MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Lecture 21 Capacity Expansion
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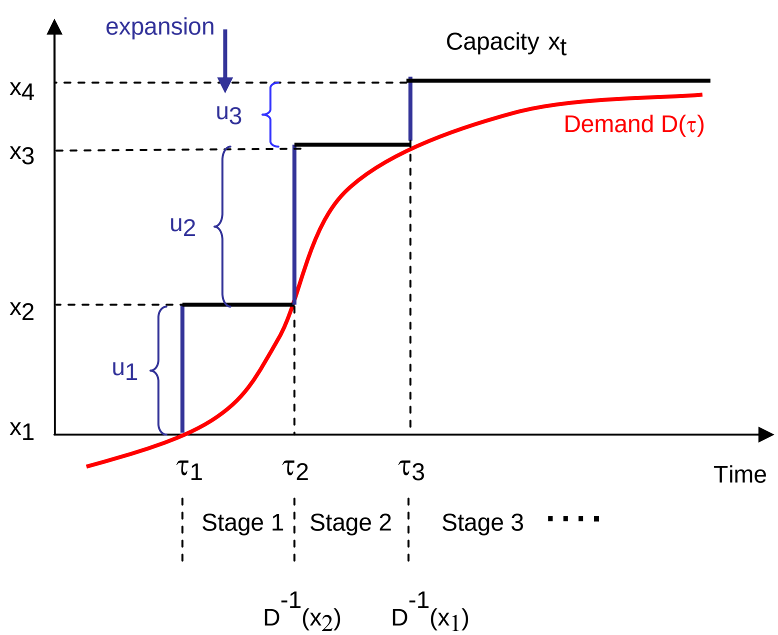
Problem Formulation

Capacity expansion problems are concerned with the **timing** of facility expansions to meet increasing demand. Since demand is difficult to forecast and expansion plans may need to change over time, dynamic programming offers a convenient way to solve these problems.

Basic problem features:

- Determine an expansion strategy for a facility (e.g. a water treatment plant) which starts with an **initial capacity** x_1 at time τ_1.
- **Capacity expansions** $u_1, u_2, \ldots, u_t, \ldots, u_T$ occur at discrete times $\tau_1, \tau_2, \ldots, \tau_t, \ldots, \tau_T$ (to be determined). These increase capacity to $x_2, x_3, \ldots, x_{t+1}, \ldots, x_{T+1}$, respectively.
- Expansion t defines the start of a new **stage** that extends from time τ_t to time τ_{t+1}.
- Capacities are **constant** over each stage
- The **demand** $D(\tau)$ at any time τ can never exceed the capacity at that time.
- The objective is to **minimize cost** while satisfying demand.
- Cost usually increases less at higher capacities (**economy of scale**). Basic problem **tradeoff** is between cost of borrowing capital for expansion vs. economy of scale obtained with larger expansions



Following the conventions of dynamic programming, the optimization problem is:

$$\underset{u_t,\ldots u_T}{Min}\ F_t(x_t, u_t, \ldots, u_T) \qquad t = 1,\ldots,T$$

where the cost-to-go just before the stage *t* expansion is the sum of the remaining expansion costs:

$$: \qquad F_t(x_t, u_t, \ldots, u_T) = \sum_{i=t}^{T} f_i(x_i, u_i)$$

Minimization is subject to the state equation:

$$x_{i+1} = x_i + u_i \quad ; \quad i = t, \ldots, T$$

and the following constraints on the decision variables:

$$x_i \geq D(\tau_i) \quad ; \quad i = t, \ldots, T$$

$$0 \leq x_i \leq x_{max} \quad ; \quad i = t, \ldots, T$$

$$0 \leq u_i \leq u_{max} \quad ; \quad i = t, \ldots, T$$

Decision rule $u_t(x_t)$ at each *t* is obtained by finding sequence of exansions $u_t, \ldots, u_T$ that minimizes $F_t(x_t, u_t, \ldots, u_T)$ for a given capacity x_t at τ_t and a given demand function $D(\tau)$.

Objective Function and Dynamic Programming Recursion

The cost of expansions at different times is expressed as a present value:

$$f_t(x_t, u_t) = (1+r)^{-(\tau_i - \tau_1)} c_t(u_t) = (1+r)^{-[D^{-1}(x_t) - D^{-1}(x_1)]} c_t(u_t)$$

where:

$\tau_t = D^{-1}(x_t)$ is obtained by inverting the demand function

. c_t = undiscounted cost of expansion *t*

This implies:

$$F_t(x_t, u_t, \ldots, u_T) = \sum_{i=t}^{T} (1+r)^{-[D^{-1}(x_i) - D^{-1}(x_1)]} c_i(u_i) \qquad , \qquad V_{T+1}(x_{T+1}) = 0$$

The dynamic programming backward recursion is then:

$$V_t(x_t) = \underset{u_t}{Min}\left[f_t(u_t, x_t) + V_{t+1}(x_t + u_t) \right] \quad ; \quad V_{T+1}(x_{T+1}) = 0$$

The expansion and capacity variables u_t and x_t are discretized and the optimum decision rule $u_t(x_t)$ is identified by searching through all feasible u_t at Stage *t*, for specified x_t and $V_{t+1}(x_t + u_t)$, moving from Stage $T + 1$ backwards to Stage 1

Example: Water Supply System Expansion

Consider capacity expansion of a water supply system that has initial capacity (0.7 mgd) that just meets initial demand in Year 2000.

Expansion cost (exhibits economy of scale):

Capacity (MGD)	Cost(10^6 \$)
0.5	20
1.0	30
1.5	36
2.0	40
2.5	42

Projected Demand:

Year	Demand (MGD)
2000	0.7
2005	0.9
2010	1.1
2015	1.7
2020	2.2
2025	2.5
2030	2.7
2035	2.8

Costs/demands at intermediate capacities/years obtained with linearly interpolation.

- Allow up to two more expansions after the initial expansion in 2000 (so problem has 3 stages).
- Annual interest rate = 0.08
- Discretize state and control variable ranges into intervals of 0.1 mgd.
- Assume projected demand is always satisfied.

Solve with MATLAB program Lecture06_21.m.

Optimum solution: $u_1 = 0.2$, $u_2 = 0.2$, $u_3 = 1.7$