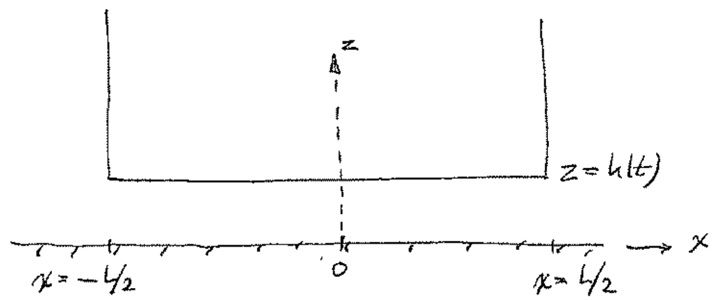
Problem Set No. 2 ; Problem 1 Solution

Separating a flat object from a smooth surface



* Mass conservation : $\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$ $\qquad \underline{\underline{h/L \ll 1}}$

$u \sim U, \quad w \sim W \quad \Rightarrow \quad W \sim U\frac{h}{L} \sim \frac{h}{T} \quad \Rightarrow \quad T \sim \frac{L}{U}$

* x-momentum eqn : $\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}p_x + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial z^2}\right)$

typical inertia terms $\frac{\partial u}{\partial t} \sim \frac{U^2}{L} \sim u\frac{\partial u}{\partial x}, w\frac{\partial u}{\partial z}$

typical viscous term $\mu\frac{\partial^2 u}{\partial z^2} \sim \mu\frac{U}{h^2}$

Neglect inertia terms if $\frac{\mu}{\rho}\frac{U}{h^2} \gg \frac{U^2}{L} \Rightarrow \underline{\underline{\frac{h^2 U}{L\nu} \ll 1}}$

i.e., slow, viscous flow in a narrow gap.

$\therefore \quad \underline{\underline{p_x = \mu\frac{\partial^2 u}{\partial z^2}}} \quad \left(\Rightarrow p \sim \mu\frac{UL}{h^2}\right)$

* z-momentum eqn Neglecting inertia terms again,

$$p_z = \mu\frac{\partial^2 w}{\partial z^2}$$

$p_z \sim \frac{p}{h} \sim \frac{\mu UL}{h^2}\frac{1}{h}, \quad \mu\frac{\partial^2 w}{\partial z^2} \sim \mu\frac{W}{h^2} \sim \mu\frac{Uh/L}{h^2} \sim \mu\frac{U}{hL}$

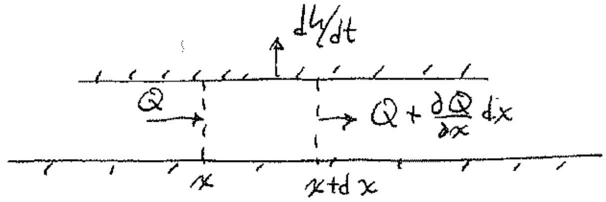
$p_z / \mu\frac{\partial^2 w}{\partial z^2} \sim L^2/h^2 \gg 1 \quad \Rightarrow \quad \underline{\underline{p_z \approx 0}} \quad$ i.e. $\quad \underline{\underline{p \approx p(x)}}$

* Since $p_x = G(x)$, may solve for $u(z,x)$:

$$\underline{u(z,x) = \frac{G(x)}{2\mu}\, z(z-h)} \qquad (u = 0 \text{ at } z = 0, h)$$

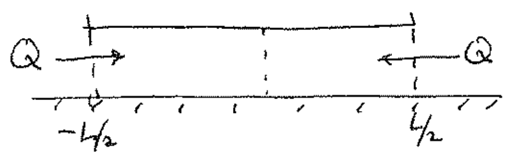
so locally we have Poiseuille flow

* Impose mass conservation:



$$\frac{\partial}{\partial t}(\rho h\, dx) = -\frac{\partial Q}{\partial x}dx \Rightarrow \rho\frac{dh}{dt} = -\frac{\partial Q}{\partial x} \Rightarrow$$

$$\underline{Q = -\rho x \frac{dh}{dt}}$$



$$Q\left(-\tfrac{L}{2}\right) = \rho \frac{L}{2}\frac{dh}{dt}$$
$$Q\left(\tfrac{L}{2}\right) = -\rho \frac{L}{2}\frac{dh}{dt}$$

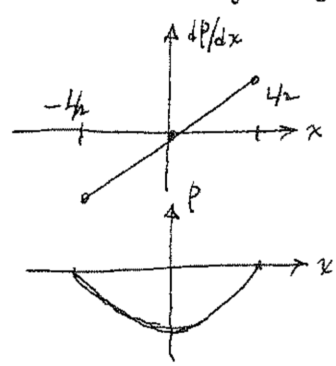
* Also, from $u(z,x) = \dfrac{dP/dx}{2\mu}\, z(z-h)$,

$$Q = \rho\int_0^h u\, dz = \rho\frac{dP/dx}{2\mu}\int_0^h (z^2 - zh)\, dz = \underline{-\rho\frac{dP/dx}{12\mu}h^3 = Q}$$

$$\therefore \quad -\rho\frac{dP/dx}{12\mu}h^3 = -\rho x\frac{dh}{dt} \Rightarrow \frac{dP}{dx} = \frac{12\mu}{h^3}x\frac{dh}{dt}$$

So, taking $p = 0$ at $x = \pm L/2$,

$$\underline{p(x) = \frac{12\mu}{h^3}\frac{dh}{dt}\left(\frac{x^2}{2} - \frac{L^2}{8}\right)}$$



* Total force (directed downwards)

$$F = \frac{12\mu}{h^3}\frac{dh}{dt}\int_{-L/2}^{L/2}\left(\frac{x^2}{2} - \frac{L^2}{8}\right)dx$$

$$= \frac{12\mu}{h^3}\frac{dh}{dt}\left(\frac{x^3}{6} - \frac{L^2 x}{8}\right)\Bigg|_{-L/2}^{L/2} = \frac{12\mu}{h^3}\frac{dh}{dt}\left(-\frac{L^3}{8} + \frac{L^3}{24}\right)$$

$$\Rightarrow \quad F = \mu\frac{L^3}{h^3}\frac{dh}{dt} \quad \Rightarrow \quad \underline{\frac{dh}{dt} = \frac{Fh^3}{\mu L^3}}$$

* Suppose the block falls under its own weight

$$F = -W$$

Hence,

$$\frac{dh}{dt} = -\frac{Wh^3}{\mu L^3}$$

Integrating,

$$\int_{h_0}^{h}\frac{dh}{h^3} = -\frac{W}{\mu L^3}\int_0^t dt$$

$$\Rightarrow \quad \frac{1}{2}\left(\frac{1}{h_0^2} - \frac{1}{h^2}\right) = -\frac{W}{\mu L^3}t \quad \Rightarrow$$

$$h^2 = \frac{1}{\frac{1}{h_0^2} + \frac{2W}{\mu L^3}t} \quad \Rightarrow \quad \underline{h(t) = \frac{1}{\sqrt{\frac{1}{h_0^2} + \frac{2W}{\mu L^3}t}}}$$

Problem Set No. 2 ; Problem 2 Solution

Spreading of lava on a horizontal plane

* From lubrication approximation, local radial velocity

$$u(r,z,t) = -\frac{g}{2\nu}\frac{\partial h}{\partial r} z(2h-z)$$

Conservation of mass requires that

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\int_0^h u\,dz\right) = 0$$

But $\int_0^h u\,dz = -\frac{g}{2\nu}\frac{\partial h}{\partial r}\int_0^h z(2h-z)\,dz =$

$$= -\frac{g}{2\nu}\frac{\partial h}{\partial r}\left(h z^2\Big|_0^h - \frac{z^3}{3}\Big|_0^h\right) = -\frac{g}{3\nu}h^3\frac{\partial h}{\partial r}$$

$\therefore$ evolution equation for $h(r,t)$ is

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right)$$

B.C.

$$\left.\begin{aligned} h(R(t),t) &= 0 \\ \partial h/\partial r\,(0,t) &= 0 \end{aligned}\right\}$$

* Consider $\frac{\partial}{\partial t}\int_0^{R(t)} r h(r,t)\,dr = \int_0^{R(t)} r\frac{\partial h}{\partial t}\,dr + \frac{\partial R}{\partial t} R\,\cancel{h(R,t)}^{\,0}$

$$= \int_0^R \cancel{r}\frac{g}{3\nu}\frac{1}{\cancel{r}}\frac{\partial}{\partial r}\left(r h^3\frac{\partial h}{\partial r}\right)dr = \frac{g}{3\nu}\cancel{R h^3 \frac{\partial h}{\partial r}\Big|_R}^{\,0} = 0$$

$$\therefore \int_0^R r h(r,t)\,dr = \text{const}$$

Let $2\pi\int_0^{R(t)} r h(r,t)\,dr = V$

* Similarity solution

$$h = \lambda^a h', \quad r = \lambda^b r', \quad t = \lambda^c t'$$

$$\frac{\partial h}{\partial t} = \frac{g}{3\nu}\frac{1}{r}\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r}\right) \Rightarrow \lambda^{a-c}\frac{\partial h'}{\partial t'} = \frac{g}{3\nu}\lambda^{4a-2b}\frac{1}{r'}\frac{\partial}{\partial r'}\left(r' h'^3 \frac{\partial h'}{\partial r'}\right)$$

$$2\pi \int_0^R r h(r,t)\,dr = V \Rightarrow 2\pi \lambda^{2b+a}\int_0^{R'} r' h(r',t')\,dr' = V$$

Require
$$\left.\begin{aligned} a-c &= 4a-2b \\ 2b+a &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} b &= -\frac{a}{2} \\ c &= -4a \end{aligned}$$

i.e., $h = \lambda^a h', \quad r = \lambda^{-\frac{a}{2}} r', \quad t = \lambda^{-4a} t'$

Note that $$\frac{r}{t^{1/8}} = \cancel{\lambda^{-\frac{a}{2}+\frac{a}{2}}}\,1\; \frac{r'}{t'^{1/8}}$$

Therefore, the similarity variable is $\eta = A\dfrac{r}{t^{1/8}}$

and $t^{1/4} h = \cancel{\lambda^{-a}\lambda^{a}}\,1\; h'$ $\quad Bf(\eta) = t^{1/4} h$

So, we write

$$h = \frac{B f(\eta)}{t^{1/4}}, \quad \eta = A\frac{r}{t^{1/8}}$$

$$\frac{\partial h}{\partial r} = \frac{B}{t^{1/4}}\frac{A}{t^{1/8}} f'(\eta), \quad \frac{\partial h}{\partial t} = -\frac{1}{8}\frac{B}{t^{5/4}}\left(2f(\eta) + \eta f'(\eta)\right)$$

Manipulation

$$\frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r} \right) = \frac{\partial}{\partial r}\left(r \frac{B^3}{t^{3/4}} f^3 \frac{BA}{t^{3/8}} f'(\eta) \right)$$

$$= \frac{B^4 A}{t^{9/8}} f^3 f' + \frac{r B^4 A^2}{t^{5/4}} \frac{d}{d\eta}\left(f^3 f'(\eta)\right)$$

$$\left(\mathbb{C} \equiv \frac{g}{3\nu} \right)$$

$$\therefore \quad \frac{\partial h}{\partial t} = \mathbb{C} \frac{1}{r} \frac{\partial}{\partial r}\left(r h^3 \frac{\partial h}{\partial r} \right) \Rightarrow$$

$$-\frac{1}{8} \frac{B}{t^{5/4}} \left(2 f(\eta) + \eta f'(\eta) \right) = \mathbb{C} \left\{ \frac{B^4 A}{t^{9/8}} \frac{1}{r} f^3 f' + \frac{B^4 A^2}{t^{5/4}} \frac{d}{d\eta}\left(f^3 f'\right) \right\}$$

Since $r = \dfrac{\eta t^{1/8}}{A}$,

we have

$$-\frac{1}{8} \cancel{B} \left(2f + \eta f' \right) = \mathbb{C} \left\{ \frac{B^{\cancel{4}\,3} A^2}{\eta} f^3 f' + B^{\cancel{4}\,3} A^2 \frac{d}{d\eta}\left(f^3 f'\right) \right\}$$

Also, $\quad 2\pi \displaystyle\int_0^R r\, h(r,t)\, dr = V \Rightarrow$

$$2\pi \left(\frac{t^{1/8}}{A} \right)^2 \frac{B}{t^{1/4}} \int_0^{\eta_R} \eta f(\eta)\, d\eta = V \Rightarrow 2\pi \frac{B}{A^2} \int_0^{\eta_R} \eta f(\eta)\, d\eta = V$$

* Choose (for simplicity) $\quad B^3 A^2 = \dfrac{1}{8\mathbb{C}} \Rightarrow A^2 = \dfrac{1}{8\mathbb{C} B^3}$

$$2\pi \frac{B}{A^2} \frac{1}{V} = 2\pi \frac{B^4 \mathbb{C}}{V} = 1 \Rightarrow$$

$$B^4 = \frac{V}{2\pi \mathbb{C}}$$

* <u>Solution</u>

$$-2f - \eta \frac{df}{d\eta} = \frac{1}{\eta} f^3 \frac{df}{d\eta} + \frac{d}{d\eta}\left(f^3 \frac{df}{d\eta}\right)$$

$$\Rightarrow \quad 2\eta f + \eta^2 \frac{df}{d\eta} + f^3 \frac{df}{d\eta} + \eta \frac{d}{d\eta}\left(f^3 \frac{df}{d\eta}\right) = 0$$

$$\Rightarrow \quad \frac{d}{d\eta}\left\{ \eta^2 f + \eta f^3 \frac{df}{d\eta} \right\} = 0$$

$$\Rightarrow \quad \eta^2 f + \eta f^3 \frac{df}{d\eta} = 0$$

$$\Rightarrow \quad \underline{\underline{\eta + f^2 \frac{df}{d\eta} = 0}} \quad \text{with } \underline{\underline{BC}} \quad \frac{df}{d\eta} = 0 \ (\eta = 0)$$

$$f(\eta_R) = 0$$

May integrate analytically :

$$f^2 df = -\eta\, d\eta \quad \Rightarrow \quad \frac{1}{3} d\left(f^3\right) = -\frac{1}{2} d\eta^2$$

$$\frac{f^3}{3} = -\frac{\eta^2}{2} + \text{const} = -\frac{1}{2}\left(\eta^2 - \eta_R^2\right)$$

$$\Rightarrow \quad \underline{\underline{f(\eta) = \left\{\frac{3}{2}\left(\eta_R^2 - \eta^2\right)\right\}^{1/3}}}$$

* To <u>determine</u> η_R impose $\displaystyle\int_0^{\eta_R} \eta\, f(\eta)\, d\eta = 1$

$$\Rightarrow \quad \left(\frac{3}{2}\right)^{1/3} \int_0^{\eta_R} \eta \left(\eta_R^2 - \eta^2\right)^{1/3} d\eta = 1$$

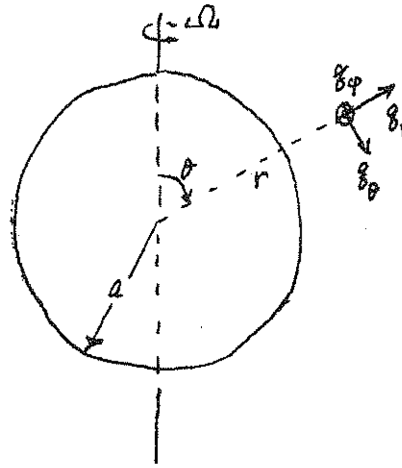
* Since $\eta_R = A \dfrac{R}{t^{1/8}} \Rightarrow R \propto t^{1/8}$ as t increases

$h = \dfrac{f(\eta)}{t^{1/4}} \Rightarrow h \sim \dfrac{f(0)}{t^{1/4}}$ as t increases

[illegible]

Problem Set No. 2 ; Problem 3 Solution

Rigid sphere rotating in a viscous fluid at low Re



* It is convenient to use spherical polar coordinates r (radial), θ (polar), φ (azimuthal) (see Appendix 2 of Batchelor's book)

* Owing to symmetry, the velocity field $\underset{\sim}{q} = (q_r, q_\theta, q_\varphi)$ and the pressure p are independent of φ

* At low Reynolds number, $\Omega a^2/\nu \ll 1$, inertia terms are expected to be negligible; this in turn implies that $\underset{\sim}{q} = q_\varphi \hat{e}_\varphi$, with $q_\varphi = q_\varphi(r,\theta)$

Indeed, the continuity equation

$$\nabla \cdot \underset{\sim}{q} = \frac{1}{r^2}\left(\frac{\partial}{\partial r}(r^2 q_r)\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\, q_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial q_\varphi}{\partial\varphi} = 0$$

is automatically satisfied;

moreover, the only surviving inertia terms in the r- and θ-mom. eqn's

$$\frac{q_\varphi^2}{r} \sim \frac{\Omega^2 a^{\cancel{2}}}{\cancel{a}} \sim \Omega^2 a,$$

are much smaller than the typical viscous term

$$\nu \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial q_\varphi}{\partial r}\right) \sim \frac{\nu q_\varphi}{a^2} \sim \frac{\nu \Omega \cancel{a}}{\cancel{a}\, a}$$

kept in the φ-mom. equation, since $\frac{\nu\Omega}{a} \gg \Omega^2 a$

$\Rightarrow \Omega a^2/\nu \ll 1$

Therefore, φ-momentum eqⁿ $\Rightarrow$

$$\nu \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial g_\varphi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial g_\varphi}{\partial\theta}\right) - \frac{g_\varphi}{r^2\sin^2\theta} \right\} = 0 \quad (*)$$

* This (linear) equation for g_φ has a separable solution:

$$g_\varphi = f(r)\sin\theta \quad (**)$$

(This is motivated by the fact that on the sphere $(r=a)$,

$$g_\varphi \hat{\underset{\sim}{e}}_\varphi = \underset{\sim}{\Omega} \times a\hat{\underset{\sim}{e}}_r = \Omega \cdot a \sin\theta\, \hat{\underset{\sim}{e}}_\varphi)$$

Upon substitution of $(**)$ into $(*)$, we find that $f(r)$ satisfies:

$$\frac{d}{dr}\left(r^2 \frac{df}{dr}\right) - 2f(r) = 0$$

This ode has solutions of the form r^n for $n=-2,1$

Since $g_\varphi \to 0$ as $r\to\infty$, only $n=-2$ is acceptable

Hence,
$$g_\varphi = \frac{A}{r^2}\sin\theta$$

No slip at $r=a$ $\Rightarrow$ $\Omega a = \dfrac{A}{a^2} \Rightarrow A = \Omega a^3$

So, finally
$$g_\varphi = \frac{\Omega a^3}{r^2}\sin\theta$$

* On the surface of the sphere

$$\tau_{r\varphi}\Big|_{r=a} = \mu\left(\frac{\partial g_\varphi}{\partial r} - \frac{g_\varphi}{r}\right)\Big|_{r=a} = -3\mu\Omega\sin\theta$$

Therefore, the moment exerted by the fluid on the sphere is given by

$$M = \int_0^{\pi} \tau_{r\varphi}\Big|_a \, a\sin\theta \cdot 2\pi a^2 \sin\theta \, d\theta = -8\pi\mu a^3 \Omega$$

* An alternative way to deduce M is by using the fact that

$$\begin{pmatrix} \text{power required to} \\ \text{maintain steady} \\ \text{rotation of sphere} \end{pmatrix} = (-M)\cdot\Omega = \begin{pmatrix} \text{rate of viscous} \\ \text{dissipation in the} \\ \text{fluid} \end{pmatrix}$$

From problem 2 of PS # 1, we know that

$$\begin{pmatrix} \text{rate of viscous} \\ \text{dissipation} \end{pmatrix} = \iiint_V \Phi \, dV \quad \text{where } \Phi = \frac{1}{2}\mu \left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i}\right)^2$$

Here, $$\Phi = 4\mu e_{r\varphi}^2 = \mu r^2 \left(\frac{\partial}{\partial r}\left(\frac{q_\varphi}{r}\right)\right)^2 = \mu\left(\frac{\partial q_\varphi}{\partial r} - \frac{q_\varphi}{r}\right)^2$$

$$= 9\mu \Omega^2 a^6 \frac{\sin^2\theta}{r^6}$$

Now,

$$\iiint_V \Phi \, dV = 9\mu\Omega^2 a^6 \, 2\pi \int_0^{\pi} d\theta \int_a^{\infty} dr \, \frac{\sin^2\theta}{r^6} \, r^2 \sin\theta$$

$$= 8\pi\mu a^3 \Omega^2$$

Hence $\quad M = -8\pi\mu a^3\Omega \quad$ (as before)