1.63J/2.26J Advanced Fluid Dynamics Spring 2014

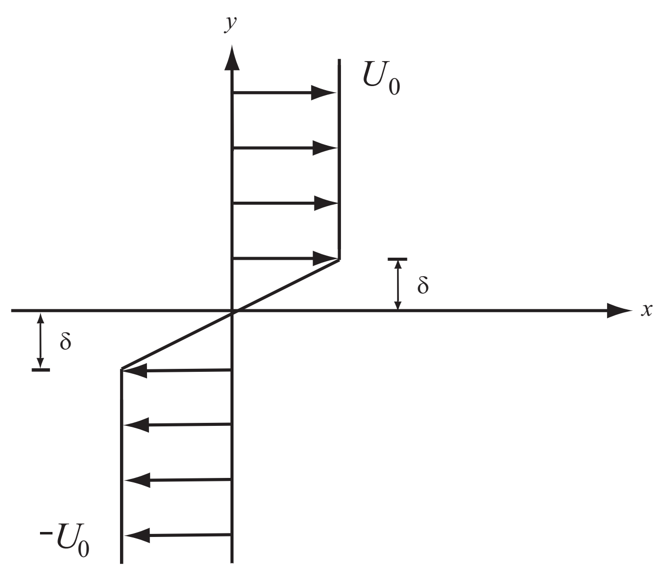
Problem Set No. 6

Out: Wednesday, April 30, 2014
Due: Monday, May 12, 2014 (in class, or before 5:00pm in Room 3-362)
Recitation: 4:30–5:30pm, Wednesday, May 7, 2014 in Room 1-150

Problems 1 and 2 are for Module I (Hydrodynamic Stability); Problems 3 and 4 are for Module II (Biolocomotion)

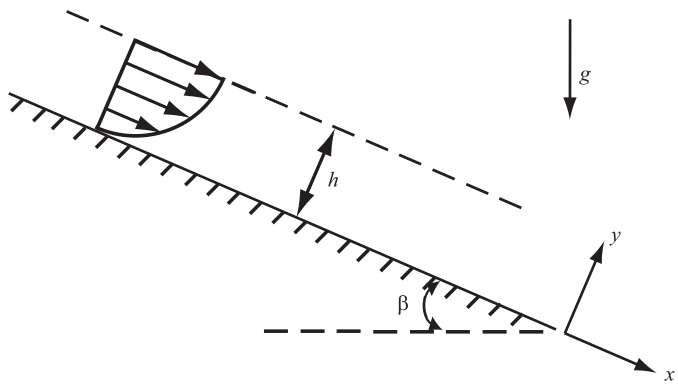
Problem 1

Consider the piecewise-linear-profile model for a shear layer shown below. Examine the stability of this profile to infinitesimal perturbations on the basis of linear inviscid stability theory. Determine the wavenumber of the perturbation corresponding to the maximum growth rate.



Problem 2

Earlier in the term, we examined the stability of a thin film flowing down an inclined plane on the assumption that all inertia terms are negligible, in which case the uniform film was found to be stable. Here, we re-visit the stability of a uniform film, assuming finite Reynolds number.



Recall that the basic flow is given by

$$U(y) = \frac{g\,\sin\beta}{\nu}\left(hy - \frac{y^2}{2}\right), \qquad P(y) = P_0 - \rho g\,\cos\beta\,(y-h), \tag{1}$$

and the depth-average velocity is

$$U_0 = \frac{gh^2\,\sin\beta}{3\nu}. \tag{2}$$

(a) Show from the full Navier–Stokes equations that infinitesimal disturbances are governed by the Orr–Sommerfeld equation

$$\nu\left(\frac{d^2}{dy^2} - k^2\right)^2 \hat{v} = ik\left\{(U-c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} - \frac{d^2U}{dy^2}\hat{v}\right\} \tag{3}$$

for the vertical velocity perturbation $v = \hat{v}\exp\{ik(x-ct)\}$, subject to the no-slip bottom boundary conditions

$$\hat{v}(0) = \frac{d\hat{v}}{dy}(0) = 0. \tag{4}$$

(b) By linearizing the kinematic and the normal- and tangential-stress conditions on the free surface $y = h + \eta(x,t)$, obtain the following two conditions for $\hat{v}$ on $y = h$:

$$k\hat{v} + \frac{1}{k}\frac{d^2\hat{v}}{dy^2} - \frac{d^2U}{dy^2}\frac{\hat{v}}{k(U-c)} = 0 \qquad (y = h), \tag{5a}$$

$$\frac{d^3\hat{v}}{dy^3} - \left\{\frac{ik(U-c)}{\nu} + 3k^2\right\}\frac{d\hat{v}}{dy} + \frac{ik \;\; g\cos\beta}{\nu(U-c)}\hat{v} = 0 \qquad (y = h). \tag{5b}$$

(c) The boundary-value problem consisting of the Orr–Sommerfeld equation (3), subject to the four boundary conditions (4) and (5), is an eigenvalue problem, $c = c_r + ic_i$ being the eigenvalue, for given k real. Using the film depth h as length scale and the velocity on the free surface $U(h)$ as velocity scale, write this boundary-value problem in the dimensionless form:

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 \hat{v} = ikR\left\{(U-c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} - \frac{d^2U}{dy^2}\hat{v}\right\}, \tag{6}$$

$$\hat{v}(0) = \frac{d\hat{v}}{dy}(0) = 0, \tag{7}$$

$$\left(\frac{d^2}{dy^2} + k^2 + \frac{2}{1-c}\right)\hat{v} = 0 \qquad (y = 1), \tag{8a}$$

$$\frac{d^3\hat{v}}{dy^3} - \left\{ikR(1-c) + 3k^2\right\}\frac{d\hat{v}}{dy} + \frac{2ik}{1-c}\cot\beta\hat{v} = 0 \qquad (y = 1), \tag{8b}$$

$$R = \frac{U(h)h}{\nu} = \frac{g\sin\beta\, h^3}{2\nu^2}$$

being the Reyonlds number.

(d) Assume $R = O(1)$ and consider disturbances of very long wavelength ($k \ll 1$ or, in dimensional variables, $\lambda \gg h$). Solve the eigenvalue problem (6)–(8) by a long-wave perturbation expansion:

$$\hat{v} = \hat{v}^{(0)} + ik\,\hat{v}^{(1)} + (ik)^2\hat{v}^{(2)} + \ldots, \tag{9a}$$

$$c = c^{(0)} + ik\,c^{(1)} + (ik)^2c^{(2)} + \ldots\,. \tag{9b}$$

From the $O(1)$ problem, show that

$$\hat{v}^{(0)} = y^2, \quad c^{(0)} = 2. \tag{10}$$

(e) Finally, solve the $O(k)$ problem and show that

$$c^{(1)} = \frac{8}{15}\left(R - \frac{5}{4}\,\text{cot}\beta\right); \tag{11}$$

hence, conclude that instability to long-wave perturbations arises if

$$R > \frac{5}{4}\,\text{cot}\beta \;\Rightarrow\; h^3 > \frac{5\nu^2}{2g}\frac{\text{cos}\beta}{\text{sin}^2\beta}. \tag{12}$$

Interpret this stability condition physically and compare to the case $R = 0$ discussed earlier in the term.

Problem 3

Consider a "snail" driven by an out-of-plane wave as discussed in lecture. For an arbitrary (periodic) wave form $h(x,t)$, compute the efficiency of the crawler in terms of the integrals $I_j = \int_0^1 \frac{dx}{h^j}$.

Problem 4

In lecture we computed the digging speed and efficiency of a "pushmepullyou" burrower. Compute the swimming speed and efficiency of the analogous "pushmepullyou" in a fluid with viscosity μ. You may assume low Reynolds number in this calculation. Which would you expect to be faster, the digger or the swimmer? Why? Which would you expect to be more efficient?

Bonus question: Compute swimming speed and efficiency for a "pushmepullyou" in a fluid, operating in a high Reynolds number regime ($Re \gg 1$).