} = -g\beta h\frac{\partial \bar{\zeta}}{\partial x} + a\frac{\bar{\tau}_x^S}{\rho} \tag{7.8.22}$$

$$\frac{\partial \bar{V}}{\partial t} + f\bar{U} = -g\beta h\frac{\partial \bar{\zeta}}{\partial y} + a\frac{\bar{\tau}_y^S}{\rho} \tag{7.8.23}$$

$$\frac{\partial \bar{\zeta}}{\partial t} + \frac{\partial \bar{U}}{\partial x} + \frac{\partial \bar{V}}{\partial y} = 0 \tag{7.8.24}$$

and take on the appearance of the governing equations for a single layer fluid.

For this reduction to be possible, conditions (7.8.19) and (7.8.20) imply that

$$a\zeta + \frac{h'\rho}{h\rho'}\zeta + \frac{h'}{h}\epsilon\zeta' = \beta\left(a\zeta - a\zeta' + \zeta'\right) \tag{7.8.25}$$

Equating the coefficients of ζ and ζ' separately, we get

$$\beta a = a + \frac{h'\rho}{h\rho'}, \tag{7.8.26}$$

$$\beta(-a+1) = \frac{h'\epsilon}{h} \tag{7.8.27}$$

A quadratic equation is found for β by eliminating a

$$\beta^2 - \beta\left(1 + \frac{h'\epsilon}{h} + \frac{h'\rho}{h\rho'}\right) + \frac{h'\epsilon}{h} = 0$$

i.e.,

$$\beta^2 - \left(\frac{h'}{h} + 1\right)\beta + \frac{h'\epsilon}{h} = 0 \tag{7.8.28}$$

There are two solutions for β

$$\left\{\begin{array}{c}\beta_1\\ \beta_2\end{array}\right\} = \frac{h+h'}{2h}\left[1 \pm \sqrt{1 - \frac{4\epsilon hh'}{(h+h')^2}}\right], \tag{7.8.29}$$

to be refered to as two normal modes. The solutions corresonding to β_1 and β_2 will be refered to as Mode 1 and Mode 2, respectively.

Let us see what the corresponding a's are. Using (7.8.27), we have

$$a = 1 - \frac{h'\epsilon}{h\beta}. \tag{7.8.30}$$

Since from (7.8.28),

$$\beta_1\beta_2 = \frac{h'\epsilon}{h}$$

it follows that

$$a = 1 - \frac{\beta_1\beta_2}{\beta}.$$

Therefore,

$$a_1 = 1 - \beta_2, \quad a_2 = 1 - \beta_1 \tag{7.8.31}$$

Let us summarize the normal form equations for Mode k with $k = 1, 2$. Defining,

$$\bar{U}_k = a_k U + U', \quad \bar{V}_k = a_k V + V' \tag{7.8.32}$$

$$\bar{\zeta}_k = a_k\zeta + (1 - a_k)\zeta' \tag{7.8.33}$$

$$(\bar{\tau}_x)_k = a_k\tau_x^S, \qquad (\bar{\tau}_y)_k = a_k\tau_y^S. \tag{7.8.34}$$

then

$$\frac{\partial \bar{U}_k}{\partial t} - f\bar{V}_k = -g\beta_k h\frac{\partial \bar{\zeta}_k}{\partial x} + \frac{(\bar{\tau}_x)_k}{\rho} \tag{7.8.35}$$

$$\frac{\partial \bar{V}_k}{\partial t} + f\bar{U}_k = -g\beta_k h\frac{\partial \bar{\zeta}_k}{\partial y} + \frac{(\bar{\tau}_y)_k}{\rho} \tag{7.8.36}$$

$$\frac{\partial \bar{\zeta}_k}{\partial t} + \frac{\partial \bar{U}_k}{\partial x} + \frac{\partial \bar{V}_k}{\partial y} = 0 \tag{7.8.37}$$

Note that for mode k the effective gravity is $g\beta_k$, and the effective forcing stresses are $a_k\tau_x^S$ and $a_k\tau_y^S$. Solution for the set of three equations is obvioully a simpler task. For a general wind stress field (τ_x^S, τ_y^S), we first solve for $(\bar{U}_1, \bar{V}_1, \bar{\zeta}_1)$ and $(\bar{U}_2\bar{V}_2, \bar{\zeta}_2)$ separately from (7.8.35) to (7.8.37). Afterwards (U, V, ζ) and (U', V', ζ') can be solved from the algebraic equations (7.8.32) and (7.8.33). For example, From

$$\bar{\zeta}_1 = a_1\zeta + (1 - a_1)\zeta', \quad \bar{\zeta}_2 = a_2\zeta + (1 - a_2)\zeta' \tag{7.8.38}$$

we get for the sea surface,

$$\zeta = \frac{\begin{vmatrix} \bar{\zeta}_1 & 1 - a_1 \\ \bar{\zeta}_2 & 1 - a_2 \end{vmatrix}}{\begin{vmatrix} a_1 & 1 - a_1 \\ a_2 & 1 - a_2 \end{vmatrix}} = \frac{(1 - a_2)\bar{\zeta}_1 - (1 - a_1)\bar{\zeta}_2}{a_1 - a_2} \tag{7.8.39}$$

and for the interface

$$\zeta' = \frac{\begin{vmatrix} a_1 & \bar{\zeta}_1 \\ a_2 & \bar{\zeta}_2 \end{vmatrix}}{\begin{vmatrix} a_1 & 1 - a_1 \\ a_2 & 1 - a_2 \end{vmatrix}} = \frac{-a_2\bar{\zeta}_1 + a_1\bar{\zeta}_2}{a_1 - a_2} \tag{7.8.40}$$

Similarly, from

$$\bar{U}_1 = a_1U + U', \quad \bar{U}_2 = a_2U + U' \tag{7.8.41}$$

we get, for the upper layer

$$U = \frac{\begin{vmatrix} \bar{U}_1 & 1 \\ \bar{U}_2 & 1 \end{vmatrix}}{\begin{vmatrix} a_1 & 1 \\ a_2 & 1 \end{vmatrix}} = \frac{\bar{U}_1 - \bar{U}_2}{a_1 - a_2} \tag{7.8.42}$$

and for the lower layer

$$U' = \frac{\begin{vmatrix} a_1 & \bar{U}_1 \\ a_2 & \bar{U}_2 \end{vmatrix}}{\begin{vmatrix} a_1 & 1 \\ a_2 & 1 \end{vmatrix}} = \frac{-a_2\bar{U}_1 + a_1\bar{U}_2}{a_1 - a_2} \tag{7.8.43}$$

The formulas for V and V' are similar, with U's replaced by V's.

7.8.3 Normal form for small density difference

From here we shall only be interested in small density differences, then

$$\beta_1 = 1 + \frac{h'}{h} - \frac{\epsilon h'}{h + h'} + O(\epsilon^2) \tag{7.8.44}$$

$$\beta_2 = \frac{\epsilon h'}{h + h'} + O(\epsilon^2) \tag{7.8.45}$$

or

$$a_1 = 1 - \frac{\epsilon h'}{h + h'} + O(\epsilon^2), \tag{7.8.46}$$

$$a_2 = -\frac{h'}{h} + \frac{\epsilon h'}{h + h'} + O(\epsilon^2) \tag{7.8.47}$$

Taking only the leading order terms, we have

$$a_1 = 1, \quad a_2 = -\frac{h'}{h}. \tag{7.8.48}$$

From (7.8.44)

$$\beta_1 = \frac{h + h'}{h}, \quad \beta_2 = \frac{\epsilon h'}{h + h'} \tag{7.8.49}$$

We also get from (7.8.42) and (7.8.43)

$$\zeta = \frac{(1 + h'/h)\,\bar{\zeta}_1 + (1 - 1)\bar{\zeta}_2}{1 + h'/h} = \bar{\zeta}_1. \tag{7.8.50}$$

$$\zeta' = \frac{(h'/h)\bar{\zeta}_1 + \bar{\zeta}_2}{1 + h'/h} \tag{7.8.51}$$

$$U = \frac{\bar{U}_1 - \bar{U}_2}{1 + h'/h}, \quad U' = \frac{(h'/h)\bar{U}_1 + \bar{U}_2}{1 + h'/h} \tag{7.8.52}$$

$$V = \frac{\bar{V}_1 - \bar{V}_2}{1 + h'/h}, \quad V' = \frac{(h'/h)\bar{V}_1 + \bar{V}_2}{1 + h'/h} \tag{7.8.53}$$

For Mode 1 ($\bar{\zeta}_2 = \bar{U}_2 = \bar{V}_2 = 0$), we get from (7.8.32),(7.8.33) and (7.8.34),

$$\begin{aligned} \bar{U}_1 &= U + U' \qquad \bar{V}_1 = V + V' \\ \bar{\zeta}_1 &= \zeta, \qquad (\bar{\tau}_x)_1 = \tau_x^S \qquad (\bar{\tau}_y)_1 = \tau_y^S. \end{aligned} \tag{7.8.54}$$

Let $H = h + h'$ denote the total depth, the normal mode equations are

$$\frac{\partial \bar{U}_1}{\partial t} - f\bar{V}_1 = -gH\frac{\partial \bar{\zeta}_1}{\partial x} + \frac{\tau_x^S}{\rho} \tag{7.8.55}$$

$$\frac{\partial \bar{V}_1}{\partial t} + f\bar{U}_1 = -gH\frac{\partial \bar{\zeta}_1}{\partial y} + \frac{\tau_y^S}{\rho} \tag{7.8.56}$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \bar{U}_1}{\partial x} + \frac{\partial \bar{V}_1}{\partial y} = 0. \tag{7.8.57}$$

In view of (7.8.54), the two layers move together as if a single layer of homogeneous fluid. This is called the Barotropic (surface) mode.

For Mode 2 ($\bar{\zeta}_1 = \bar{U}_1 = \bar{V}_1 = 0$), we recall from (7.8.35),

$$a_2 = -\frac{h'}{h} + O(\epsilon), \quad \beta_2 = \frac{\epsilon h'}{h + h'} \tag{7.8.58}$$

Then from (7.8.32) to (7.8.34),

$$\begin{aligned} \bar{U}_2 &= -\frac{h'}{h} U + U', \qquad \bar{V}_2 = -\frac{h'}{h} V + V' \\ \bar{\zeta}_2 &= -\frac{h'}{h}\zeta + \left(1 + \frac{h'}{h}\right)\zeta' \end{aligned} \tag{7.8.59}$$

and

$$\beta_2 = \frac{\epsilon h'}{h + h'} \tag{7.8.60}$$

from (7.8.45). The normal form equations are

$$\frac{\partial \bar{\zeta}_2}{\partial t} + \frac{\partial \bar{U}_2}{\partial x} + \frac{\partial \bar{V}_2}{\partial y} = 0 \tag{7.8.61}$$

$$\frac{\partial \bar{U}_2}{\partial t} - f\bar{V}_2 = -g\frac{\epsilon h h'}{h + h'}\frac{\partial \bar{\zeta}_2}{\partial x} - \frac{h'}{h}\tau_x^S, \tag{7.8.62}$$

$$\frac{\partial \bar{V}_2}{\partial t} + f\bar{U}_2 = -g\frac{\epsilon h h'}{h + h'}\frac{\partial \bar{\zeta}_2}{\partial y} - \frac{h'}{h}\tau_x^S, \tag{7.8.63}$$

Note that the effective gravity

$$g\beta_2 = g\frac{\epsilon h'}{h + h'} \tag{7.8.64}$$

is much smaller than g. Moreover, the effective wind stresses are

$$-\frac{h'}{h}\tau_x^S, \quad -\frac{h'}{h}\tau_y^S \tag{7.8.65}$$

hence are oppposite to the surface wind stresses.

7.8.4 Free modes without wind

Let us consider free waves in the absence of wind forcing $((\tau_x)_k, (\tau_y)_k) = 0)$. It is then possible to consider one mode and assume that the other modes to be zero. In particular we shall examine (eigen, or natural) modes of sinusoidal wave form

$$(\zeta_k, U_k, V_k) \propto e^{ik_x x + ik_y y - i\omega t} \tag{7.8.66}$$

Barotropic (surface) mode:

Let us focus on mode 1 alone and assume mode 2 to be absent, i.e.,

$$\bar{U}_2 = \bar{V}_2 = \bar{\zeta}_2 = 0 \tag{7.8.67}$$

then

$$\begin{aligned}
\bar{U}_2 &= -(h'/h)U + U' = 0 \\
\bar{V}_2 &= -(h'/h)V + V' = 0 \\
\bar{\zeta}_2 &= -(h'/h)\zeta + (1 + h'/h)\,\zeta' = 0,
\end{aligned} \tag{7.8.68}$$

hence

$$U' = U\frac{h'}{h} \qquad V' = V\frac{h'}{h}, \qquad \zeta' = \frac{h'}{h+h'}\zeta. \tag{7.8.69}$$

Again the depth-averaged velocities in both layers are in phase and are numerically the same. The interface displacement is in phase with and smaller than the free surface displacement in proportion to the vertical distance above the seabed.

Moreover, (7.8.55) to (7.8.57) become for the upper layer:

$$\frac{\partial}{\partial t}\left(\frac{h+h'}{h}U\right) - f\left(\frac{h+h'}{h}V\right) = -g(h+h')\frac{\partial \zeta}{\partial x} \tag{7.8.70}$$

$$\frac{\partial}{\partial t}\left(\frac{h+h'}{h}V\right) + f\left(\frac{h+h'}{h}U\right) = -g(h+h')\frac{\partial \zeta}{\partial y} \tag{7.8.71}$$

$$\frac{\partial \zeta}{\partial t} + \frac{h+h'}{h}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0. \tag{7.8.72}$$

or

$$\frac{\partial U}{\partial t} - fV = -gh\frac{\partial \zeta}{\partial x} \tag{7.8.73}$$

$$\frac{\partial V}{\partial t} + fU = -gh\frac{\partial \zeta}{\partial y} \tag{7.8.74}$$

$$\frac{\partial \zeta}{\partial t} + \frac{h+h'}{h}\left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) = 0. \tag{7.8.75}$$

It is easy to get the equations for the lower layer.

$$\frac{\partial U'}{\partial t} - fV' = -gh'\frac{\partial \zeta'}{\partial x} \tag{7.8.76}$$

$$\frac{\partial V'}{\partial t} + fU' = -gh'\frac{\partial \zeta'}{\partial y} \tag{7.8.77}$$

$$\frac{\partial \zeta'}{\partial t} + \frac{h+h'}{h'}\left(\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y}\right) = 0. \tag{7.8.78}$$

Consider sinusidodal waves and the equations for the upper layer,

$$-i\omega U - fV = -ghik_x\zeta$$

$$-i\omega V - fU = -ghik_y\zeta$$

$$-i\omega\zeta + \frac{h+h'}{h}(ik_xU + ik_yV) = 0$$

For nontrivbil solution th eeigenvalue condition is easily found to be

$$\omega^2 = f^2 + g(h+h')k^2 \tag{7.8.79}$$

which can be written as

$$\omega\sqrt{\frac{h+h'}{g}} = \pm\left[\left(f\sqrt{\frac{h+h'}{g}}\right)^2 + k^2(h+h')^2\right]^{1/2} \tag{7.8.80}$$

This is the dispersion relation for the barotropic (surface wave) mode, and is plotted in Figure 7.8.2

Baroclinic (internal wave) mode

Now consider the unforced Mode 2 and assume Mode 1 to be absent,

$$\bar{U}_1 = \bar{V}_1 = \bar{\zeta}_1 = 0 \tag{7.8.81}$$

Since

$$a_1 = 1 \tag{7.8.82}$$

we have, from (7.8.32) to (7.8.34)

$$U + U' = 0, \quad V + V' = 0. \tag{7.8.83}$$

$$\zeta = -\frac{\epsilon h'}{h+h'}\zeta' \tag{7.8.84}$$

Note that the horizontal velocities in the two layers are oppositive in phase, so are the free surface and the interface. The last result implies the free surface behaves like a rigid lid; the interface oscillates with much greater vigor.

Using these results, we get

$$U_2 = -\frac{h'}{h}U + U' = \left(1 + \frac{h'}{h}\right)U' \tag{7.8.85}$$

$$V_2 = \left(1 + \frac{h'}{h}\right)V' \tag{7.8.86}$$

$$\zeta_2 = \left(1 + \frac{h'}{h}\right)\zeta' \tag{7.8.87}$$

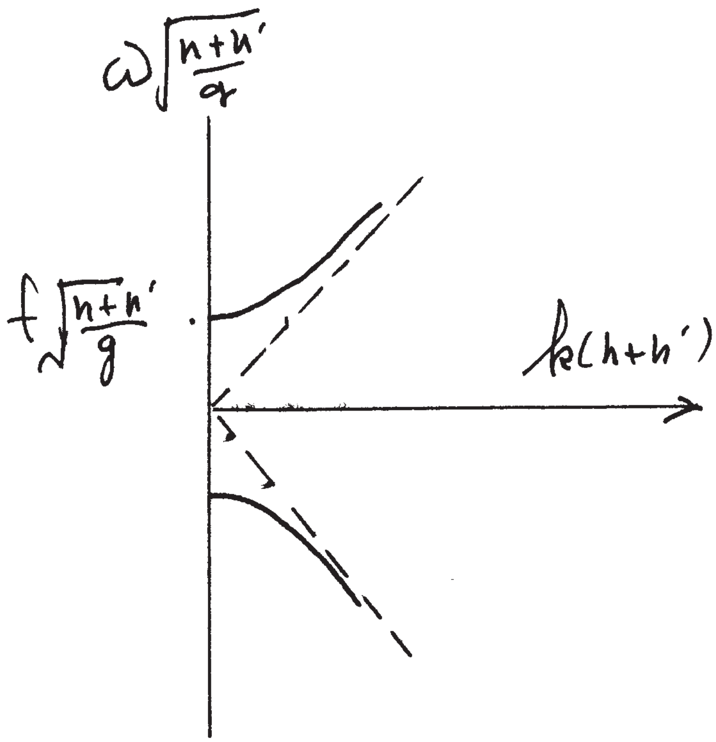

Figure 7.8.2: Dispersion relation for the barotropic mode in a two-layered sea of contant depth.

Putting all this in (7.8.35) to (7.8.37), we get

$$\frac{\partial}{\partial t}\left(1+\frac{h'}{h}\right)U' - f\left(1+\frac{h'}{h}\right)V'$$
$$= -g\frac{\epsilon h'h}{h+h'}\frac{h+h'}{h}\frac{\partial \zeta'}{\partial x}. \tag{7.8.88}$$

Therefore, for the lower layer,

$$\frac{\partial U'}{\partial t} - fV' = -g\frac{\epsilon h'h}{h+h'}\frac{\partial \zeta'}{\partial x}. \tag{7.8.89}$$

Similarly,

$$\frac{\partial V'}{\partial t} + fU' = -\frac{g\epsilon h'h}{h+h'}\frac{\partial \zeta'}{\partial y} \tag{7.8.90}$$

Also from (7.8.32),

$$\left(1+\frac{h'}{h}\right)\frac{\partial \zeta'}{\partial t} + \left(1+\frac{h'}{h}\right)\left(\frac{\partial U'}{\partial x}+\frac{\partial V'}{\partial y}\right) = 0. \tag{7.8.91}$$

hence,

$$\frac{\partial \zeta'}{\partial t} + \frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = 0. \tag{7.8.92}$$

We can also write the equations for the upper layer,

$$\frac{\partial U}{\partial t} - fV = g\frac{\epsilon h' h}{h + h'}\frac{\partial \zeta'}{\partial x}. \tag{7.8.93}$$

,

$$\frac{\partial V}{\partial t} + fU = \frac{g\epsilon h' h}{h + h'}\frac{\partial \zeta'}{\partial y} \tag{7.8.94}$$

and

$$\frac{\partial \zeta'}{\partial t} - \frac{\partial U}{\partial x} - \frac{\partial V}{\partial y} = 0. \tag{7.8.95}$$

Note the sign changes in the momentum and mass conservation equations.

For sinusoical waves we have

$$-i\omega U - fV = g' h i k_x \zeta'$$

$$-i\omega V - fU = g' h i k_y \zeta'$$

$$-i\omega \zeta' - (ik_x U + ik_y V) = 0$$

with

$$g' = g\epsilon \frac{h'}{h + h'} \tag{7.8.96}$$

The eigenvalue condition acan be easily found to be

$$\omega^2 = f^2 + g' h k^2 \tag{7.8.97}$$

which can be written as

$$\omega\sqrt{\frac{h}{g'}} = \pm\left(f\sqrt{\frac{h}{g'}} + k^2 h^2\right)^{1/2} \tag{7.8.98}$$

This dispersion relation is similar to Figure 7.8.2. Since the reduced gravity is very small. the baroclinic mode has a much longer natural period.

Mode 2 is called the barolinic (internal wave) mode.