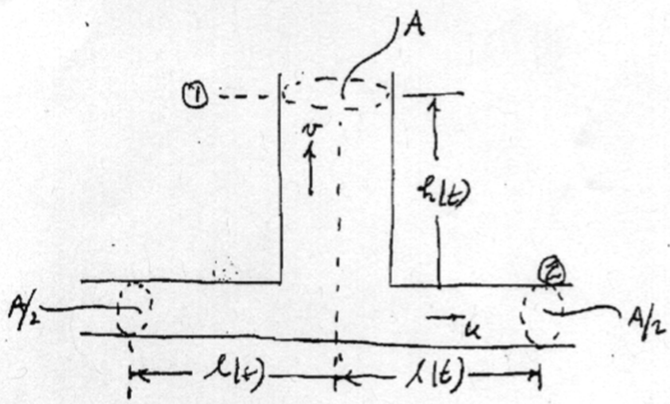
Problem Set No. 1 ; Problem 1 Solution

Draining of a T tube



Mass conservation $\Rightarrow$

$$A \cdot h(t) + \frac{A}{2} \cdot 2l = A \cdot H$$

$$\Rightarrow \quad \underline{\underline{h + l = H}}$$

$$\underbrace{\frac{dh}{dt}}_{v} = -\underbrace{\frac{dl}{dt}}_{-u} \Rightarrow v = -u$$

1./ Use ~~conservation of energy~~ $\quad \frac{d}{dt}(T+V) = 0$

$$\frac{H}{2} \cdot (\rho g A H) = \frac{h}{2} \cdot (\rho g A h) + \frac{1}{2}(\rho \cdot A h)\left(\frac{dh}{dt}\right)^2 + 2 \cdot \frac{1}{2}\left(\rho \cdot \frac{A}{2} \cdot l\right)\left(\frac{dl}{dt}\right)^2$$

$$\Rightarrow \quad \frac{1}{2}\rho g A H^2 = \frac{1}{2}\rho g A h^2 + \frac{1}{2}\rho A H\left(\frac{dh}{dt}\right)^2$$

$$\Rightarrow \quad \frac{dh}{dt} = \pm \sqrt{\frac{g}{H}}\sqrt{H^2 - h^2}$$ take (−) sign, $dh/dt < 0$

$$\therefore \quad \underline{\underline{h = H \cos \sqrt{g/H}\, t}}$$

2./ Use momentum equations directly

In horizontal portion : $\frac{\partial u}{\partial x} = 0, \quad u = u(t) \quad : \quad \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

$$\Rightarrow \quad \frac{p'}{\rho} = (l - x)\frac{\partial u}{\partial t}$$

In vertical portion : $\frac{\partial v}{\partial y} = 0, \quad v = v(t) \quad : \quad \frac{\partial v}{\partial t} = -g - \frac{1}{\rho}\frac{\partial p}{\partial y}$

$$\Rightarrow \quad \frac{p}{\rho} = (h - y)\left(\frac{\partial v}{\partial t} + g\right)$$

But $\quad p = p'$ when $x = y = 0 \Rightarrow$

$$l \frac{\partial u}{\partial t} = h\left(\frac{\partial v}{\partial t} + g\right)$$

Also, $\quad u = -\frac{dh}{dt}, \quad v = \frac{dk}{dt}, \quad l + h = H$

Hence, $\quad H\frac{d^2h}{dt^2} + gh = 0 \quad \Rightarrow \quad \underline{h = H\cos\sqrt{g/H}\;t}$

3.| Integrate momentum equation along the tube

$$\rho\left(\frac{\partial \underline{q}}{\partial t} + \underline{q}\cdot\nabla\underline{q}\right) = -\nabla p - \rho g\,\hat{e}_y$$

$$\underline{q}\cdot\nabla\underline{q} = \nabla\left(\tfrac{1}{2}\underline{q}^2\right) - \cancelto{0}{\underline{q}\times(\nabla\times\underline{q})}$$

$$\Rightarrow \quad \frac{\partial \underline{q}}{\partial t} + \left(\frac{1}{\rho}\nabla p + \nabla\tfrac{1}{2}\underline{q}^2 + g\,\hat{e}_y\right) = 0$$

$$\Rightarrow \quad \frac{\partial \underline{q}}{\partial t} + \nabla\left(\tfrac{1}{2}\underline{q}^2 + \frac{p}{\rho} + gy\right) = 0$$

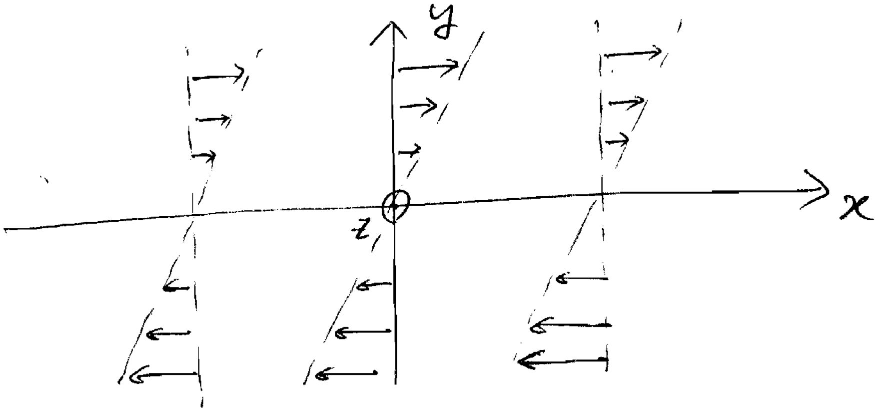
Integrate along the tube from ① → ② : $\displaystyle\int_{①}^{②} \frac{\partial \underline{q}}{\partial t}\cdot d\underline{s} + \left(\tfrac{1}{2}\underline{q}^2 + \frac{p}{\rho} + gy\right)\Big|_{①}^{②} =$

$$(-h)\left(+\frac{\partial v}{\partial t}\right) + l\left(\frac{\partial u}{\partial t}\right) + \left\{\tfrac{1}{2}q^2 + \frac{p}{\rho} + gy\right\}\Big|_{①}^{②} = 0$$

$$\Rightarrow \quad -H\frac{d^2h}{dt^2} - gh = 0 \quad \Rightarrow \quad \underline{\frac{d^2h}{dt^2} + \frac{g}{H}h = 0.}$$

Problem set No 1 : Problem 2 solution

parallel shear flow $\underset{\sim}{q} = (sy, 0, 0)$



(a) Vorticity $\underset{\sim}{\zeta} = \nabla \times \underset{\sim}{q} = (0, 0, -s)$

$$\Rightarrow [\Omega] = \left[\Omega_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right)\right] = \left[\frac{1}{2}\underset{\sim}{\zeta}\right] = \begin{bmatrix} 0 & s/2 & 0 \\ -s/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rate of strain

$$\Rightarrow [E] = \left[e_{ij} = \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)\right] = \begin{bmatrix} 0 & s/2 & 0 \\ s/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $d\underset{\sim}{x} = (dx, dy, dz)$ is along a principal direction,

then $[E]\{d\underset{\sim}{x}\} = e\{d\underset{\sim}{x}\}$

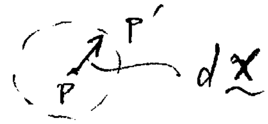
$$\rightarrow \begin{bmatrix} -e & s/2 & 0 \\ s/2 & -e & 0 \\ 0 & 0 & -e \end{bmatrix} \begin{Bmatrix} dx \\ dy \\ dz \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\rightarrow \det [\quad] = 0 \rightarrow e = s/2, -s/2, 0$

$e_1 = s/2$; $dy/dx = 1$ ($\pi/4$ direction)

$e_2 = -s/2$; $dy/dx = -1$ ($-\pi/4$ direction)

(b) $[A] = [E] + [\Omega]$

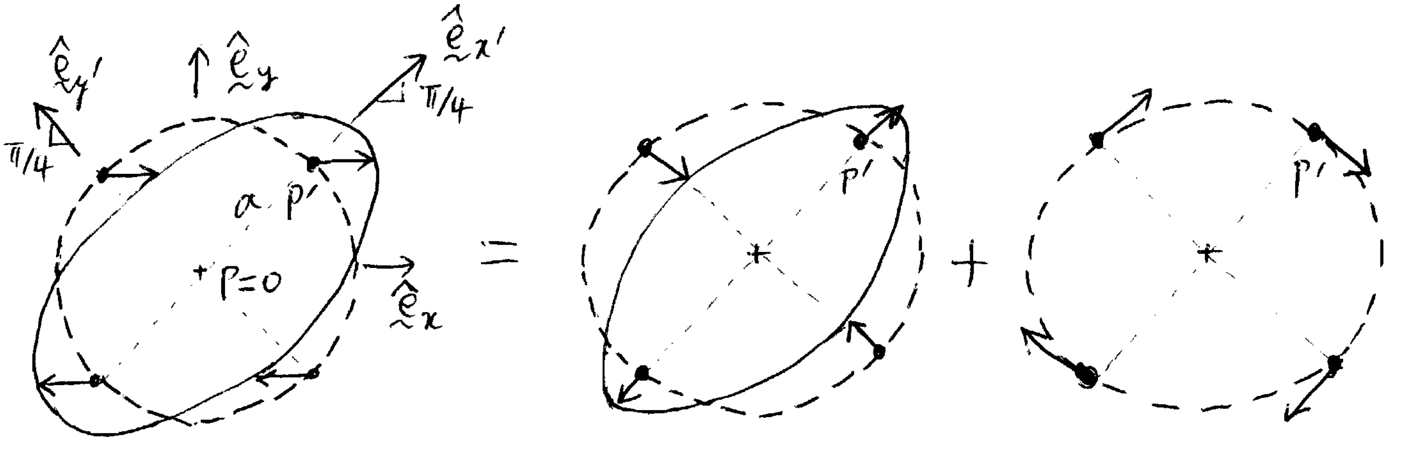


$$\Rightarrow \underset{\sim}{q}_{P'} = \underset{\sim}{q}_P + [A]\{d\underset{\sim}{X}\}$$

$$= \underset{\sim}{q}_P + [E]\{d\underset{\sim}{X}\} + [\Omega]\{d\underset{\sim}{X}\}$$

$$= \underbrace{\begin{Bmatrix} sy \\ 0 \\ 0 \end{Bmatrix}}_{\text{translation}} + \underbrace{\begin{bmatrix} 0 & s/2 & 0 \\ s/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\{d\underset{\sim}{X}\}}_{\text{strain}} + \underbrace{\begin{bmatrix} 0 & -s/2 & 0 \\ s/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\{d\underset{\sim}{X}\}}_{\text{rotation}}$$

e.g., A circular fluid element centered at the origin in 2D (radius $= a$)



shear flow — strain — rotation

At P',

$$(sy)\,dt\,\underset{\sim}{\hat{e}}_x = \underbrace{\frac{s}{2} a\,dt\,\underset{\sim}{\hat{e}}_{x'} + -\frac{s}{2} a\,dt\,\underset{\sim}{\hat{e}}_{y'}}$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$s\frac{a}{\sqrt{2}}dt\,\underset{\sim}{\hat{e}}_x \qquad\qquad \frac{s}{2}a\,dt\cdot(\underset{\sim}{\hat{e}}_{x'} - \underset{\sim}{\hat{e}}_{y'})$$

$$\| \qquad\qquad\qquad\qquad \|$$

$$\frac{sa}{2}dt\cdot\sqrt{2}\,\underset{\sim}{\hat{e}}_x \qquad\qquad \frac{s}{2}a\,dt\cdot\sqrt{2}\,\underset{\sim}{\hat{e}}_x$$

Problem Set No. 1 ; Problem 3 Solution

$$\rho \frac{D}{Dt} q_i = \rho f_i - \frac{\partial p}{\partial x_i} + \frac{\partial \tau_{ij}}{\partial x_j}$$

1.) Take dot product with $\underset{\sim}{q}$:

$$\rho \frac{D}{Dt}\left(\frac{1}{2} q_i q_i\right) = \rho f_i q_i - q_i \frac{\partial p}{\partial x_i} + q_i \frac{\partial \tau_{ij}}{\partial x_j}$$

$$= \rho f_i q_i - \frac{\partial}{\partial x_i}\left(p q_i\right) + \cancel{p \frac{\partial q_i}{\partial x_i}}^{\,0} + \frac{\partial}{\partial x_j}\left(q_i \tau_{ij}\right) - \tau_{ij} \frac{\partial q_i}{\partial x_j}$$

$$= \rho f_i q_i + \frac{\partial}{\partial x_j}\left(q_i \tau_{ij}\right) + \frac{\partial}{\partial x_j}\left(-p \delta_{ij} q_i\right) - \tau_{ij} \frac{\partial q_i}{\partial x_j}$$

$$\therefore \quad \rho \frac{D}{Dt}\left(\frac{1}{2} q_i q_i\right) = \rho f_i q_i + \frac{\partial}{\partial x_j}\left(\sigma_{ij} \cdot q_i\right) - \tau_{ij} \frac{\partial q_i}{\partial x_j}$$

2/ Rate of change of KE = rate of work done by body force

+ work done by surface stress

+ 'dissipation' rate

3/ Dissipation $\quad \Phi = \tau_{ij} \dfrac{\partial q_i}{\partial x_j} = \mu\left(\dfrac{\partial q_i}{\partial x_j} + \dfrac{\partial q_j}{\partial x_i}\right) \dfrac{\partial q_i}{\partial x_j}$

$$= \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)\left\{\frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} - \frac{\partial q_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)\right\}$$

$$= \frac{1}{2}\mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)^2 > 0 \quad \text{always!}$$

4./ Within a fixed container with solid walls,

$$\underbrace{\rho \frac{D}{Dt}\left(\tfrac{1}{2} q_i q_i\right)} = \frac{\partial}{\partial x_j}\left(\sigma_{ij} q_i\right) - \Phi$$

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2} q_i q_i\right) + q_j \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} q_i q_i\right) = \frac{\partial}{\partial t}\left(\tfrac{1}{2} q_i q_i\right) + \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} q_i q_i q_j\right)$$

Integrating over the volume V,

$$\rho \frac{\partial}{\partial t} \iiint_V \tfrac{1}{2} q_i q_i \, dV + \iiint_V dV \frac{\partial}{\partial x_j}\left\{\left(\tfrac{1}{2}\rho\, q_i q_i q_j - \sigma_{ij} q_i\right)\right\}$$

$$= - \iiint_V \Phi \, dV$$

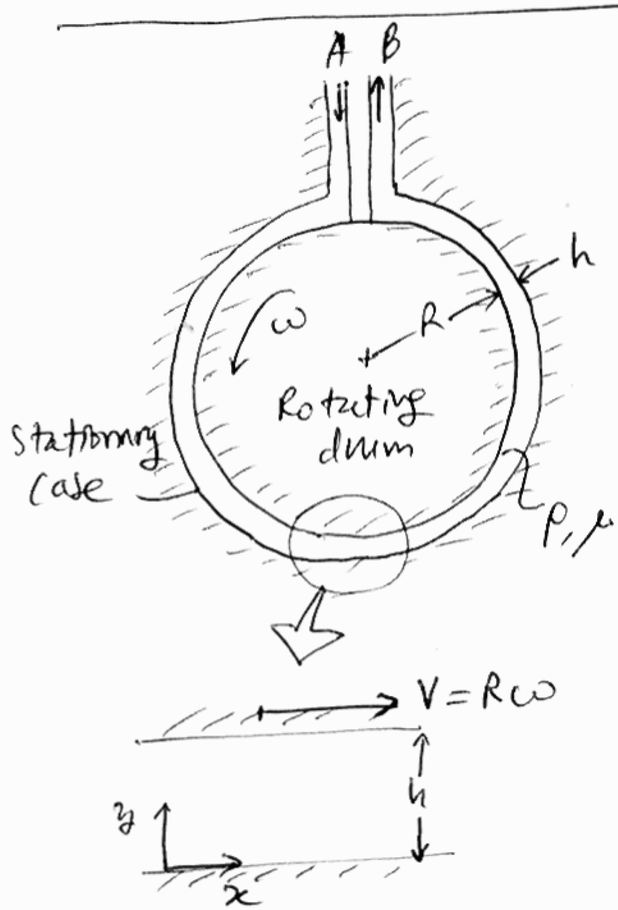
By Gauss thm,

$$\iiint_V dV \frac{\partial}{\partial x_j}\left\{ \quad \right\} = \iint_S dS\, n_j \left\{ \quad \right\}_j =$$

$$= \iint_S dS\, n_j \left(\tfrac{1}{2}\rho\, q_i q_i q_j - \sigma_{ij} q_i\right) = 0 \text{ by no slip}$$

$$\therefore \quad \rho \frac{\partial}{\partial t} \iiint_V \tfrac{1}{2} q_i q_i \, dV = - \iiint_V \Phi \, dV$$

Problem set No 1; Problem 4 solution



$$\begin{cases} P_A - P_B = \Delta P > 0 \\ \text{length of annulus} = l \\ l \gg h \end{cases}$$

Assumption

- Incompressible
- Steady
- No body force
- Fully-developed (No entry & exit effects)
- 2D

$\Rightarrow$ ((Mass Conservation))

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{u}{l} \sim \frac{v}{h} \Rightarrow \frac{v}{u} \sim \frac{h}{l} \ll 1$$

$$\Rightarrow u \gg v \Rightarrow \underline{\frac{\partial u}{\partial x} = 0} \Rightarrow \underline{\underline{u = u(y)}}$$

((x-Mom))

$$\rho\left(\cancel{\frac{\partial u}{\partial t}} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}}\right) = -\frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right)$$

((y-Mom))

$$\rho\left(\cancel{\frac{\partial v}{\partial t}} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}}\right) = -\frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}} + \cancel{\frac{\partial^2 v}{\partial y^2}}\right)$$

$$\Rightarrow \frac{\partial P}{\partial y} = 0 \Rightarrow \underline{\underline{P = P(x)}}$$

Finally, from x-Mom eq.,

$$\underbrace{\frac{dP}{dx}}_{\text{function of } x} = \mu \underbrace{\frac{d^2 u}{dy^2}}_{\text{function of } y} = \text{constant} = -\frac{\Delta P}{l}$$

((B.C.)) $\begin{cases} u(y=0) = 0 \\ u(y=h) = V = R\omega \end{cases}$

$$\Rightarrow \underset{\text{velocity}}{u(y)} = \underbrace{\frac{V}{h} y}_{\text{Couette}} + \underbrace{\frac{1}{2\mu}\Big(\overbrace{-\frac{dP}{dx}}^{= \frac{\Delta P}{l}}\Big)(hy - y^2)}_{\text{Poiseuille}}$$

(a)

$$\to \underset{\underline{\underline{\text{flow rate } [\frac{m^3/s}{m}]}}}{Q} = \int_0^h u(y)\,dy = \underbrace{\frac{V}{2} h}_{\text{Couette}} + \underbrace{\frac{1}{12\mu}\Big(\overbrace{-\frac{dP}{dx}}^{= \frac{\Delta P}{l}}\Big) h^3}_{\text{Poiseuille}}$$

<u>Shear stress in x-direction</u>

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right)$$

Power delivered to the drum by the fluid

$= \text{Power-out}$

$= \int_S \left[-\tau_{xy} \cdot u \right]_{y=h} ds$

$= \int_0^l \left[-\mu \frac{du}{dy} \cdot u \right]_{y=h} dx$

$= -\mu l \left[\frac{du}{dy} \cdot u \right]_{y=h}$

$= -\mu \frac{l V^2}{h} + \frac{h}{2} \left(-\frac{dP}{dx} \right) l \cdot V$

$= \frac{hV}{2} \cdot \Delta P - \mu \frac{l V^2}{h}$

$= \left[Q - \frac{h^3 \Delta P}{12 l \mu} \right] \Delta P - \mu \frac{l V^2}{h}$

$= \underbrace{Q \cdot \Delta P}_{\text{Power-in}} - \underbrace{\left[\frac{h^3 (\Delta P)^2}{12 \mu l} + \mu \frac{l V^2}{h} \right]}_{\text{Total viscous dissipation rate}}$ unit [W/m]

(b)

Viscous dissipation in 2-D

$$\Phi = \tau_{ij}\frac{\partial q_i}{\partial x_j} = \mu\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)\frac{\partial q_i}{\partial x_j}$$

$$= 2\mu\left[\cancel{\left(\frac{\partial u}{\partial x}\right)^2} + \cancel{\left(\frac{\partial v}{\partial y}\right)^2}\right] + \mu\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial u}{\partial y}\right)^2$$

$$= \mu\left(\frac{du}{dy}\right)^2 = \mu\left[\frac{V}{h} + \frac{1}{2\mu}\frac{\Delta P}{l}(h-2y)\right]^2, \text{ where } V = R\omega$$

⇒ **Total viscous dissipation rate**

$$= \int_V \Phi \, dV$$

$$= \int_0^h \int_0^l \mu\left(\frac{du}{dy}\right)^2 dx\, dy = \mu l \underline{\int_0^h \left(\frac{du}{dy}\right)^2 dy}$$

can do direct calculation

$$= \mu l \int_0^h \left[\frac{d}{dy}\left(u\frac{du}{dy}\right) - u\underbrace{\frac{d^2u}{dy^2}}_{=\frac{1}{\mu}\frac{dP}{dx}}\right] dy$$

$$= \mu l\left[u\frac{du}{dy}\right]_{y=h} - \cancel{\mu l\left[u\frac{du}{dy}\right]_{y=0}} + \cancel{l}\cdot\frac{\Delta P}{\cancel{l}}\cdot\underbrace{\int_0^h u\, dy}_{=Q}$$

$$\because u(y=0) = 0$$

$$= \Delta P \cdot Q - \left\{-\mu l \cdot \left[u\frac{du}{dy}\right]_{y=h}\right\}$$

$$= (\text{Power-in}) - (\text{Power-out}) = \mu\frac{lV^2}{h} + \frac{h^3(\Delta P)^2}{12\mu l}$$