1.63J / 2.26J Advanced Fluid Dynamics

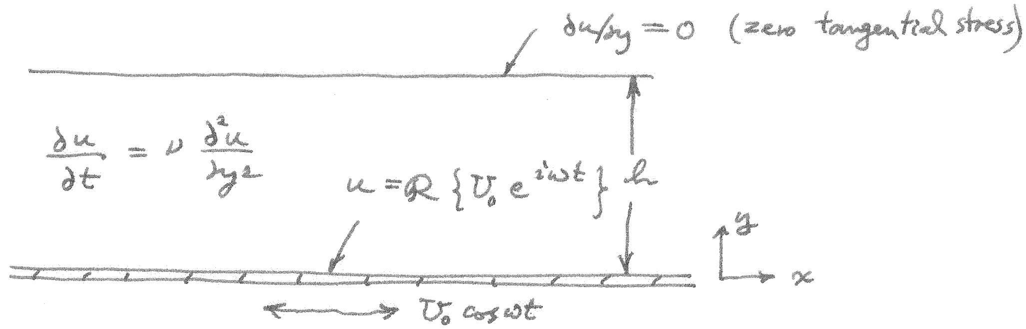
Take-Home Exam; Solutions

Wednesday, May 14, 2014

Problem 1

By symmetry, $\underline{q} = u(y,t)\,\hat{\underline{e}}_x$

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$$



Solve by letting $\quad u = \mathcal{R}\{f(y)\, e^{i\omega t}\}$,

where $f(y)$ satisfies:

$$\frac{d^2 f}{dy^2} - \frac{i\omega}{\nu} f = 0 \quad (0 < y < h)$$

$$\left.\begin{aligned} f(0) &= U_0 \\ \frac{df}{dy}(h) &= 0 \end{aligned}\right\}$$

f is a linear combination of $\exp\left\{\pm \frac{(1+i)}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}\, y\right\}$

In compact form,

$$f = U_0 \frac{\cosh k(y-h)}{\cosh kh}, \quad \underline{\underline{k = \frac{1+i}{\sqrt{2}}\sqrt{\frac{\omega}{\nu}}}}$$

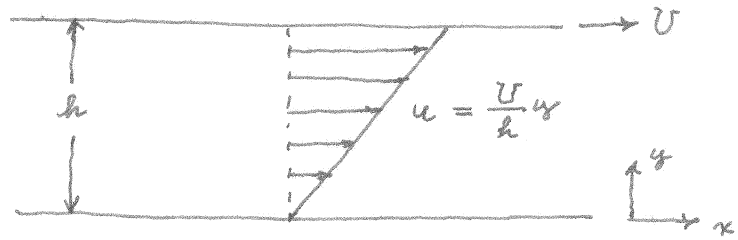
$$\Rightarrow \quad \underline{\underline{u = \mathcal{R}\left\{ U_0 \frac{\cosh k(y-h)}{\cosh kh}\, e^{i\omega t}\right\}}}$$

As $\underline{\underline{\omega h^2/\nu \gg 1}}$ $(|kh| \gg 1)$: the motion is confined in (thin) Stokes layer $O((\nu/\omega)^{1/2})$ close to the plate

$\underline{\underline{\omega h^2/\nu \ll 1}}$ $(|kh| \ll 1)$: $\cosh k(y-h) \sim \cosh kh \sim 1$

the fluid oscillates with the plate, essentially as a rigid body

Problem 2



In general, the dye concentration is governed by the convection - diffusion equation:

$$\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}(uC) = D\left(\frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2}\right)$$

For $t \gg h/U$, we expect the horizontal extent $= O(L)$ of C to be much larger than h: $\underline{h/L = \epsilon \ll 1}$

Non-dimensionalize: $x \to Lx$, $y \to hy$, $u \to Uu$, $t \to \frac{L}{U} t$

$$\Rightarrow \quad \left.\begin{aligned} \epsilon Pe \left(\frac{\partial C}{\partial t} + \frac{\partial}{\partial x}(uC)\right) &= \frac{\partial^2 C}{\partial y^2} + \epsilon^2 \frac{\partial^2 C}{\partial x^2} \\ \frac{\partial C}{\partial y} &= 0 \qquad (y = 0, 1) \end{aligned}\right\}$$

where $u = y$ and $\underline{Pe = \frac{Uh}{D}}$ is assumed $O(1)$

To investigate the long-time evolution of C we use <u>multiple-time-scale analysis</u>

$$t_{\substack{lat\\diff.}} \sim \frac{h^2}{D} \Rightarrow \underline{\tau = t/\epsilon}, \quad t_{\substack{long\\diff.}} \sim \frac{L^2}{D} \Rightarrow \underline{T = \epsilon t}$$

$$C(x, y, t) \Rightarrow C(x, y; \tau, t, T)$$

$$\frac{\partial}{\partial t} \to \frac{1}{\epsilon}\frac{\partial}{\partial \tau} + \frac{\partial}{\partial t} + \epsilon \frac{\partial}{\partial T}$$

Therefore, C satisfies

$$\left.\begin{aligned} Pe\,\frac{\partial C}{\partial \tau} + \epsilon Pe\left(\frac{\partial C}{\partial t} + u(y)\frac{\partial C}{\partial x}\right) + \epsilon^2 Pe\,\frac{\partial C}{\partial T} &= \frac{\partial^2 C}{\partial y^2} + \epsilon^2 \frac{\partial C}{\partial x^2} \\ \frac{\partial C}{\partial y} = 0 \quad (y = 0, 1) & \end{aligned}\right\}$$

and IC : $\quad C = C_0 \quad (\tau = 0)$

<u>$O(1)$</u> :

$$\left.\begin{aligned} Pe\,\frac{\partial C}{\partial \tau} &= \frac{\partial^2 C}{\partial y^2} \\ \frac{\partial C}{\partial y} &= 0 \quad (y = 0, 1) \\ C &= C_0 \quad (\tau = 0) \end{aligned}\right\}$$

Lateral diffusion only (May solve by separation of variables)

As $\tau \gg 1$ (ie, $t_{dim} \gg h^2/D \sim h/U$), $\quad \underline{C \longrightarrow C_1(x; t, T)}$

Expand : $\quad C = C_1(x; t, T) + \epsilon C_2(x, y; t, T) + \epsilon^2 C_3 + \dots$

<u>$O(\epsilon)$</u> :

$$\left.\begin{aligned} \frac{\partial^2 C_2}{\partial y^2} &= Pe\left(\frac{\partial C_1}{\partial t} + u(y)\frac{\partial C_1}{\partial x}\right) \\ \frac{\partial C_2}{\partial y} &= 0 \quad (y = 0, 1) \end{aligned}\right\}$$

To ensure that this forced BVP has solution, the RHS must satisfy the <u>solvability condition</u>:

$$\int_0^1 (RHS)\,dy = 0 \quad \Rightarrow \quad \frac{\partial C_1}{\partial t} + \langle u \rangle \frac{\partial C_1}{\partial x} = 0$$

$$\Rightarrow \quad \underline{C_1 = C_1(\xi; T), \quad \xi = x - \langle u \rangle t}$$

where $\quad \langle u \rangle = \int_0^1 u\,dy = \int_0^1 y\,dy = \frac{1}{2}$

May now solve for $C_2(x, y; t, T)$:

$$\frac{\partial^2 C_2}{\partial y^2} = Pe\left(\underbrace{\cancel{\frac{\partial C_1}{\partial t}}}_{-\langle u\rangle \frac{\partial C_1}{\partial x} = -\frac{1}{2}\frac{\partial C_1}{\partial \xi}} + u(y)\overbrace{\cancel{\frac{\partial C_1}{\partial x}}}^{\partial C_1/\partial \xi}\right) = Pe\,\tilde{u}(y)\frac{\partial C_1}{\partial \xi} \qquad \left(\tilde{u} = y - \tfrac{1}{2}\right)$$

$\therefore$ $C_2 = Pe \dfrac{\partial C_1}{\partial \xi} B(y)$, where $B(y)$ satisfies

$$\left.\begin{aligned} \frac{d^2 B}{dy^2} &= y - \frac{1}{2} \\ \frac{dB}{dy} &= 0 \quad (y = 0, 1) \end{aligned}\right\} \Rightarrow B(y) = \frac{y^3}{6} - \frac{y^2}{4} + \frac{1}{24}$$

s.t $\langle B\rangle = 0 \Rightarrow \langle C\rangle = C_1$

i.e, $$\underline{\underline{C_2 = Pe \frac{\partial C_1}{\partial \xi} \frac{(4y^3 - 6y^2 + 1)}{24}}}$$

$\underline{\underline{O(\epsilon^2)}}$:

$$\frac{\partial^2 C_3}{\partial y^2} = Pe \frac{\partial C_1}{\partial T} - \frac{\partial^2 C_1}{\partial \xi^2} + Pe\left(\underbrace{\cancel{\frac{\partial C_2}{\partial t}}}_{-Pe\langle u\rangle \frac{\partial^2 C_1}{\partial \xi^2} B(y)} + u(y)\overbrace{\cancel{\frac{\partial C_2}{\partial x}}}^{Pe \frac{\partial^2 C_1}{\partial \xi^2} B(y)}\right)$$

$$\Rightarrow \left.\begin{aligned} \frac{\partial^2 C_3}{\partial y^2} &= Pe \frac{\partial C_1}{\partial T} - \left(1 - Pe^2 \tilde{u}(y) B(y)\right)\frac{\partial^2 C_1}{\partial \xi^2} \\ \frac{\partial C_3}{\partial y} &= 0 \quad (y = 0, 1) \end{aligned}\right\}$$

Again, applying solvability condition,

$$\int_0^1 dy\, RHS = 0 \Rightarrow \underline{\underline{Pe \frac{\partial C_1}{\partial T} = E \frac{\partial^2 C_1}{\partial \xi^2}}},$$

where $$\underline{\underline{E = 1 - Pe^2 \langle \tilde{u} B\rangle}}$$

$$\langle \tilde{u} B\rangle = \int_0^1 dy \left(y - \frac{1}{2}\right)\left(\frac{y^3}{6} - \frac{y^2}{4} + \frac{1}{24}\right) = -\frac{1}{120}$$

Finally, for plane Couette flow,

(effective diffusivity) $$\boxed{E = 1 + \frac{Pe^2}{120}}$$ (in dimensionless form)

or, in dimensional variables, $C_1 = \langle C \rangle$ satisfies

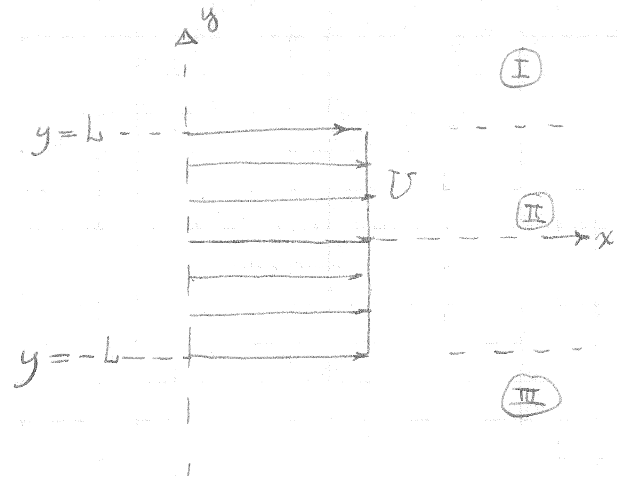
$$\frac{\partial}{\partial t}\langle C \rangle + \overset{\frac{1}{2}U}{\langle u \rangle}\frac{\partial \langle C \rangle}{\partial x} = DE \frac{\partial^2}{\partial x^2}\langle C \rangle$$

where $$DE = \overbrace{D}^{\text{molecular}} + \overbrace{\frac{1}{120}\frac{U^2 h^2}{D}}^{\text{due to shear}}$$ effective diffusivity

For $t_{dim} \gg h/U \sim h^2/D$,

The depth-averaged concentration $\langle C \rangle$ will translate (convect) with the average flow speed $\frac{1}{2}U = \langle u \rangle$ and diffuse in streamwise direction with $D_{eff} = D + \frac{1}{120}\frac{U^2h^2}{D}$

Problem 3 Stability of 'top-hat' jet profile



For 2-D inviscid perturbations, streamfunction

$$\psi = f(y)\, e^{ik(x-ct)},$$

where $f(y)$ satisfies the Rayleigh equation:

$$\left(U(y)-c\right)\left\{\frac{d^2f}{dy^2} - k^2 f\right\} - \frac{d^2U}{dy^2} f = 0$$

Divide the flow into 3 regions:

$$\text{I} : \quad \frac{d^2f}{dy^2} - k^2 f = 0 \Rightarrow f = C_I e^{-|k|y} \quad (f\to 0, y\to\infty)$$

$$\text{II} : \quad \frac{d^2f}{dy^2} - k^2 f = 0 \Rightarrow f = C_{II}^s \sinh ky + C_{II}^c \cosh ky$$

$$\text{III} : \quad \frac{d^2f}{dy^2} - k^2 f = 0 \Rightarrow f = C_{III} e^{|k|y} \quad (f\to 0, y\to -\infty)$$

At the 'interface' $y=L$ kinematic $\Rightarrow \left[\frac{f}{U-c}\right]_{y=L^-}^{y=L^+} = 0$

$$\Rightarrow \frac{C_I e^{-|k|L}}{-c} = \frac{1}{(U-c)}\left\{C_{II}^s \sinh kL + C_{II}^c \cosh kL\right\}$$

Dynamic (pressure) $\Rightarrow \left[(U-c)\frac{df}{dy} - \frac{dU}{dy} f\right]_{y=L^-}^{y=L^+} = 0$

$$\Rightarrow c|k| C_I e^{-|k|L} = (U-c)k\left\{C_{II}^s \cosh kL + C_{II}^c \sinh kL\right\}$$

Similarly, at the 'interface' $y = L$:

kinematic $\Rightarrow \left\{-\mathcal{C}_{II}^{s} \sinh kL + \mathcal{C}_{II}^{c} \cosh kL\right\}\dfrac{1}{U-c} = \dfrac{\mathcal{C}_{III}\, e^{-|k|L}}{-c}$

dynamic $\Rightarrow (U-c)\,k\left\{\mathcal{C}_{II}^{s}\cosh kL - \mathcal{C}_{II}^{c}\sinh kL\right\} = -c|k|\,\mathcal{C}_{III}\, e^{-|k|L}$

Therefore, $\mathcal{C}_{I}$, $\mathcal{C}_{II}^{s}$, $\mathcal{C}_{II}^{c}$, $\mathcal{C}_{III}$ satisfy the linear homogeneous system:

$$\begin{bmatrix} \dfrac{(U-c)}{c}e^{-|k|L} & \sinh kL & \cosh kL & 0 \\ -c|k|\,e^{-|k|L} & (U-c)k\cosh kL & (U-c)k\sinh kL & 0 \\ 0 & -\sinh kL & \cosh kL & \dfrac{(U-c)}{c}e^{-|k|L} \\ 0 & (U-c)k\cosh kL & -(U-c)k\sinh kL & c|k|\,e^{-|k|L} \end{bmatrix}\begin{Bmatrix} \mathcal{C}_{I} \\ \mathcal{C}_{II}^{s} \\ \mathcal{C}_{II}^{c} \\ \mathcal{C}_{III} \end{Bmatrix} = 0$$

May divide disturbances into

(i) <u>symmetric</u>: $\mathcal{C}_{II}^{s} = 0$, $\mathcal{C}_{I} = \mathcal{C}_{III}$

These satisfy
$$\begin{bmatrix} \dfrac{(U-c)}{c}e^{-|k|L} & \cosh kL \\ -c|k|\,e^{-|k|L} & (U-c)k\sinh kL \end{bmatrix}\begin{Bmatrix} \mathcal{C}_{I} \\ \mathcal{C}_{II}^{c} \end{Bmatrix} = 0$$

$$\Rightarrow \underline{(U-c)^2 \sinh|k|L + c^2\cosh kL = 0}$$

Since both terms are positive for c real, for this equation to be satisfied, c has to be complex $\Rightarrow$ <u>unstable for all</u> k

(ii) <u>antisymmetric</u> : $C_{II}^{c} = 0$, $C_{I} = -C_{III}$

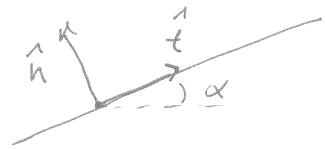
These satisfy
$$\begin{bmatrix} \frac{U-c}{c} e^{-|k|L} & \sinh kL \\ -c|k| e^{-|k|L} & (U-c) k \cosh kL \end{bmatrix} \begin{Bmatrix} C_{I} \\ C_{II}^{s} \end{Bmatrix} = 0$$

$$\Rightarrow \quad \underline{(U-c)^2 \cosh kL + c^2 \sinh |k| L = 0}$$

Again, for this equation to be satisfied (for k real), c has to be complex $\Rightarrow$ <u>instability for all k</u>

$$\alpha(t) = \alpha_0 \cos(\omega t)$$

$$x(t) = \frac{L_0}{2}\left[1 + \sin(\omega t)\right]$$

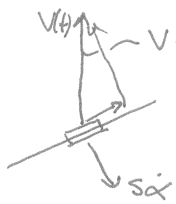


$$\dot{\alpha} = -\omega\alpha_0 \sin(\omega t)$$

$$\dot{x} = \frac{L_0\omega}{2}\cos(\omega t)$$

$$\hat{t} = \cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}$$

$$\hat{n} = -\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}$$



$$\underline{dF} = -C_\perp s\dot{\alpha}\,\hat{n}\,ds - C_\perp V\cos\alpha\,\hat{n}\,ds - C_\parallel V\sin\alpha\,\hat{t}\,ds$$

$$\underline{F} = -C_\perp \frac{L_0^2}{2}\dot{\alpha}\,\hat{n} - C_\perp V\cos\alpha\, L_0\,\hat{n} - C_\parallel V\sin\alpha\, L_0\,\hat{t}$$

$$= -\left(C_\perp \frac{L_0^2}{2}\dot{\alpha} + C_\perp V\cos\alpha\, L_0\right)\left(-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}\right)$$

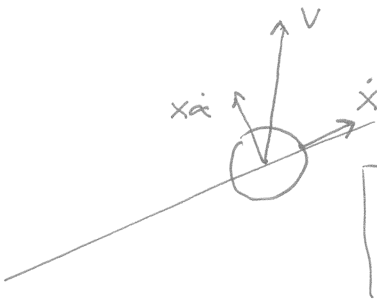
$$- C_\parallel V\sin\alpha\, L_0\left(\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}\right)$$

$F_x = 0$ by symmetry

$$F_y = 0 = -\left(C_\perp \frac{L_0^2}{2}\dot{\alpha} + C_\perp V\cos\alpha\, L_0\right)\cos\alpha - \cancelto{\text{small}}{C_\parallel V\sin\alpha\, L_0\sin\alpha}$$

$$\Rightarrow \quad V = \frac{-\cancel{C_\perp}\frac{L_0^{\cancel{2}}}{2}\dot{\alpha}}{\cancel{C_\perp}\cos^2\alpha\,\cancel{L_0}} = \omega\alpha_0\sin(\omega t)\frac{L_0}{2}$$

$$\boxed{\bar{V} = 0}$$



$$\boxed{\underline{V}_{sphere} = V(t)\,\hat{y} + \dot{x}\,\hat{t} + x\dot{\alpha}\,\hat{n}}$$

To find $V(t)$:

$$\underline{V}_{sphere} = V(t)\,\hat{y} + \dot{x}\,(\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y}) + x\dot{\alpha}(-\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y})$$

$$F_y = 0 = -\left(C_\perp \frac{L_0^2}{2}\dot{\alpha} + C_\perp V\cos\alpha\, L_0\right)\cos\alpha - C_\parallel V \sin^2\alpha\, L_0 - 6\pi\mu R\, \underline{V}_{sph}\cdot\hat{y}$$

$$= -C_\perp \frac{L_0^2}{2}\dot{\alpha} - C_\perp V L_0\left(1-\frac{\alpha^2}{2}\right)^2 - C_\parallel V\alpha^2 L_0$$

$$- 6\pi\mu R\,(V + \dot{x}\alpha + x\dot{\alpha})$$

$$V(t)\left[C_\perp L_0(1-\alpha^2) + C_\parallel \alpha^2 L_0 + 6\pi\mu R\right] = -C_\perp\frac{L_0^2}{2}\dot{\alpha} - 6\pi\mu R\,(\dot{x}\alpha + x\dot{\alpha})$$

$$V(t) = \frac{-\cancel{C_\perp}\frac{L_0^{\cancel{2}}}{\cancel{2}}\dot{\alpha} - \frac{6\pi\mu R}{CL_0}(\dot{x}\alpha + x\dot{\alpha})}{2\cancel{C_\perp}\cancel{L_0}(1-\alpha^2) + \cancel{C_\parallel}\alpha^2\cancel{L_0} + \frac{6\pi\mu R}{CL_0}} \qquad \frac{6\pi\mu R}{CL_0} = k$$

$$= \frac{-L_0\dot{\alpha} - k(\dot{x}\alpha + x\dot{\alpha})}{(2 - 2\alpha^2 + \alpha^2 + k)} = \frac{-L_0\dot{\alpha} - k(\dot{x}\alpha + x\dot{\alpha})}{(k+2)\left(1 - \frac{\alpha^2}{k+2}\right)}$$

$$(2 - 2\alpha^2 + \alpha^2 + k) = k + 2 - \alpha^2$$

$$V(t) = \frac{-1}{(k+2)}\left[L_0\dot{\alpha} + k(\dot{x}\alpha + x\dot{\alpha})\right]\left(1 + \frac{\alpha^2}{k+2}\right)$$

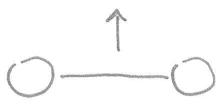
$$\overline{V(t)} = \frac{-1}{(k+2)^2}\left[\cancelto{0}{\overline{L_0\alpha^2\dot{\alpha}}} + k\,\overline{\alpha^2(\dot{x}\alpha + x\dot{\alpha})}\right]$$

$$= \frac{-k}{(k+2)^2}\left[\alpha_0^2\cos^2(\omega t)\left\{\frac{L_0\omega}{2}\cos(\omega t)\,\alpha_0\cos(\omega t)\right.\right.$$

$$\left. + \frac{L_0}{2}\left[1 + \sin(\omega t)\right](-\omega\alpha_0)\sin(\omega t)\right.$$

$$\overline{V(t)} = \frac{-k}{(k+2)^2}\alpha_0^3\frac{L_0\omega}{2}\underbrace{\left[\frac{3\pi}{4}-\frac{\pi}{4}\right]}_{\pi/2} = \boxed{-(L_0\omega)\,\alpha_0^3\,\frac{k\pi}{4(k+2)^2}}$$

Efficiency: $\eta = \dfrac{F\cdot V}{\Phi}$



$C_\perp$ or $C_\parallel$ depending on which way we drag.

$$\underline{F}\cdot\underline{V} = 2(6\pi\mu R V)V + (2L_0 C V)V$$

$$= \boxed{(12\pi\mu R + 2L_0 C)(L_0\omega)^2\alpha_0^6\left[\frac{k\pi}{4(k+2)^2}\right]^2}$$

$$\Phi = 2\,\Phi_{sphere} + 2\,\Phi_{bar}$$

$$\Phi_{sphere} = 6\pi\mu R\,V(t)\cdot V(t)$$

small

$$|V(t)_{sphere}|^2 = \left|(\cancel{V(t)} + \dot{x}\sin\alpha + x\dot\alpha\cos\alpha)\hat{y} + (\dot{x}\cos\alpha - x\dot\alpha\sin\alpha)\hat{x}\right|^2$$

$$\approx \left|(\dot{x}\alpha + \cancel{x\dot\alpha})\hat{y} + (\dot{x} - \cancel{x\dot\alpha}\alpha)\hat{x}\right|^2 \approx \dot{x}^2$$

$$\approx \left(\frac{L_0\omega}{2}\right)^2\cos^2(\omega t)$$

$$\Phi_{sphere}(t) = 6\pi\mu R\left(\frac{L_0\omega}{2}\right)^2\cos^2(\omega t)$$

$$\boxed{\overline{\Phi}_{sphere} = 6\pi^2\mu R\left(\frac{L_0\omega}{2}\right)^2}$$

$$\Phi_{bar}(t) = F(t)\cdot V(t)$$

$$d\underline{F} = +C_\perp\,ds\,\hat{n}\left[s\dot\alpha + V\cos\alpha\right] + C_\parallel\,ds\,\hat{t}\,V\sin\alpha$$

$$\underline{V} = s\dot\alpha\,\hat{n} + V\cos\alpha\,\hat{n} + V\sin\alpha\,\hat{t}$$

$$d\underline{F}\cdot\underline{V} = C_\perp(s\dot\alpha + \overset{O(\alpha^3)}{V}\cos\alpha)^2\,ds + C_\parallel(V\sin\alpha)^2\,ds \approx C_\perp\dot\alpha^2 s^2\,ds$$

$$\Phi_{bar}(t) \approx C_\perp \dot{\alpha}^2 \frac{L_o^3}{3} = \frac{C_\perp L_o^3}{3} \omega^2 \alpha_o^2 \underbrace{\sin^2(\omega t)}_{\pi}$$

$$\eta = \frac{(12\pi\mu R + 2L_o C)\cancel{(L_o\omega)^2}\,\alpha_o^6 \left[\frac{k\pi}{4(k+2)^2}\right]^2}{12\pi^2\mu R\left(\frac{\cancel{L_o\omega}}{2}\right)^2 + \underbrace{\frac{2}{3}\pi\, C_\perp L_o^3 \cancel{\omega^2}\alpha_o^2}_{\text{small}}}$$

$$= \frac{(12\pi\mu R + 2L_o C)\,\alpha_o^6 \left[\frac{k\pi}{4(k+2)^2}\right]^2}{3\pi^2\mu R}$$

$$= \left(4 + \frac{2\cdot 2L_o C}{2\cdot 3\pi^2\mu R}\right)\alpha_o^6\left[\frac{k\pi}{4(k+2)^2}\right]^2$$

$$= \left(4 + \frac{4}{k\pi}\right)\alpha_o^6\left[\frac{k\pi}{4(k+2)^2}\right]^2 = \boxed{\frac{\alpha_o^6}{4}\left(1 + \frac{1}{k\pi}\right)\left[\frac{k\pi}{(k+2)^2}\right]^2}$$