MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Lecture 3, General Optimization Concepts 1, Sept. 14, 2006

Problem Formulation:

$$\underset{x_1,x_2,\dots,x_n}{\text{Maximize}} \quad F(x_1,x_2,\dots,x_n)$$

such that :

$$g_i(x_1,x_2,\dots,x_n)=0 \qquad i=1,\dots,r \qquad \text{Strict equality constraints}$$
$$g_i(x_1,x_2,\dots,x_n)\le 0 \qquad i=r+1,\dots,m \qquad \text{Inequality constraints}$$

Basic components:

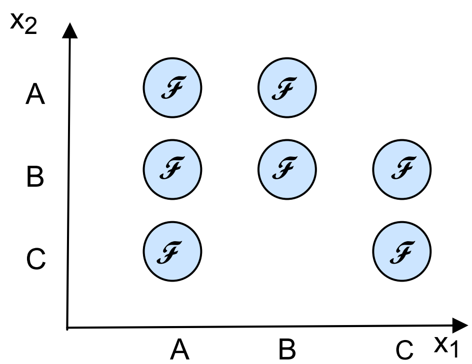
- n **decision variables** $x \rightarrow x_i = [x_1, x_2, \dots, x_n]$, collectively define a decision strategy.
- **Scalar objective function** $F(x) \rightarrow F(x_1, x_2, \dots, x_n)$ measures performance of decision strategy
- r **equality constraints** $g_i(x)$, $i=1,\dots,r$
- n-r **inequality constraints** $g_i(x)$, $i= r+1,\dots,m$

Note:

- Minimization of $F(x)$ is maximization of $-F(x)$
- $g(x) > 0$ is same as $-g(x) < 0$

Feasible region $\mathscr{F}$: Set of x that satisfies constraints (depends only on $g_i(x)$).

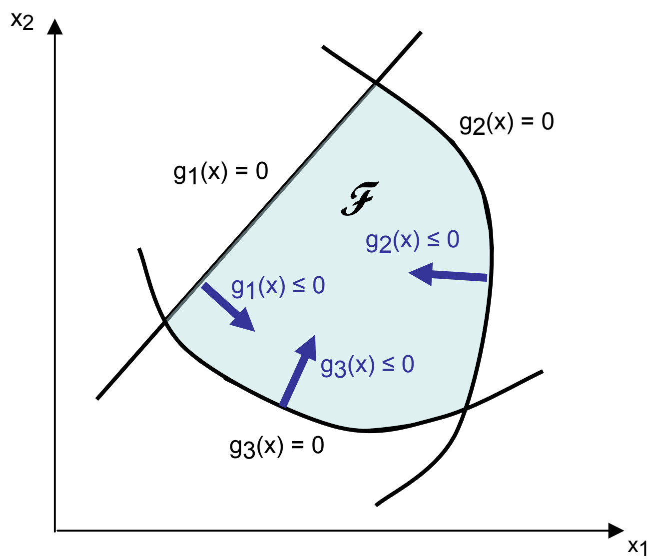
Discrete optimization: $\mathscr{F}$ consists of a finite number of feasible solutions



$\mathscr{F}$ consists of 7 discrete x_1 and x_2 values, indicated by circles

$$g_1(x) = -1 \text{ for } (x_1, x_2) = \{AA, AB, AC, BA, BB, CB, CC\}$$
$$g_1(x) = +1 \text{ otherwise}$$

Continuous (non-discrete) optimization: $\mathcal{F}$ consists of an infinite number of feasible solutions



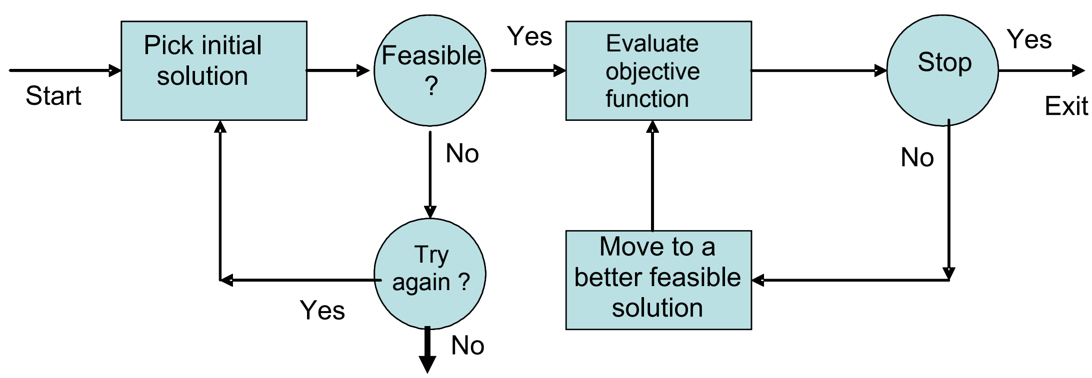
$\mathcal{F}$ is bounded by curves corresponding to $g_i(x) = 0$.
Interior of $\mathcal{F}$ is set of points that satisfy $g_i(x) < 0$.

Solving Optimization Problems

Objective in optimization is to find the **best decision strategy** among all feasible possibilities:

→ We seek a **global optimum**

Most common way to find optimum for large problems is to use an **iterative search**:



An iterative search algorithm needs:

- A method for selecting an **initial feasible solution -** Can be formulated as a secondary optimization problem
- A **stopping criterion** that detects following:
  1. No feasible solution – no way to satisfy all **constraints**
  2. Optimal solution found – satisfies **optimality conditions**
  3. Objective function unbounded over feasible region - Objective can be **infinite** within feasible region.

- A **solution improvement mechanism** – challenging for nonlinear problems, often based on optimality conditions, sometimes *ad hoc*.

Types of search procedures:

- **Exhaustive Searches - ** For **discrete problems**:
  Move methodically through all (or sometimes a subset) of the feasible solutions to determine which has best objective value.
- **Selective Searches** – For **continuous problems:**
  Use information from current and past candidate solutions (e.g. objective value or objective gradient) to determine next feasible solution.

For now, focus on continuous problems and selective searches.

Global vs. Local Maxima for Continuous Problems

In practice, it is much easier to find **local optima:**

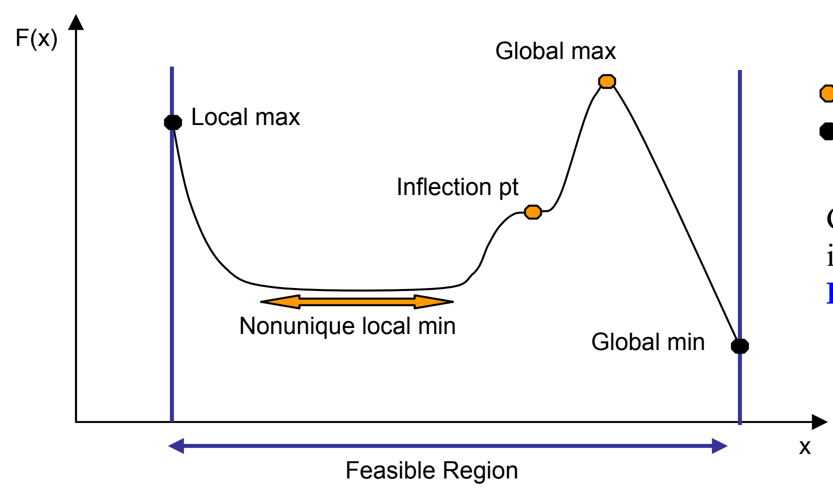
x^* is a **local maximum** if $F(x^*) \geq F(x)$ for all **feasible** x **near** x^*

x^* is a **local minimum** if $F(x^*) \leq F(x)$ for all **feasible** x **near** x^*

Two **key questions**:

1. When is a **local** optimum also **global** optimum?
2. How do we know when a particular candidate solution x^* is a **local optimum**?

What can we say about **global optimality** based on **local properties** (near x*) ?



$dF(x)/dx=0$

$dF(x)/dx \neq 0$

Global optima **cannot** be identified from local properties !
Except in certain special cases