

Problem Set 5 SolutionsRotating-disk flow

Letting

$$u \equiv q_r, \quad v \equiv q_\theta, \quad w \equiv q_z$$

and exploiting rotational symmetry  $\frac{\partial}{\partial \theta} = 0$ , and steady flow,

the Navier-Stokes equations in cylindrical polar coordinates are

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

$$u \frac{\partial u}{\partial r} - \frac{v^2}{r} + w \frac{\partial u}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left\{ \frac{\partial^2 u}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{u}{r} \right) + \frac{\partial^2 u}{\partial z^2} \right\}$$

$$u \frac{\partial v}{\partial r} + \frac{uv}{r} + w \frac{\partial v}{\partial z} = \nu \left\{ \frac{\partial^2 v}{\partial r^2} + \frac{\partial}{\partial r} \left( \frac{v}{r} \right) + \frac{\partial^2 v}{\partial z^2} \right\}$$

$$u \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + \nu \left\{ \frac{\partial^2 w}{\partial r^2} + \frac{1}{r} \frac{\partial w}{\partial r} + \frac{\partial^2 w}{\partial z^2} \right\}$$

We also have the BCs:

$$\begin{cases} u = 0, \quad v = \Omega r, \quad w = 0 & (z=0) \\ u, v \rightarrow 0 & (z \rightarrow \infty) \end{cases}$$

Similarity solution

$$\xi = \left( \frac{\Omega}{\nu} \right)^{1/2} z$$

$$u = \Omega r f(\xi), \quad v = \Omega r g(\xi), \quad w = (\nu \Omega)^{1/2} h(\xi)$$

$$p = \rho \nu \Omega P(\xi)$$

By direct substitution, we find that

$$\text{continuity} \Rightarrow 2f + h' = 0$$

$$r\text{-momentum} \Rightarrow f^2 - g^2 + hf' = f''$$

$$\theta\text{-momentum} \Rightarrow 2fg + hg' = g''$$

$$z\text{-momentum} \Rightarrow p' + hh' = h''$$

$$\text{Also, BC's} \Rightarrow \begin{cases} f=0, g=1, h=0 & (\xi=0) \\ f \rightarrow 0, g \rightarrow 0 & (\xi \rightarrow \infty) \end{cases}$$

This problem must be solved numerically

Note that  $h(\xi) \rightarrow h_\infty \neq 0$  as  $\xi \rightarrow \infty$  ( $h_\infty \approx -0.8$ )

see Schlichting p.104

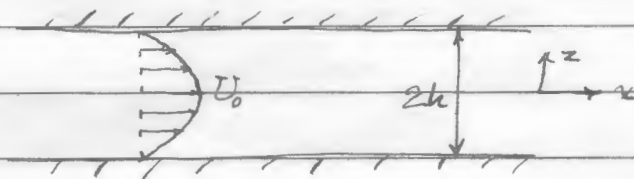
This makes sense physically because fluid is being pumped outwards in the radial direction, so there must be axial flow at infinity in order to conserve mass. Specifically, for control volume ( $0 \leq r \leq R$ ,  $0 \leq z < \infty$ ):

$$Q_{out} = 2\pi R \int_0^\infty u \, dz = 2\pi R^2 (\Omega \nu)^{1/2} \underbrace{\int_0^\infty f(\xi) \, d\xi}_{-\frac{1}{2} h(\xi) \Big|_0^\infty}$$
$$\Rightarrow Q_{out} = -\pi R^2 (\Omega \nu)^{1/2} h_\infty > 0 \quad (h_\infty < 0)$$

$$Q_{in} = -\pi R^2 w(z \rightarrow \infty) = -\pi R^2 (\Omega \nu)^{1/2} h_\infty$$

Note that  $Q_{in} = Q_{out}$ , as expected

## Dispersion of passive solvent in 2D channel



Background steady flow is  $u = U_s(z) = U_0 \left(1 - \frac{z^2}{h^2}\right)$

Solvent concentration  $C(x, z, t)$  satisfies convection-diffusion equation:

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x} (u C) = D \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2} \right) C$$

There are 3 relevant timescales:

$h^2/D$  (diffusion in  $z$ ),  $L/U_0$  (convective),  $L^2/D$  (diffusion along  $x$ )

Assuming  $\epsilon = h/L \ll 1$  (long channel),

expect  $h^2/D \ll L/U_0 \ll L^2/D$

specifically,  $\frac{h^2/D}{L/U_0} = \frac{h}{L} \left( \frac{h U_0}{D} \right) = \epsilon Pe$ ,  $Pe \equiv \frac{h U_0}{D} = O(1)$

$$\frac{L/U_0}{L^2/D} = \frac{D}{L U_0} = \epsilon / Pe$$

Non-dimensionalize as follows:

$$x = Lx', \quad z = hz', \quad t = \frac{L}{U_0} t', \quad u = U_0 u'$$

Dropping the primes,

$$u(z) = 1 - z^2$$

and

$$\epsilon Pe \left( \frac{\partial C}{\partial t} + \frac{\partial}{\partial x} (uC) \right) = \frac{\partial^2 C}{\partial z^2} + \epsilon^2 \frac{\partial^2 C}{\partial x^2} \quad (*)$$

To leading order in  $\epsilon$ ,

$$\frac{\partial^2 C}{\partial z^2} = 0 \quad \text{with} \quad \frac{\partial C}{\partial z} = 0 \quad (z = \pm 1)$$

$\Rightarrow C \approx C_0(x, t)$  independent of  $z$  (diffusion in  $z$  is complete for  $t = O(1)$ )

This suggests averaging across the channel width:

$$\langle C \rangle = \frac{1}{2} \int_{-1}^1 C(x, z, t) dz$$

Averaging (\*) and using BC  $\frac{\partial C}{\partial z} = 0$  ( $z = \pm 1$ ) yields

$$Pe \left\{ \frac{\partial \langle C \rangle}{\partial t} + \frac{\partial}{\partial x} \langle uC \rangle \right\} = \epsilon \frac{\partial^2 \langle C \rangle}{\partial x^2} \quad (**)$$

Now, evaluate  $\langle uC \rangle$  for  $\epsilon \ll 1$ :

Expand  $C = C_0(x, t) + \epsilon C_1(x, z, t) + \epsilon^2 C_2(x, z, t) + \dots$

To leading order in  $\epsilon$ ,  $\langle uC \rangle = \langle u \rangle C_0 \Rightarrow$  (seen (\*\*))

$$\frac{\partial C_0}{\partial t} + \langle u \rangle \frac{\partial C_0}{\partial x} \approx 0 \quad \text{ie, pure convection of solvent by } \langle u \rangle$$

$$\langle u \rangle = \frac{1}{2} \int_{-1}^1 dz (1 - z^2) = \frac{2}{3} \quad \text{ie, } \frac{\partial C_0}{\partial t} + \frac{2}{3} \frac{\partial C_0}{\partial x} = 0$$

Now, write  $u = \langle u \rangle + \hat{u} \Rightarrow \hat{u} = \frac{1}{3} - z^2$

Then,

$$\langle uC \rangle = \langle u \rangle C_0 + \epsilon \langle u \rangle \langle C_1 \rangle + \epsilon \langle \hat{u} C_1 \rangle + O(\epsilon^2)$$

normalize to 0; otherwise  $\epsilon \langle C_1 \rangle$  could be absorbed into  $C_0$

(\*)  $\Rightarrow$

$$\frac{\partial^2 C_1}{\partial z^2} = Pe \left( \frac{\partial C_0}{\partial t} + u \frac{\partial C_0}{\partial x} \right) = Pe \hat{u} \frac{\partial C_0}{\partial x} \quad \left( \text{since } \frac{\partial C_0}{\partial t} = -\langle u \rangle \frac{\partial C_0}{\partial x} \right)$$

$$\text{ie, } \left. \begin{aligned} \frac{\partial^2 C_1}{\partial z^2} &= Pe \frac{\partial C_0}{\partial x} \left( \frac{1}{3} - z^2 \right) \\ \text{with BC } \frac{\partial C_1}{\partial z} &= 0 \quad (z = \pm 1) \end{aligned} \right\}$$

$$\text{and } \langle C_1 \rangle = \frac{1}{2} \int_{-1}^1 dz C_1 = 0 \quad (\text{normalization})$$

We find  $C_1 = Pe \frac{\partial C_0}{\partial x} \left( \frac{1}{6} z^2 - \frac{z^4}{12} - \frac{21}{540} \right)$

Hence,

$$\langle uC \rangle = \langle u \rangle C_0 + \epsilon \langle \hat{u} C_1 \rangle + O(\epsilon^2)$$

$$\begin{aligned} \langle \hat{u} C_1 \rangle &= Pe \frac{\partial C_0}{\partial x} \frac{1}{2} \int_{-1}^1 dz \left( \frac{1}{3} - z^2 \right) \left( \frac{1}{6} z^2 - \frac{z^4}{12} - \frac{21}{540} \right) \\ &= Pe \frac{\partial C_0}{\partial x} \left( -\frac{8}{945} \right) \end{aligned}$$

Finally, correct to  $O(\epsilon)$ ,  $C_0$  satisfies (from \*\*):

$$Pe \left\{ \frac{\partial C_0}{\partial t} + \frac{2}{3} \frac{\partial C_0}{\partial x} - \frac{8}{945} \epsilon Pe \frac{\partial^2 C_0}{\partial x^2} \right\} = \epsilon \frac{\partial^2 C_0}{\partial x^2}$$

$\Rightarrow$

$$Pe \left( \frac{\partial C_0}{\partial t} + \frac{2}{3} \frac{\partial C_0}{\partial x} \right) = \epsilon \left( 1 + \frac{8}{945} Pe^2 \right) \frac{\partial^2 C_0}{\partial x^2}$$

i.e., the shear-enhanced diffusivity is given by

$$\underline{D + \frac{8}{945} \frac{k^2 U_0^2}{D}}$$