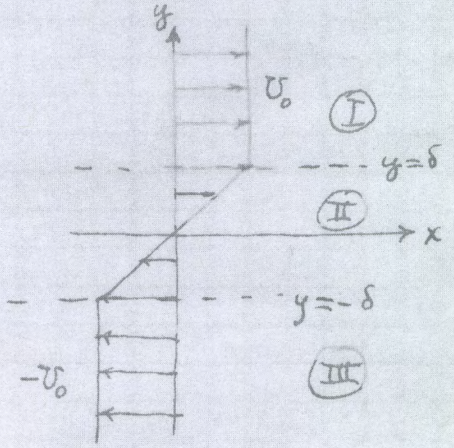
1.63J/2.26J Advanced Fluid ~~Dynamics~~

Problem Set #6

Stability of broken-line shear-layer profile



* Ignore viscosity

* Assume 2-D perturbations, described by the streamfunction $\psi(x,y,t)$

* Introduce normal modes:

$$\psi(x,y,t) = \varphi(y)\exp\{ik(x-ct)\}$$

* $\varphi(y)$ satisfies the Rayleigh equation:

$$(U-c)\left\{\frac{d^2\varphi}{dy^2} - k^2\varphi\right\} - \frac{d^2U}{dy^2}\varphi = 0$$

Here, however, $d^2U/dy^2 = 0$, so the governing equation reduces to

$$\frac{d^2\varphi}{dy^2} - k^2\varphi = 0 \Rightarrow \varphi = C^+e^{ky} + C^-e^{-ky}$$

* Consider 3 regions:

$$\left.\begin{aligned}
&\text{I}: && \varphi_{I} = C_{I}\, e^{-|k|y}\\
&\text{II}: && \varphi_{II} = C_{II}^{+}\, e^{ky} + C_{II}^{-}\, e^{-ky}\\
&\text{III}: && \varphi_{III} = C_{III}\, e^{|k|y}
\end{aligned}\right\}$$

* At the 'interfaces' the following conditions apply (as explained in class)

(i) kinematic $\Rightarrow$ φ is continuous

(ii) dynamic $\Rightarrow$ $(U-c)\dfrac{d\varphi}{dy} - \dfrac{dU}{dy}\varphi$ is continuous

* Apply these conditions

$\boxed{y=\delta}$:

$$C_I e^{-|k|\delta} = C_{II}^+ e^{k\delta} + C_{II}^- e^{-k\delta} \qquad (1)$$

$$(U_0-c)(-|k|)\, C_I e^{-|k|\delta} = (U_0-c)\left\{ C_{II}^+ k e^{k\delta} - C_{II}^- k e^{-k\delta} \right\} - \frac{U_0}{\delta}\left(C_{II}^+ e^{k\delta} + C_{II}^- e^{-k\delta} \right) \qquad (2)$$

$\boxed{y=-\delta}$:

$$C_{III} e^{-|k|\delta} = C_{II}^+ e^{-k\delta} + C_{II}^- e^{k\delta} \qquad (3)$$

$$(-U_0-c)(|k|)\, C_{III} e^{-|k|\delta} = (-U_0-c)\left\{ C_{II}^+ k e^{-k\delta} - C_{II}^- k e^{k\delta} \right\} - \frac{U_0}{\delta}\left(C_{II}^+ e^{-k\delta} + C_{II}^- e^{+k\delta} \right) \qquad (4)$$

Introduce dimensionless wavenumber : $\alpha \equiv k\delta$

" speed : $c = U_0 \hat{c}$

Then, equations (1) – (4) may be written as

$$\begin{bmatrix} e^{-|\alpha|} & -e^{\alpha} & -e^{-\alpha} & 0 \\ -|\alpha|(1-\hat{c})\, e^{-|\alpha|} & e^{\alpha}\left(1-\alpha(1-\hat{c})\right) & e^{-\alpha}\left(1+\alpha(1-\hat{c})\right) & 0 \\ 0 & -e^{-\alpha} & -e^{\alpha} & e^{-|\alpha|} \\ 0 & e^{-\alpha}\left(1+\alpha(1+\hat{c})\right) & e^{\alpha}\left(1-\alpha(1+\hat{c})\right) & -|\alpha|(1+\hat{c})\, e^{-|\alpha|} \end{bmatrix} \times \begin{Bmatrix} C_I \\ C_{II}^+ \\ C_{II}^- \\ C_{III} \end{Bmatrix} = 0$$

* For non-trivial solutions, require

$$\det [\quad] = 0$$

Expanding the determinant, after some algebra, one finds (for $\alpha > 0$) that

$$\boxed{\hat{c}^2 = \frac{(1-2\alpha)^2 - e^{-4\alpha}}{4\alpha^2}}$$

As $\alpha \to \infty$, $\hat{c}^2 \to 1 \Rightarrow$ stable

As $\alpha \to 0$, $\hat{c}^2 \sim \frac{1}{4\alpha^2} \left\{ 1 - 4\alpha + 4\alpha^2 - 1 + 4\alpha - 8\alpha^2 + \dots \right\}$

$$\Rightarrow \hat{c}^2 \sim -1 \Rightarrow \text{unstable}$$

It turns out that $\hat{c}^2 < 0$ for $\alpha < \alpha_{crit} = 0.64$

i.e, instability for $\alpha < \alpha_{crit} = 0.64$

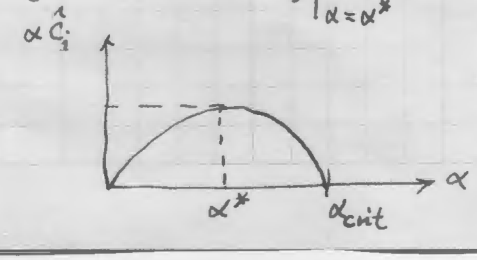
Growth rate: $\alpha \hat{c}_i \qquad (\hat{c} = \hat{c}_r + i\,\hat{c}_i)$

$$\alpha^2 \hat{c}^2 = \frac{1}{4} \left\{ (1-2\alpha)^2 - e^{-4\alpha} \right\}$$

Max. growth rate is realized at $\alpha = \alpha^*$ s.t.

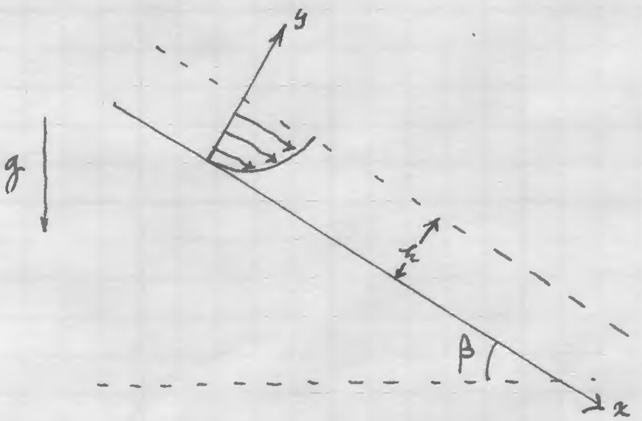
$$\frac{d}{d\alpha} \left\{ (1-2\alpha)^2 - e^{-4\alpha} \right\} \Big|_{\alpha=\alpha^*} = 0 \Rightarrow 1 - 2\alpha^* = e^{-4\alpha^*}$$

Solve numerically for $\alpha^* \neq 0$



Stability of 2-D laminar flow down an inclined plane

* Basic flow (exact solution of the N-S equations)



$$\left.\begin{aligned} \nu \frac{d^2U}{dy^2} + g\sin\beta &= 0 \\ -\frac{1}{\rho}\frac{dP}{dy} - g\cos\beta &= 0 \end{aligned}\right\}$$

$$\underline{\underline{BC}} \begin{cases} U = 0 & (y=0) \\ \dfrac{dU}{dy} = 0 & (y=h) \end{cases}$$

$\therefore$ the basic state is
$$\begin{cases} U(y) = -\dfrac{g\sin\beta}{\nu}\left(\dfrac{y^2}{2} - hy\right) \\ P(y) = P_0 - \rho g\cos\beta\,(y-h) \end{cases}$$

* Perturbation equations

$$u = U(y) + u', \quad v = v', \quad p = P(y) + p'$$

Assume normal modes : $f(x,y,t) = \hat{f}(y)\,e^{ik(x-ct)}$

Assume k real, then $c = c_r + ic_i$ (temporal stability)

Then, after linearizing,

$$\nabla\cdot\underline{u} = 0 \Rightarrow ik\hat{u} + \frac{d\hat{v}}{dy} = 0$$

$$\frac{D\underline{u}}{Dt} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u} \Rightarrow \begin{cases} ik(U-c)\hat{u} + \dfrac{dU}{dy}\hat{v} = -\dfrac{1}{\rho}ik\hat{p} + \nu\left(\dfrac{d^2}{dy^2} - k^2\right)\hat{u} \\ ik(U-c)\hat{v} = -\dfrac{1}{\rho}\dfrac{d\hat{p}}{dy} + \nu\left(\dfrac{d^2}{dy^2} - k^2\right)\hat{v} \end{cases}$$

$$\Rightarrow \nu\left(\frac{d^2}{dy^2} - k^2\right)^2\hat{v} = ik\left\{(U-c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} - \frac{d^2U}{dy^2}\hat{v}\right\}$$

Orr-Sommerfeld eq'n

In dimensionless variables,

$$R = \frac{U_0 h}{\nu}, \quad U_0 = \frac{g\sin\beta}{2\nu} h^2 \left(= U(y=h)\right), \quad R = \frac{g\sin\beta\, h^3}{2\nu^2}$$

$$\left(\frac{d^2}{dy^2} - k^2\right)^2 \hat{v} = ikR\left\{ (U-c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} - \frac{d^2 U}{dy^2}\hat{v} \right\}$$

* Boundary conditions

On the bottom $(y=0)$ $\quad \hat{u}(0) = \hat{v}(0) \Rightarrow \hat{v}(0) = \left.\frac{d\hat{v}}{dy}\right|_{y=0} = 0$

At the free surface

(i) kinematic BC

(ii) normal stress balance (ignore surface tension)

(iii) tangential stress vanishes

(i) if free surface is at $y = h + \zeta(x,t)$, then

$$\zeta_t + u\zeta_x = v \Rightarrow \zeta_t + U(h)\zeta_x \approx v$$

$$\Rightarrow ik\left(U_0 - c\right)\hat{\zeta} - \hat{v} = 0 \quad (y=h)$$

(ii) $\quad -P_0 = -p + 2\mu \frac{\partial v}{\partial y} \qquad p = P + p'$

$$p\big|_{y=h+\zeta} = P\big|_{y=h+\zeta} + p'\big|_{y=h+\zeta} \Rightarrow P\big|_{y=h+\zeta} \approx P_0 - \rho g\cos\beta\,\zeta + p'\big|_{y=h}$$

$$\therefore \quad \rho g\cos\beta\,\hat{\zeta} - \hat{p} + 2\rho\nu\frac{d\hat{v}}{dy} = 0 \quad (y=h)$$

(iii) $\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} = 0$ at $y = h + \zeta$

$$\Rightarrow \zeta \left.\frac{d^2 U}{dy^2}\right|_0 + \frac{\partial u'}{\partial y} + \frac{\partial v'}{\partial x} = 0 \quad \text{at } y = h$$

Using $\dfrac{\partial u'}{\partial x} + \dfrac{\partial v'}{\partial y} = 0 \Rightarrow ik\hat{u} + \dfrac{d\hat{v}}{dy} = 0 \Rightarrow \hat{u} = -\dfrac{d/dy\, \hat{v}}{ik}$

$$\therefore \left.\frac{d^2 U}{dy^2}\right|_0 \hat{\zeta} + \frac{d\hat{u}}{dy} + ik\hat{v} = 0$$

$$\Rightarrow \left(\text{since } \hat{\zeta} = \frac{\hat{v}}{ik(U_0 - c)} \right)$$

$$\left.\frac{d^2 U}{dy^2}\right|_0 \frac{\hat{v}}{ik(U_0 - c)} - \frac{d^2/dy^2\, \hat{v}}{ik} + ik\hat{v} = 0 \quad (y = h) \quad (*)$$

Also, from (ii), substituting $\hat{\zeta} = \dfrac{\hat{v}}{ik(U_0 - c)}$,

$$\frac{\hat{p}}{\rho} = -(U - c)\hat{u} + \frac{i}{k}\frac{dU}{dy}\hat{v} - \frac{i\nu}{k}\left(\frac{d^2}{dy^2} - k^2\right)\hat{u}$$

$$= \frac{(U - c)}{ik}\frac{d\hat{v}}{dy} + \frac{i}{k}\frac{dU}{dy}\hat{v} + \frac{\nu}{k^2}\left(\frac{d^2}{dy^2} - k^2\right)\frac{d\hat{v}}{dy}$$

Hence, (ii) $\Rightarrow$

$$\frac{\cancel{\rho} g\cos\beta}{ik(U_0 - c)}\hat{v} - \frac{(U - c)}{ik}\frac{d\hat{v}}{dy} - \frac{i}{k}\frac{dU}{dy}\hat{v} - \frac{\nu}{k^2}\left(\frac{d^2}{dy^2} - k^2\right)\frac{d\hat{v}}{dy} + 2\nu\frac{d\hat{v}}{dy} = 0$$

$$\Rightarrow -\frac{\nu}{k^2}\frac{d^3\hat{v}}{dy^3} - \left[\frac{U - c}{ik} - 3\nu\right]\frac{d\hat{v}}{dy} - \left(\frac{i}{k}\underbrace{\cancel{\frac{dU}{dy}}}_{0 \text{ at } y = h} - \frac{g\cos\beta}{ik(U_0 - c)}\right)\hat{v} = 0$$

$$\Rightarrow \frac{d^3\hat{v}}{dy^3} - \left[\frac{ik(U - c)}{\nu} + 3k^2\right]\frac{d\hat{v}}{dy} - \frac{k^2 g\cos\beta}{ik\nu(U_0 - c)}\hat{v} = 0 \quad (y = h) \quad (**)$$

* So, have a BVP consisting of O-S eqn (4th order) subject to 4 BC for $\hat{v}$ (2 at $y=0$ and 2 at $y=h$)

* In dimensionless variables $U(y) = 2y - y^2$

O-S $\Rightarrow$ $$\left(\frac{d^2}{dy^2} - k^2\right)^2 \hat{v} = ikR\left\{(U-c)\left(\frac{d^2}{dy^2} - k^2\right)\hat{v} - \frac{d^2U}{dy^2}\hat{v}\right\}$$

BC at $y=0$: $\hat{v} = \frac{d\hat{v}}{dy} = 0 \quad (y=0)$

BC at $y=1$: $$\begin{cases} \left(\frac{d^2}{dy^2} + k^2 + \frac{2}{1-c}\right)\hat{v} = 0 & (y=1) \\ \frac{d^3\hat{v}}{dy^3} - \left\{ikR(1-c) + 3k^2\right\}\frac{d\hat{v}}{dy} + \frac{2ik}{1-c}\cot\beta = 0 & (y=1) \end{cases}$$

* Since this is a homogeneous BVP, in general we have the trivial solution unless, for any given k, c has a special value(s) ie, we are dealing with EVP, $c(k;R)$ being the eigenvalue parameter.

* Long-wave expansion

Assume $R \sim O(1)$, k small (long waves)

Expand $$\begin{cases} \hat{v} = \hat{v}^{(0)} + ik\hat{v}^{(1)} + (ik)^2\hat{v}^{(2)} + \dots \\ c = c^{(0)} + ikc^{(1)} + (ik)^2 c^{(2)} + \dots \end{cases}$$

Stability is decided by sign of kc_i ($\mathrm{Im}\{c\}$)

$$e^{ik(x-ct)} = e^{ik(x-c_r t)}\, e^{kc_i t} \qquad kc_i = \cancelto{0}{kc_i^{(0)}} + \underline{k^2 c_r^{(1)}} + \dots$$

$\underline{O(1)}$:

$$\left.\begin{aligned} &\frac{d^4 \hat{v}^{(0)}}{dy^4} = 0 \\ &\hat{v}^{(0)} = \frac{d\hat{v}^{(0)}}{dy}(0) = 0 \\ &\left(\frac{d^2}{dy^2} + \frac{2}{1-c}\right)\hat{v}^{(0)} = 0 \quad (y=1) \\ &\frac{d^3\hat{v}^{(0)}}{dy^3} = 0 \quad (y=1) \end{aligned}\right\}$$

$$\hat{v}^{(0)} = By^2 + Cy^3 \qquad BC \Rightarrow C = 0 \text{ and}$$

$$2B + \frac{2B}{1-c^{(0)}} = 0 \Rightarrow \underline{c^{(0)} = 2}$$

Normalize by taking $\hat{v} = 1$ at $y = 1$

Hence $\hat{v}^{(0)} = 1$, $\underline{\hat{v}^{(k)} = 0 \ (y=1)}$

$$\therefore \quad \underline{\hat{v}^{(0)} = y^2, \quad c^{(0)} = 2}$$

$\underline{O(k)}$:

$$\frac{d^4\hat{v}^{(1)}}{dy^4} = R\left\{(U - c^{(0)})\frac{d^2}{dy^2}\hat{v}^{(0)} - \frac{d^2U}{dy^2}\hat{v}^{(0)}\right\}$$

$$= R\left\{(2y - y^2 - 2)2 - (-2)y^2\right\} = R\left[4y - 4\right]$$

and $\hat{v}^{(1)} = \frac{d}{dy}\hat{v}^{(1)} = 0 \quad (y=0)$

$$\Rightarrow \hat{v}^{(1)} = Ky^2 + Ly^3 - \frac{Ry^4}{6} + \frac{Ry^5}{30}$$

Surface BC's $\Rightarrow$ Expand $\frac{1}{1-c} \approx \frac{1}{1-c^{(0)} - ikc^{(1)} + \dots} \approx \frac{1}{(1-c^{(0)})\left(1 - \frac{ikc^{(1)}}{1-c^{(0)}}\right)}$

$$\approx \frac{1}{1-c^{(0)}}\left(1 + \frac{ikc^{(1)}}{1-c^{(0)}} + \dots\right) = -1\left(1 - ikc^{(1)} + \dots\right)$$

$$= -1 + ikc^{(1)} + \dots$$

Hence,

$$\left.\frac{d^2\hat{v}^{(1)}}{dy^2}\right|_{y=1} + 2c^{(1)}\cancelto{1}{\hat{v}^{(0)}(1)} - 2\cancelto{0}{\hat{v}^{(1)}(1)} = 0$$

$$\Rightarrow \frac{d^2\hat{v}^{(1)}}{dy^2} + 2c^{(1)} = 0 \quad (y=1)$$

Also,

$$\frac{d^3\hat{v}^{(1)}}{dy^3} + R\frac{d\hat{v}^{(0)}}{dy} - 2\cot\beta = 0 \quad (y=1)$$

and $\hat{v}^{(1)} = 0 \quad (y=1)$ Normalization

By substituting the known form of $\hat{v}^{(1)}(y)$,

$$\left.\begin{aligned} K + L - \frac{R}{6} + \frac{R}{30} &= 0 \\ 2K + 6L - \frac{4}{3}R + 2c^{(1)} &= 0 \\ 6L - 2\cot\beta &= 0 \end{aligned}\right\} \rightarrow L = \frac{1}{3}\cot\beta$$

Solve for K and $c^{(1)}$ $\Rightarrow$ $\underline{c^{(1)} = \frac{8}{15}\left(R - \frac{5}{4}\cot\beta\right)}$

* Therefore $\underline{c_i = k\frac{8}{15}\left(R - \frac{5}{4}\cot\beta\right) + \cdots}$

i.e., instability if $\underline{R > \frac{5}{4}\cot\beta}$

$$\Rightarrow h^3 > \frac{5\nu^2}{2g}\frac{\cos\beta}{\sin^2\beta}$$ Restricts film thickness

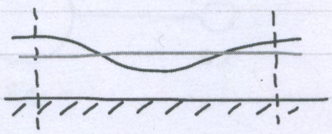
As $\beta\uparrow$ instability is easier to induce

$\left[e^{ik(x-c_r t)} \quad e^{kc_i t} \quad kc_i > 0 \Rightarrow \text{instability} \right]$

Problem 3 Biolocomotion

Homework solutions

Snail efficiency: $\eta = \dfrac{E \cdot V_s}{\Phi}$



E = Force to drag snail in flat position (red)

$= \mu V_s \lambda / H$

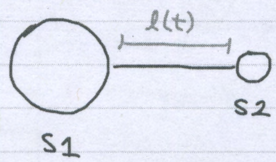
$$\Phi = \int_0^\lambda \int_0^h 2\mu \left[\overset{\text{small}}{\cancel{\left(\frac{\partial u}{\partial x}\right)^2}} + \cancel{\left(\frac{\partial v}{\partial y}\right)^2} + \frac{1}{2}\left(\cancel{\frac{\partial v}{\partial x}} + \frac{\partial \hat{u}}{\partial \hat{y}}\right)^2 \right] d\hat{x}\, d\hat{y}$$

$$= \int_0^1 \int_0^h \cancel{2}\mu \frac{1}{\cancel{2}} \left(\frac{V_w}{H}\right)^2 \left(\frac{\partial u}{\partial y}\right)^2 \lambda H \, dx\, dy$$

$$\eta = \frac{\cancel{\mu} V_s^2 \cancel{\lambda} \cancel{H}}{\cancel{H} \cancel{\mu} V_w^2 \cancel{\lambda}} \int_0^1 \int_0^h \underbrace{\left(\frac{\partial u}{\partial y}\right)^2}_{\text{computed in lecture}} dx\, dy$$

Problem 4 Biolocomotion

Pushmepullyou HW



Kinematics

$S1: \; R_1 = a_0 + \delta_a \cos(\omega t)$

$S2: \; R_2 = a_0 - \delta_a \cos(\omega t)$

$l(t) = l_0 - \delta_l \sin(\omega t)$

Drag Force

$$\underline{F}_i = 6\pi\mu R_i(t) V_i(t)$$

Equilibrium

$$F_1 + F_2 = 0$$

$$\left.\begin{aligned} &6\pi\mu R_1(t) V_1(t) + 6\pi\mu R_2(t) V_2(t) = 0 \\ \text{kinematic constraint} \rightarrow\; &V_2(t) = V_1(t) + \dot{l} \end{aligned}\right\} \text{solve for } V_1(t)$$

Efficiency: $\eta = \dfrac{\underline{F}\cdot\underline{V}}{\Phi}$ $\qquad \underline{F}\cdot\underline{V} = \underbrace{2}_{\text{two spheres}}(6\pi\mu a_0)\underbrace{\bar{V}}_{\text{average swimming velocity}}{}^2$

$$\Phi = \Phi_{translation} + \Phi_{dilation}$$

$$\Phi_{translation} = \overset{\text{two spheres}}{2} \int_{cycle} 6\pi\mu R_i(t) V_i^2(t)\, dt$$

$$\Phi_{dilation} = 2\int_{cycle} \sigma \cdot \dot{\forall}\, dt = 2\int_{cycle} \frac{4}{3}\mu \frac{\dot{\forall}^2}{\forall}\, dt$$