1.63J / 2.26J Advanced Fluid Dynamics

Take-Home Quiz Solutions

Monday, April 7, 2014

Problem 1 Vorticity transport

(a) If $Re \gg 1$, the transport is predominantly convective and vorticity, generated at the gauze will be mostly convected downstream. There will also be some vorticity upstream due to diffusion, but as we move upstream the vorticity magnitude will decay fast. Thus, we expect

$$x_{up} \ll x_{down}$$

(b) In this 2D flow, the vorticity transport equation for $\underset{\sim}{\zeta} = (0, 0, \zeta)$ takes the form

$$\underline{q} \cdot \nabla \zeta = \nu \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)$$

Expect
$$\underline{q} \approx U \hat{\underset{\sim}{e}}_x$$

$$\therefore \quad U \frac{\partial \zeta}{\partial x} = \nu \left(\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} \right)$$

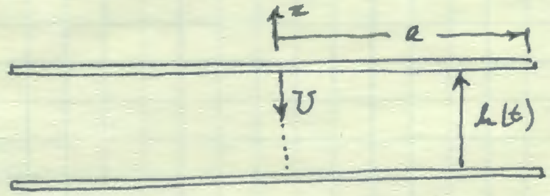
Downstream $\quad x \sim x_{down}, \quad y \sim L \quad$ where $x_{down} \gg L$

$$\Rightarrow \quad U / x_{down} \sim \nu / L^2 \quad \Rightarrow \quad \underline{\underline{x_{down} \sim Re \cdot L}}$$

Upstream $\quad x \sim x_{up}, \quad y \sim L \quad$ where $x_{up} \ll L$

$$\Rightarrow \quad \frac{U}{x_{up}} \sim \frac{\nu}{x_{up}^2} \quad \Rightarrow \quad \underline{\underline{x_{up} \sim \frac{\nu}{U} = L / Re}}$$

Problem 2 Axisymmetric flow due to falling disk



(a) By symmetry, the flow is axi-symmetric:

$$\underset{\sim}{q} = u\,\hat{\underset{\sim}{e}}_r + w\,\hat{\underset{\sim}{e}}_z$$

with $\quad w(z=h) = -U = \dfrac{dh}{dt}$

- Continuity : $\dfrac{\partial w}{\partial z} + \dfrac{1}{r}\dfrac{\partial}{\partial r}(ur) = 0 \Rightarrow \dfrac{W}{u} \sim \dfrac{h}{a} \ll 1$

ie, the flow is essentially radial

- Radial momentum: typical inertia $\sim \rho\, u^2/a$

dominant viscous $\sim \mu \dfrac{u}{h^2}$

$$\frac{\text{inertia}}{\text{viscous}} \sim \frac{u h^2}{\nu a} \sim \frac{Uh}{\nu} \quad \text{ie, assuming } \underline{\underline{Re = \frac{Uh}{\nu} \ll 1}}$$

neglect inertia terms

$$\therefore \quad \left.\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu \frac{\partial^2 u}{\partial z^2} &= 0 \\ u = 0 \quad (z &= 0, h) \end{aligned}\right\}$$

- Axial momentum : $\dfrac{1}{\rho}\dfrac{\partial p}{\partial z} \sim \nu \dfrac{\partial^2 w}{\partial z^2} \rightarrow \dfrac{1}{\rho}\dfrac{\partial p}{\partial z} \sim \dfrac{\nu U}{h^2}$

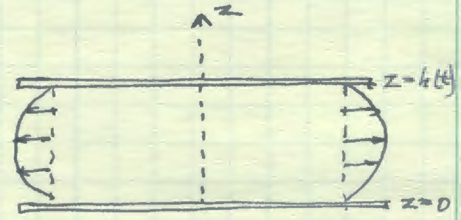
Hence, $\quad \dfrac{1}{\rho}\dfrac{\partial p}{\partial r} \sim \nu \dfrac{U}{h^2}\dfrac{a}{h} \gg \dfrac{1}{\rho}\dfrac{\partial p}{\partial z}$

ie, $\quad p = p(r)$, to a first approximation,

and $\dfrac{\partial p}{\partial r}$ is independent of z

$$\therefore \quad \left.\begin{aligned} -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\frac{\partial^2 u}{\partial z^2} &= 0 \\ u = 0 \; (z &= 0, h) \end{aligned}\right\} \Rightarrow \underline{\underline{u = \frac{1}{2\mu}\frac{\partial p}{\partial r}(z-h)z}}$$

(b) Mass conservation now says:



$$\frac{d}{dt}\left(\rho \pi r^2 h\right) = \rho\, 2\pi r \int_0^h u\, dz$$

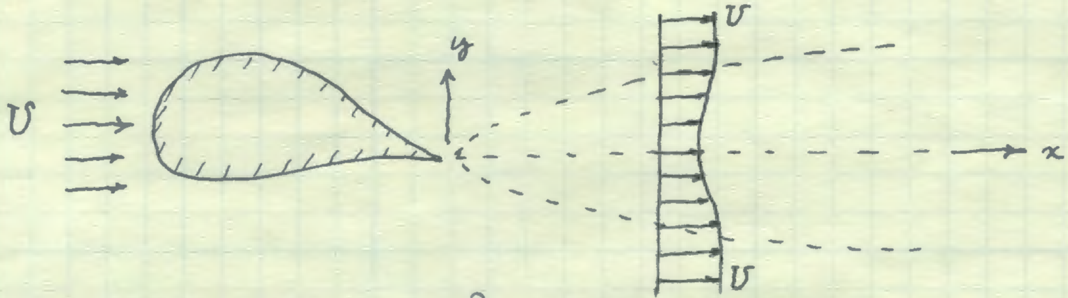
$$\Rightarrow \quad r \underbrace{\frac{dh}{dt}}_{-U} = \frac{1}{6\mu}\frac{\partial p}{\partial r} h^3 \quad \rightarrow \quad \boxed{\frac{\partial p}{\partial r} = -6\mu \frac{Ur}{h^3}}$$

Hence, upon integrating in r and setting $p(r=a)=0$,

$$\boxed{p = -3\mu \frac{U}{h^3}\left(r^2 - a^2\right)}$$

Finally, $\quad \text{Drag} = 2\pi \int_0^a p(r)\, r\, dr = \boxed{\frac{3\pi}{2}\mu \frac{U a^4}{h^3}}$

Problem 3 2D laminar far wake



Assuming $\frac{\partial}{\partial x} \ll \frac{\partial}{\partial y}$, the (steady) momentum BL eq'n

$$\Rightarrow \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x}\Big|_{outer} + \nu \frac{\partial^2 u}{\partial y^2}$$

(a) In the far wake,

$$\underline{q} \approx (U - u_1)\, \hat{\underline{e}}_x, \quad u_1/U \ll 1$$

so BL eq'n may be approximated as $\quad \partial p/\partial x\big|_{outer} = 0$

$$U \frac{\partial u_1}{\partial x} = \nu \frac{\partial^2 u_1}{\partial y^2} \quad (*) \qquad \text{with } \underline{\text{BC}} \quad \frac{\partial u_1}{\partial y}\Big|_0 = 0, \quad u_1\Big|_{\pm\infty} = 0$$

(b) Upon integrating (*) in y $\Rightarrow$ $U \frac{d}{dx} \int_{-\infty}^{\infty} u_1\, dy = \nu \frac{\partial u_1}{\partial y}\Big|_{-\infty}^{\infty} = 0$

$$\Rightarrow \quad \int_{-\infty}^{\infty} u_1\, dy = \text{const} \quad (**)$$

(c) Look for similarity solution of (*), subject to (**)

$$x \rightarrow \lambda x, \quad y \rightarrow \lambda^{\alpha} y, \quad u_1 \longrightarrow \lambda^{\beta} u_1$$

$$(**) \Rightarrow \alpha + \beta = 0, \quad (*) \Rightarrow 2\alpha = 1 \quad \therefore \alpha = -\beta = \tfrac{1}{2}$$

Hence, $\quad \eta = A\, y/x^{1/2}, \quad u_1 = \frac{C}{x^{1/2}} f(\eta)$

$$(*) \Rightarrow \quad -\tfrac{1}{2} U \left(f + \eta \frac{df}{d\eta} \right) = \nu A^2 \frac{d^2 f}{d\eta^2} \qquad \underline{\text{Choose}} \quad A^2 = U/2\nu$$

Then, $\quad \frac{d^2 f}{d\eta^2} = -\frac{d}{d\eta}(\eta f) \Rightarrow \eta f = -\frac{df}{d\eta} \Rightarrow f = e^{-\eta^2/2}$

$$\Rightarrow \quad u_1 = \frac{C}{x^{1/2}} \exp\left(-U y^2/4\nu x \right)$$

The constant C may be related to the drag on the body (see Batchelor, pp. 348–352).