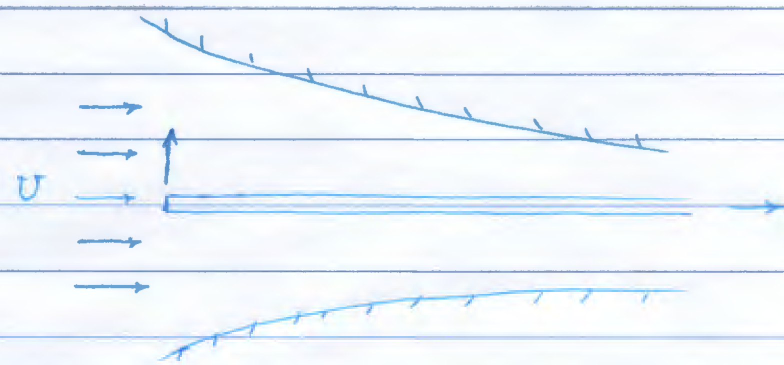
1.63J / 2.26J Advanced Fluid Dynamics

Problem Set No. 4 Solutions

Problem 1

$U = \lambda x$



(a) 2D steady BL equations $\Rightarrow$

$$U\frac{\partial U}{\partial x} = \lambda^2 x$$

$$\left.\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \cancel{-\frac{1}{\rho}\frac{\partial p}{\partial x}} + \nu\frac{\partial^2 u}{\partial y^2} \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \end{aligned}\right\}$$

with BC $\quad u = v = 0 \quad (y=0) \quad$ No-slip

$\quad u \to U \quad (y \to \infty) \quad$ matching with outer flow

Using streamfunction $\psi(x,y)$: $\quad u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$

$$\left.\begin{aligned} \frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} &= \lambda^2 x + \nu\frac{\partial^3 \psi}{\partial y^3} \\ \psi = \frac{\partial \psi}{\partial y} &= 0 \quad (y=0) \\ \frac{\partial \psi}{\partial y} &\to \lambda x \quad (y \to \infty) \end{aligned}\right\}$$

(b) Look for possible similarity solution:

$$x = \alpha x', \quad y = \alpha^{\beta} y', \quad \psi = \alpha^{\gamma}\psi'$$

$$\Rightarrow \quad 2\gamma - 1 - 2\beta = \gamma - 3\beta = 1, \quad \gamma - \beta = 1$$

$$\therefore \quad \gamma = 1, \quad \beta = 0$$

Hence, y and Ψ/x remain invariant under transformation; this suggests similarity solution in the form:

$$\Psi = Ax\, f(By)$$

By direct substitution into BL equation and BC:

$$(BA)^2 x\, f'^2 - (BA)^2 x f f'' = \lambda^2 x + \nu A x B^3 f'''$$

$\Rightarrow$

$$f'^2 - f f'' = \left(\frac{\lambda}{AB}\right)^2 x + \left(\nu \frac{B}{A}\right) f'''$$

$$ABx\, f' \to \lambda x \quad (y \to \infty) \Rightarrow f' \to \frac{\lambda}{AB} \quad (y \to \infty)$$

Choose

$$\frac{\lambda}{AB} = 1 \;, \quad \nu \frac{B}{A} = 1 \Rightarrow A = \sqrt{\nu \lambda} \;, \; B = \sqrt{\lambda/\nu}$$

Finally,

$$\left.\begin{aligned} &f''' + f f'' - f'^2 + 1 = 0 \\ &f = f' = 0 \quad (y = 0) \\ &f' \to 1 \quad (y \to \infty) \end{aligned}\right\}$$

with

$$\Psi = \sqrt{\nu\lambda}\; x\; f(\eta) \;, \quad \eta = \sqrt{\lambda/\nu}\; y$$

Clearly, BL thickness is $O\left(\nu/\lambda\right)^{1/2}$, independent of x

Problem 2 — Streaming due to oscillating circular cylinder

As discussed in class, the streaming (mean-flow) velocity induced by an oscillatory flow

$$U(x,t) = \Re\left\{ U_0(x)\, e^{-i\omega t} \right\}$$

is

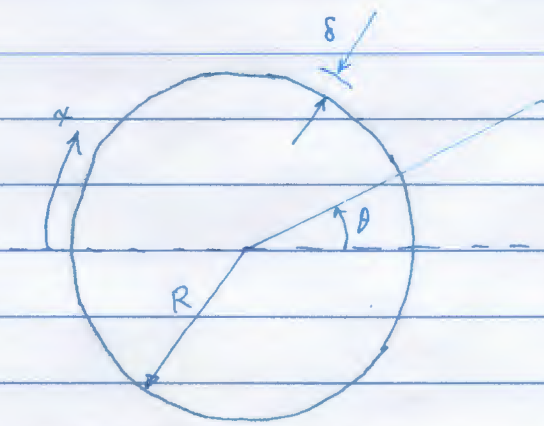
$$\langle u_2 \rangle\Big|_\infty = -\frac{3}{4\omega}\,\Re\left\{ (1-i)\, U_0 \frac{dU_0^*}{dx} \right\} \quad \text{at the edge of Stokes layer}$$

$$\langle u_2 \rangle\Big|_0 = \frac{1}{\omega}\,\Re\left\{ \frac{(1+i)}{2\sqrt{2}}\, y \sqrt{\frac{\omega}{\nu}}\, U_0 \frac{dU_0^*}{dx} \right\} \quad \text{close to the wall, } y/\delta \ll 1$$

In the problem at hand,

$$U_0(x) = -2U_\infty \sin\theta \quad , \quad x = R(\pi-\theta) \Rightarrow \theta = \pi - \frac{x}{R}$$

$$\Rightarrow \quad U_0(x) = -2U_\infty \sin\left(\pi - \frac{x}{R}\right) = -2U_\infty \sin\frac{x}{R}$$



$0 \leq x \leq 2\pi R$

$-\pi \leq \theta \leq \pi$

Here $U_0(x)$ is real $\Rightarrow$ $$U_0 \frac{dU_0}{dx} = 4\frac{U_\infty^2}{R} \sin\frac{x}{R}\cos\frac{x}{R} = 2\frac{U_\infty^2}{R}\sin\frac{2x}{R}$$

Therefore,

$$\left.\langle u_2\rangle\right|_\infty = -\frac{3}{2}\frac{U_\infty^2}{\omega R}\sin\frac{2x}{R} \longrightarrow \frac{3}{2}\frac{U_\infty^2}{\omega R}\sin 2\theta$$

$$\left.\langle u_2\rangle\right|_0 = \frac{U_\infty^2}{\omega R}\sqrt{\frac{\omega}{2\nu}}\, y \sin\frac{2x}{R} \longrightarrow -\frac{U_\infty^2}{\omega R}\sqrt{\frac{\omega}{2\nu}}\, y \sin 2\theta$$