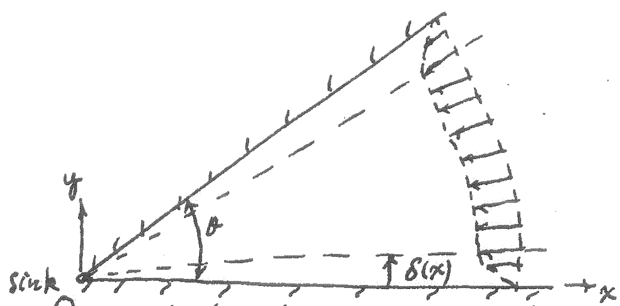
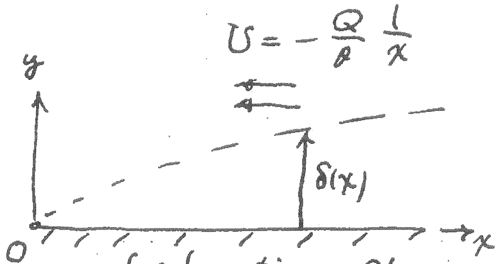
Problem Set 3 Solution

Steady boundary layer along a wall towards a corner sink



(schematic of the full flow)



(schematic of BL on horizontal wall)

* Boundary-layer equations

$$\left.\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{(continuity)} \\ &u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{\cancel{U\frac{dU}{dx}}}_{-\frac{Q^2}{\theta^2 x^3}} + \nu\frac{\partial^2 u}{\partial y^2} \quad (x\text{-mom.}) \end{aligned}\right\}$$

Boundary conditions :
$$\begin{cases} u = v = 0 & (y = 0) \\ u \to U = -\dfrac{Q}{\theta}\dfrac{1}{x} & (y/\delta \to \infty) \end{cases}$$

In terms of streamfunction $\psi(x,y)$,

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

continuity is automatically satisfied, and

$$x\text{-mom} \Rightarrow \quad \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = -\frac{Q^2}{\theta^2 x^3} + \nu \frac{\partial^3 \psi}{\partial y^3}$$

and BC's $\Rightarrow$
$$\begin{cases} \psi = \dfrac{\partial \psi}{\partial y} = 0 & (y = 0) \\ \dfrac{\partial \psi}{\partial y} \to -\dfrac{Q}{\theta}\dfrac{1}{x} & (y/\delta \to \infty) \end{cases}$$

* Similarity solution

$$x = \lambda^{\alpha} x', \quad y = \lambda^{\beta} y', \quad \psi = \lambda^{\gamma} \psi'$$

Gov. eqn $\Rightarrow \quad \lambda^{2\gamma - 2\beta - \alpha} \sim \lambda^{-3\alpha} \sim \lambda^{\gamma - 3\beta}$

$$\therefore \quad -3\alpha = -3\beta + \gamma \Rightarrow \beta = \alpha + \frac{1}{3}\gamma$$

$$-3\alpha = -\alpha + 2\gamma - 2\beta \Rightarrow 2\alpha = 2\beta - 2\gamma$$

Hence $2\alpha = 2\alpha + \frac{2}{3}\gamma - 2\gamma \Rightarrow \underline{\gamma = 0, \ \alpha = \beta}$

Also $\psi_y \to -\frac{Q}{\rho}\frac{1}{x} \Rightarrow \lambda^{\gamma - \beta} \sim \lambda^{-\alpha}$ automatically met

Thus, try $\quad \underline{\psi = f(y/x), \quad \eta = y/x}$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = -\frac{\eta}{x}\frac{d}{d\eta}, \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{x}\frac{d}{d\eta}$$

After some algebra, we find that f satisfies

$$-\frac{df}{d\eta}\left(\eta \frac{df}{d\eta}\right)' + \eta \frac{df}{d\eta}\frac{d^2 f}{d\eta^2} = -\frac{Q^2}{\rho^2} + \nu \frac{d^3 f}{d\eta^3}$$

$$\Rightarrow \quad \underline{\nu \frac{d^3 f}{d\eta^3} = \frac{Q^2}{\rho^2} - \left(\frac{df}{d\eta}\right)^2}$$

with BC

$$\begin{cases} df/d\eta = 0 \quad \text{at } \eta = 0 \\ df/d\eta \to -\dfrac{Q}{\rho} \quad \text{as } y \to \infty \ (\eta \to \infty) \\ \left(d^2 f/d\eta^2 \to 0 \quad \text{as } y \to \infty\right) \text{ (too much to ask)} \end{cases}$$

Write $$\frac{df}{d\eta} = -\frac{Q}{\theta} F(\eta)$$

Then, $$-\frac{\nu\theta}{Q}\frac{d^2F}{d\eta^2} = 1 - F^2$$

with BC $\begin{cases} F(0) = 0 & \eta = 0 \\ F \to 1 & \eta \to \infty \end{cases}$

Integrate once : $-\frac{2\nu\theta}{2Q} F'^2 = F - \frac{1}{3}F^3 + C$

As $\eta \to \infty$ $F' \to 0$ $\Rightarrow$ $C = -\frac{2}{3}$

Hence $-\frac{\nu\theta}{2Q} F'^2 = F - \frac{1}{3}F^3 - \frac{2}{3} = -\frac{1}{3}(F-1)^2(F+2)$

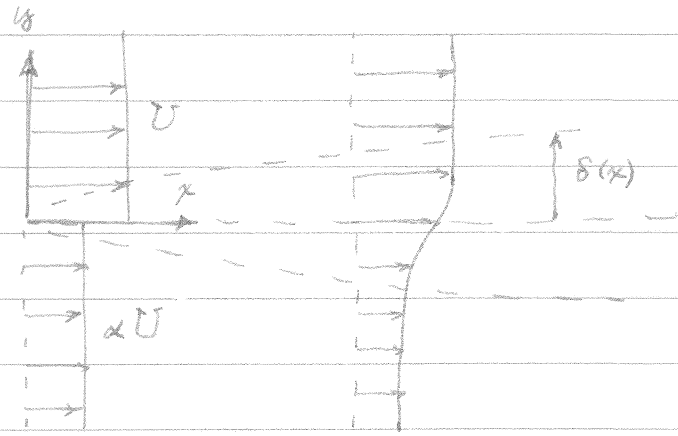
For $0 \leq F \leq 1$, RHS is always < 0 $\Rightarrow$ $Q > 0$

So, $$\frac{\nu\theta}{2Q}\left(\frac{dF}{d\eta}\right)^2 = \frac{1}{3}(F-1)^2(F+2)$$

$$u = \psi_y = \frac{1}{x}\frac{df}{d\eta} = -\frac{Q}{\theta x}F(\eta) \Rightarrow \frac{u}{U(x)} = F(\eta)$$

$$F(\eta) = 3\tanh^2\left[\frac{y}{x}\sqrt{\frac{Q}{2\nu\theta}} + \tanh^{-1}\sqrt{\frac{2}{3}}\right] - 2$$

Problem Set 3, Problem 2 ; solution



This problem is a generalization of the Blasius flat-plate boundary layer. Since the 'outer' potential flow is uniform, $U\frac{\partial U}{\partial x} = 0$, and there is no pressure gradient in the boundary layer equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

Introducing streamfunction $\psi(x,y)$: $u = \frac{\partial \psi}{\partial y}$, $v = -\frac{\partial \psi}{\partial x}$

$$\Rightarrow \quad \underline{\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y^2}\frac{\partial \psi}{\partial x} = \nu\frac{\partial^3 \psi}{\partial x^3}}$$

with <u>BC</u> (i) $\psi = 0 \quad (y=0)$ (dividing streamline is flat)

(ii) $\psi \to Uy \quad (y/\delta \to \infty)$

(iii) $\psi \to \alpha U y \quad (y/\delta \to -\infty)$

This problem can be solved by the same similarity variables as in the Blasius boundary layer:

$$\underline{\psi = \sqrt{\nu U x}\; f(\eta) \quad , \quad \eta = \frac{y}{\sqrt{\nu x/U}}} \qquad \left(\delta \sim \sqrt{\nu x/U}\right)$$

$$\Rightarrow \qquad \underline{f f'' + 2 f''' = 0} \qquad \text{(Blasius equation)}$$

$$\underline{\underline{BC}} \quad \begin{cases} f = 0 & (\eta = 0) \\ f' \to 1 & (\eta \to \infty) \\ f' \to \alpha & (\eta \to -\infty) \end{cases}$$

There is no closed-form solution of this boundary-value problem [See R.C. Lock, Quarterly Journal of Mechanics and Applied Mathematics, 4, 42-63, 1951; he shows how to obtain numerical solution and also considers the more general problem where the two streams have different densities and viscosities; eg air over water]

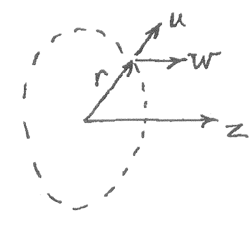
Problem Set No. 3; Problem 3 Solution

Jet from a point source

* Boundary-layer equations (axisymmetric flow) $(r, z) \longleftrightarrow (u, w)$

(continuity) $\dfrac{\partial}{\partial r}(ru) + \dfrac{\partial}{\partial z}(rw) = 0$

(Momentum) $u\dfrac{\partial w}{\partial r} + w\dfrac{\partial w}{\partial z} = \dfrac{\nu}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right)$



* Momentum flow rate at the origin

$$M = 2\pi\int_0^\infty r\,dr\,\rho w^2 = 2\pi\rho\int_0^\infty r w^2\,dr$$

* Boundary conditions are

$$\begin{cases} u = 0,\ \dfrac{\partial w}{\partial r} = 0 & \text{at } r = 0 \\ w \to 0 & \text{as } r \to \infty \end{cases}$$

Also, from the momentum equation above,

$$\int_0^\infty 2\pi r\,dr\left\{u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z}\right\} = \int_0^\infty 2\pi \cancel{r}\,dr\left\{\frac{\nu}{\cancel{r}}\frac{\partial}{\partial r}\left(r\,\partial w/\partial r\right)\right\}$$

$$\Rightarrow \underbrace{\int_0^\infty 2\pi r\,dr\,u\frac{\partial w}{\partial r}}_{\Downarrow} + \int_0^\infty 2\pi r\,dr\,w\frac{\partial w}{\partial z} = 2\pi r\nu\frac{\partial w}{\partial r}\Big|_0^\infty = 0$$

$$2\pi r u w\Big|_0^\infty - 2\pi\int_0^\infty dr\,w\underbrace{\frac{\partial}{\partial r}(ru)}_{-\frac{\partial}{\partial z}(rw)}$$

$$\therefore\ 2\int_0^\infty 2\pi r\,dr\,\frac{1}{2}\frac{\partial}{\partial z}(w^2) = 0 \Rightarrow \int_0^\infty r\,dr\,w^2 = \text{const at any } z$$

$$\Rightarrow M = 2\pi\rho\int_0^\infty r w^2\,dr = \text{const at any } z$$

* Introduce stream function $\psi(r,z)$ s.t.

$$u = \frac{1}{r}\psi_z \;, \qquad w = \frac{-1}{r}\psi_r \;;$$

Then, continuity $\frac{\partial}{\partial r}(ru) + \frac{\partial}{\partial z}(rw) = 0$ is automatically satisfied

Momentum equation $\Rightarrow$

$$-\cancel{\frac{1}{r}}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r}\left(\frac{1}{r}\psi_r\right) + \cancel{\frac{1}{r}}\psi_r\frac{\partial}{\partial z}\left(\frac{1}{r}\psi_r\right) = \frac{\nu}{\cancel{r}}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\left(\frac{-1}{r}\psi_r\right)\right)$$

Now, try the transformation

$$\psi \to \lambda^{\alpha}\psi', \qquad r \to \lambda^{\beta} r', \qquad z = \lambda^{\gamma} z'$$

Then, invariance of momentum eqn $\Rightarrow$

$$\lambda^{2\alpha-3\beta-\gamma} \sim \lambda^{\alpha-3\beta} \quad \Rightarrow \quad \alpha = \gamma$$

Also, from $\int_0^\infty r\,dr\,w^2 = \text{const} \Rightarrow \lambda^{2\beta+2\alpha-4\beta} = 1$

$$\Rightarrow \quad \alpha = \beta$$

$$\therefore \quad \alpha = \beta = \gamma$$

So, the appropriate similarity variable is of the form

$$\eta = \frac{1}{B}\frac{r}{z} \qquad \text{and} \qquad \psi = A\,z\,f(\eta)$$

* Substitute into the momentum equation (algebra)

$$\eta = \frac{1}{B}\frac{r}{z}, \qquad \psi = Az\,f(\eta)$$

$$\frac{\partial}{\partial r} = \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial r} = \frac{1}{Bz}\frac{\partial}{\partial \eta}, \qquad \frac{\partial}{\partial z} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial z} = \frac{\partial}{\partial z} - \frac{r}{Bz^2}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial z} - \frac{\eta}{z}\frac{\partial}{\partial \eta}$$

$$w = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\frac{A\cancel{z}}{\cancel{r}}\frac{1}{B\cancel{z}} f_\eta \frac{1}{Bz\eta} = \underline{\frac{-A}{B^2\eta z} f_\eta} = w$$

$$\frac{1}{r}\frac{\partial \psi}{\partial z} = \frac{1}{Bz\eta}\left\{ Af - \frac{\eta}{\cancel{z}} A\cancel{z} f_\eta \right) \Rightarrow \quad u = \underline{\frac{A}{Bz\eta}(f - \eta f_\eta)}$$

So, momentum equation $\Rightarrow$

$$-A(f-\eta f')\frac{1}{Bz}\frac{\partial}{\partial \eta}\left(\frac{A}{B^2\eta z} f'\right) + \frac{1}{Bz}\frac{\partial}{\partial \eta}(Azf)\left(\frac{\partial}{\partial z} - \frac{\eta}{z}\frac{\partial}{\partial \eta}\right)\left(\frac{A}{B^2\eta z} f'\right)$$

$$= \nu \frac{1}{Bz}\frac{\partial}{\partial \eta}\left\{ Bz\eta \frac{1}{Bz}\frac{\partial}{\partial \eta}\left(\frac{A}{B^2\eta z} f'\right)\right\}$$

Re-arranging, after some cancellation,

$$A\left\{ f\left(\frac{f'}{\eta}\right)' + \frac{f'^2}{\eta}\right\} = \nu \frac{\partial}{\partial \eta}\left\{ \eta \frac{\partial}{\partial \eta}\left(f'/\eta \right)\right\}$$

Choose $\underline{A = \nu, \quad B = 1}$

Then,
$$\underline{\left(\frac{ff'}{\eta}\right)' = \left(\eta \left(\frac{f'}{\eta}\right)'\right)'}$$

* $\underline{\text{B.C.}}$ $\quad u = 0$ at $r = 0$ $\Rightarrow$

$$\frac{1}{\eta}(f - \eta f_\eta) \to 0 \text{ as } \eta \to 0$$

ie, $\frac{f}{\eta} - f' \to 0$ as $\eta \to 0$

Also $w = \frac{\nu}{\eta z} f_\eta$ finite as $r \to 0$ $\quad \underline{f'(0) \to 0}$ and hence $\underline{f(0) = 0}$

* Solve ode for f

Integrating once, $\dfrac{ff'}{\eta} = \eta\left(\dfrac{f'}{\eta}\right)' + \text{const} = f'' - \dfrac{1}{\eta}f' + \text{const}$

BC $\Rightarrow$ $\dfrac{f}{\eta}, f' \to 0$ as $\eta \to 0$ $\Rightarrow$ const $= 0$

$\therefore$ $ff' = \eta f'' - f'$ or $\eta f'' + f' = ff' + 2f'$

$$\Rightarrow (\eta f')' = \left(\frac{f^2}{2} + 2f\right)'$$

Integrating again, $\eta f' = \dfrac{f^2}{2} + 2f = \dfrac{f}{2}(f+4)$

$$\Rightarrow \frac{d\eta}{2\eta} = \frac{df}{f(f+4)} \Rightarrow \ln\frac{f}{f+4} = 2\ln\eta + \ln C$$

$$\Rightarrow f(\eta) = \boxed{\frac{4C\eta^2}{1-C\eta^2}}$$

* Impose $M = 2\pi\rho\displaystyle\int_0^\infty w^2 r\,dr = 2\pi\rho\int_0^\infty \left(\frac{1}{r}\psi_r\right)^2 dr$

$$= 2\pi\rho\int_0^\infty \left(\frac{\nu}{\eta z}f'\right)^2 \eta z^2\,d\eta = 2\pi\rho\nu^2\int_0^\infty \frac{f'^2}{\eta}\,d\eta$$

$$= 2\pi\rho\nu^2\int_0^\infty \frac{1}{\eta}\left\{\frac{d}{d\eta}\left(\frac{4C\eta^2}{1-C\eta^2}\right)\right\}^2 d\eta = -2\pi\rho\frac{\nu^2}{3}32C = M$$

This determines $\boxed{C = -\dfrac{3M}{64\pi\rho\nu^2} < 0}$

$$\Rightarrow f(\eta) = \frac{-4K\eta^2}{1+K\eta^2} \qquad (K \equiv -C)$$

* Velocity components

$$\left.\begin{aligned} u &= \frac{1}{r}\psi_z = \frac{\nu}{z\eta}(f-\eta f') \\ w &= -\frac{1}{r}\psi_r = -\frac{\nu}{\eta z} f' \end{aligned}\right\}$$

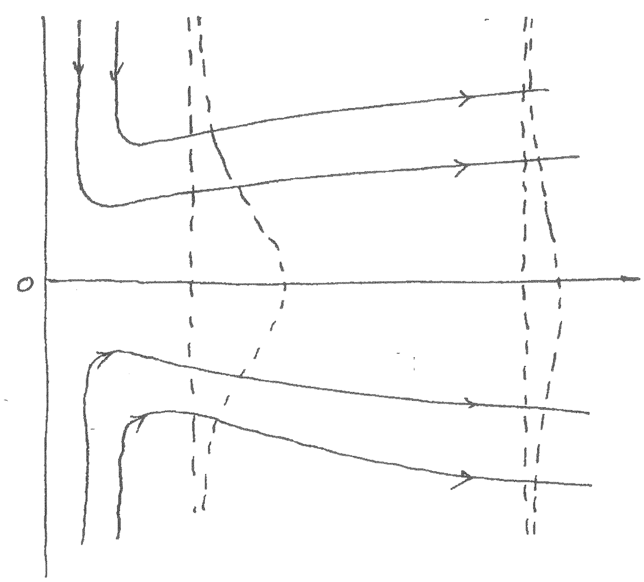
Near the axis $(r\to 0,\ \eta\to 0)$ $\quad f \longrightarrow -4K\eta^2$

$$\therefore \begin{cases} u \sim \frac{\nu}{z}(-4K+8K)\eta = 4K\nu\frac{r}{z^2} > 0 \\ w \sim \frac{\nu}{z} 8K > 0 \qquad (\text{ie, } w \gg u) \end{cases}$$

Far from the axis $(r\to\infty,\ \eta\to\infty,\ z \text{ fixed})$

$$f = \frac{-4K\eta^2}{1+K\eta^2} \sim -4\left(1 - \frac{1}{K\eta^2} + \dots\right)$$

$$\therefore \begin{cases} u \sim -\frac{4\nu}{r} < 0 \Rightarrow \text{flow is entrained from the jet edges} \\ w \sim \frac{\nu}{z}\left(\frac{8}{K\eta^4}\right) = \frac{8\nu}{K}\frac{z^3}{r^4} \qquad (\text{ie, } w \ll |u|) \end{cases}$$



$\eta = \frac{r}{z} \Rightarrow$ radius of jet grows like z

* How good is the boundary-layer approximation?

Viscous term omitted: $\nu \dfrac{\partial^2 w}{\partial z^2} = -\nu \dfrac{\partial^2}{\partial z^2}\left(\dfrac{1}{r}\dfrac{\partial \psi}{\partial r}\right) =$

$$= -\nu \frac{A}{B^2}\left(\frac{\partial}{\partial z} - \frac{\eta}{z}\frac{\partial}{\partial \eta}\right)^2 \left(\frac{f'}{\eta z}\right) \equiv \nu F(\eta)\frac{1}{z^3}$$

Viscous term included: $\nu \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial w}{\partial r}\right) =$

$$= \nu \frac{1}{B\eta z}\frac{1}{Bz}\frac{\partial}{\partial \eta}\left(B\eta z \frac{1}{Bz}\frac{\partial}{\partial \eta}\left(-\frac{A}{B^2\eta z} f'\right)\right) \equiv \nu G(\eta)\frac{1}{z^3}$$

These two terms are comparable $O\left(\frac{1}{z^3}\right)$ for $\eta = O(1)$, $z \gg 1$

Note that the Reynolds number $Re = \dfrac{U\delta}{\nu} \equiv \dfrac{W_{max}\cdot \delta}{\nu}$

$$W_{max} \sim \frac{\nu}{z} 8K, \quad \delta = O(z) \Rightarrow \underline{\underline{Re = O(1)}}$$

[In 2-D jet: $U \sim \dfrac{1}{x^{1/3}}$, $\delta \sim x^{2/3}$

$$\Rightarrow Re = \frac{U\delta}{\nu} \sim x^{1/3} \gg 1 \text{ for } x \text{ large}$$

ie, boundary-layer approximation becomes better as x increases]

* Even if we include $\nu \dfrac{\partial^2 w}{\partial z^2}$ term, solution would not be exact because pressure-gradient term $-\partial p/\partial z$ has been omitted as well.