Problem 1

\[ U = \lambda x \]

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(ii) 2D steady BL equations \[ \frac{\partial U}{\partial x} + \nu \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 U}{\partial y^2} \]

\[ \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0 \]

with BC \[ U = V = 0 \ (y = 0) \] No-slip

\[ U \to U \ (y \to \infty) \] Matching with outer flow

Using streamfunction \[ \psi(x, y) : \quad U = \frac{\partial \psi}{\partial y}, \quad V = -\frac{\partial \psi}{\partial x} \]

\[ \frac{\partial \psi}{\partial y} \frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial y} = \lambda x + \nu \frac{\partial^2 \psi}{\partial y^2} \]

\[ \psi = \frac{\partial \psi}{\partial y} = 0 \ (y = 0) \]

\[ \frac{\partial \psi}{\partial y} \to \lambda x \ (y \to \infty) \]

(b) Look for possible similarity solution:

\[ x = \kappa x', \quad y = \alpha^\beta y', \quad \psi = \alpha^{2\beta} \psi' \]

\[ 2\beta - 1 - 2\beta = \kappa - 3\beta = 1, \quad \gamma - \beta = 1 \]

\[ \therefore \quad \gamma = 1, \quad \beta = 0 \]
Hence, \( y \) and \( \psi/x \) remain invariant under transformation.

This suggests a similarity solution in the form:

\[
\psi = Ax \cdot f(By)
\]

By direct substitution into BL equation and BC:

\[
(BA)^2 \cdot f'' - (BA)^2 \cdot xf'' = A^2 x + \nu A x B^3 \cdot f''
\]

\[
\Rightarrow \quad f'' - f'' = \left( \frac{A}{BA} \right)^2 x + \left( \frac{\nu B}{A} \right) f''
\]

Choose

\[
\frac{A}{BA} = 1, \quad \frac{\nu B}{A} = 1 \Rightarrow A = \sqrt{\nu x}, \quad B = \sqrt{\nu}
\]

Finally,

\[
\begin{align*}
\psi'' + f + f'' - f' + 1 &= 0 \\
\frac{f}{\psi} &= 0 \quad (y = 0) \\
\psi' &\to 1 \quad (y \to \infty)
\end{align*}
\]

with

\[
\psi = \sqrt{\nu x} \cdot f(\eta), \quad \eta = \sqrt{\nu} y
\]

Clearly, BL thickness is \( \mathcal{O}\left(\frac{\nu}{x}\right)^n \), independent of \( x \)
Problem 2. Streaming due to oscillating circular cylinder

As discussed in class, the streaming (mean flow) velocity induced by an oscillating flow

\[ U(x,t) = R \left\{ U_0(x) e^{-i\omega t} \right\} \]

is

\[ \left\langle U_0 \right\rangle |_{\infty} = -\frac{2}{i\omega} R \left\{ (1-i) U_0 \frac{dU_0^*}{dx} \right\} \text{ at the edge of the boundary layer} \]

\[ \left\langle U_0 \right\rangle |_{0} = \frac{i}{\omega} R \left\{ \frac{(1+i)}{2\pi} \sqrt{\frac{\omega}{R}} U_0 \frac{dU_0^*}{dx} \right\} \text{ close to the wall, } \frac{\omega}{R} \ll 1 \]

In the problem at hand,

\[ U_0(x) = -2 U_0 \sin \theta, \quad x = R (\pi - \theta) \Rightarrow \theta = \pi - \frac{x}{R} \]

\[ \Rightarrow \quad U_0(x) = -2 U_0 \sin \left( \pi - \frac{x}{R} \right) = -2 U_0 \sin \frac{x}{R} \]

\[ 0 \leq x \leq 2\pi R \]

\[ -\pi \leq \theta \leq \pi \]

Here \( U_0(x) \) is real \( \Rightarrow U_0 \frac{dU_0}{dx} = \frac{4 U_0^2}{R^2} \sin \frac{x}{R} \cos \frac{x}{R} = \frac{2 U_0^2}{R} \sin \frac{2x}{R} \)
\[
\begin{align*}
\langle u_2 \rangle_\infty &= -\frac{3}{2} \frac{U_0^2}{\omega R} \sin \frac{2\alpha}{R} + \frac{3}{2} \frac{U_0^2}{\omega R} \sin \theta \\
\langle u_2 \rangle_0 &= \frac{U_0^2}{\omega R} \left[ \frac{1}{2} \frac{\omega}{\omega R} \gamma \frac{\sin \frac{3\alpha}{2}}{R} \right] - \frac{U_0^2}{\omega R} \left[ \frac{1}{2} \frac{\omega}{\omega R} \gamma \sin 2\theta \right]
\end{align*}
\]