

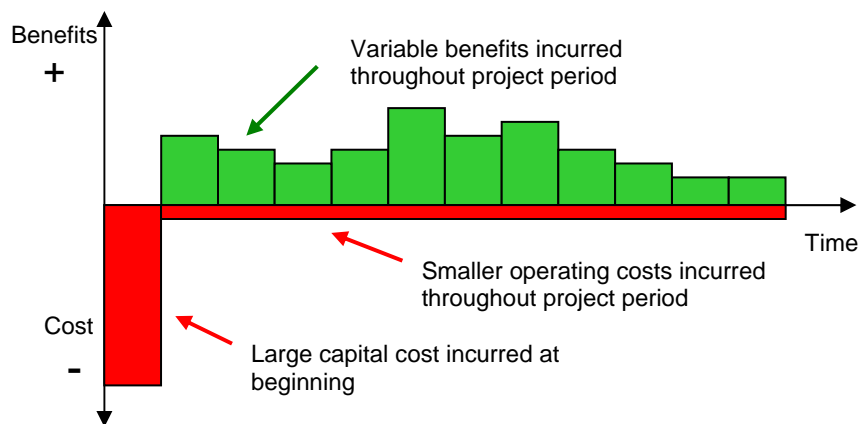
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Lecture 14 Optimization over Time, Discounting  
Oct. 26, 2006

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**Temporal allocation of benefits and costs**

Generally benefits and costs of water projects are not incurred at the same time.  
Possible benefit/cost distribution for a project with a large initial investment (e.g. capital cost):



Method used to compare benefits and costs depends on application.

**Present value:** Convert all benefits and costs to an equivalent present value at beginning of project.

**Amortization:** Convert all benefits and costs to an equal periodic (e.g. annual) payments

**Present value**

Value of initial investment  $B_0$  at interest rate  $r$  after  $t$  years is:

$$B_t = (1 + r)^t B_0$$

Conversely, a benefit  $B_t$  received (or a cost  $C_t$  incurred) in year  $t$  is equivalent to an initial investment  $B_0$  (or cost  $C_0$ ):

$$B_0 = (1 + r)^{-t} B_t \quad C_0 = (1 + r)^{-t} C_t$$

Present value of a stream of benefits (or costs) accrued over  $N$  years:

$$B_{PN} = \sum_{t=1}^N (1+r)^{-t} B_t \quad C_{PN} = \sum_{t=1}^N (1+r)^{-t} C_t$$

**Present Value Example:**

We wish to size a facility (e.g. a reservoir) that gives benefits  $B_1(x, I_1), \dots, B_N(x, I_{30})$  that are received during each of the 30 years in the project life.

$N = 30 =$  project life

$x =$  facility size

$I_t =$  specified time-dependent input (e.g. reservoir inflow).

$r =$  interest rate

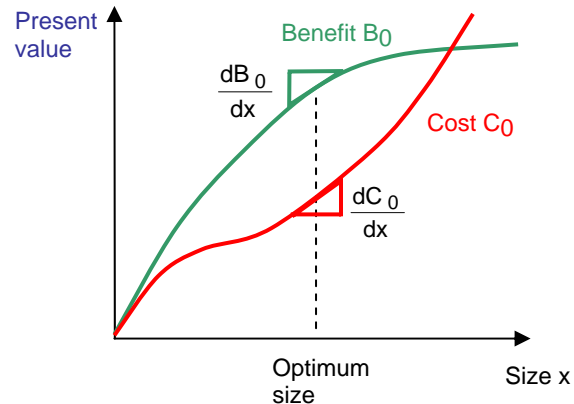
The input could be a historical time series that is assumed to represent likely fluctuations in future inputs. The benefit is credited at the end of each year.

The reservoir capital cost  $C_{cap}$  is incurred at the initial time  $t = 0$ . A constant operating cost  $C_{op}$  is incurred at the end of each of the 30 years ( $t = 1, \dots, 30$ ).

The optimal facility size maximizes the **present value** of the net benefit:

$$F(x) = B_0(x) - C_0(x) = \sum_{t=1}^{30} (1+r)^{-t} B_t(x, I_t) - C_{cap}(x) - C_{op}(x) \sum_{t=1}^{30} (1+r)^{-t}$$

The optimal solution depends on the shape of the benefit and cost functions. Often,  $B_0(x)$  is concave in  $x$  and  $C_0(x)$  is convex for sufficiently large  $x$ .



In this case if no constraints are active the optimum occurs where:

$$\frac{dF(x)}{dx} = \frac{dB_0(x)}{dx} - \frac{dC_0(x)}{dx} = 0 \quad \text{or} \quad \frac{dB_0(x)}{dx} = \frac{dC_0(x)}{dx}$$

This is the value of  $x$  where the marginal increase in present value cost just starts to exceed the marginal increase in present value benefit.

**Amortization:**

Amortization spreads an initial benefit  $B_0$  (or expense  $C_0$ ) equally over  $N$  periods (so  $B_t = B$  and  $C_t = C$ ):

$$B_0 = B \sum_{t=1}^N (1+r)^{-t} = B \underbrace{\left[ \frac{(1+r)^N - 1}{r(1+r)^N} \right]}_{\text{Capital recovery factor}} \quad C_0 = C \left[ \frac{(1+r)^N - 1}{r(1+r)^N} \right]$$

The equal installments are:

$$B = B_0 \left[ \frac{r(1+r)^N}{(1+r)^N - 1} \right] \quad C = C_0 \left[ \frac{r(1+r)^N}{(1+r)^N - 1} \right]$$

### Amortization Example:

We wish to determine optimal size  $x$  of a facility (e.g. a reservoir) based on its performance over a **typical year** that is divided into 2 seasons. The inputs  $I_1$  and  $I_2$  (e.g. reservoir inflow) for the two seasons are derived from long-term averages (i.e. climatology). The corresponding benefits are  $B_1(x, I_1)$  and  $B_2(x, I_2)$ .

The reservoir capital cost  $C_{cap}$  is incurred at the initial time  $t = 0$  and is to be amortized over a 30 year project life at an interest rate  $r$ . A constant operating cost  $C_{op}$  is incurred over the year.

The optimal facility size maximizes the **amortized net benefit** for the typical year:

$$F(x) = \sum_{t=1}^2 B_t(x, I_t) - C_{op}(x) - C_{cap}(x) \left[ \frac{r(1+r)^{30}}{(1+r)^{30} - 1} \right]$$

The economic assumptions used here are the same as those used in the present value example but the inputs are treated in a way that makes the amortization approach more convenient.

An optimization done for a **typical year** usually requires fewer decision variables and constraints than one done for the **entire project life**. However, it cannot account for the effects of **unusually wet or dry years**.

### Implications

The classical exponential discounting approach outlined above tends to **minimize the advantages of future benefits** (e.g. abundant clean water for the next generation) and to **minimize the disadvantages of future costs** (e.g. increased sea levels due to climate change).

For this reason classical discounting has been criticized by some environmental economists. Alternatives have been proposed in the economics literature but have not yet been widely accepted. Classical exponential discounting should be used with caution.

