MASSACHUSETTS INSTITUTE OF TECHNOLOGY
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1.731 Water Resource Systems

Lecture 7, Linear Programming Overview, Sept. 28, 2006

Linear Programming Problems (LPP)
Objectives and constraints are all **linear functions** of decision variables:

$$\underset{x_1, x_2, ..., x_n}{\text{Maximize}} \quad F(x_1, x_2, ..., x_n) = c_j x_j$$

such that :

$g_{Ti}(x) = A_{ij} x_j \leq b_i \qquad i = 1,..., m_T$ — **Technological** constraints (may be equalities or inequalities)

$g_{Ni}(x) = -x_{i-m} \leq 0 \qquad i = m_T + 1,..., m_T + m_N$ — **Non-negativity** constraints

Total constraints = $m_T + m_N = m$

Optimal Solutions of Linear Programming Problems:
For LPP:
Linear objective and constraint functions are both **convex** and **concave** so:

- Feasible region $\mathcal{F}$ for LP is **convex** (i.e. constructed from convex functions gi(x)≤0)
- Objective function for LP is **concave**

Therefore:
A candidate LPP solution x^* that is a **local maximum** is also a **global maximum**.

To check whether x^* is a local/global maximum use **necessary conditions**:

Focus on constraints that are **active** at x^*:

$$G^*_{Aij} x^*_j - b^*_{Ai} = 0 \qquad\qquad i \in \mathcal{C}(x^*)$$

Row i of $G^*_A = A_{ij}$ if i is a technological constraint

Row i of $b^*_{Ai} = b_i$ if i is a technological constraint

Row i of $G^*_A = -\delta_{ij}$ if i is a non-negativity constraint

Row i of $b^*_{Ai} = 0$ if i is a non-negativity constraint

1. Feasibility

x^* is chosen to be feasible

2. Stationarity

If x^* is a local maximum then:

$$\frac{\partial F(x^*)}{\partial x_j} = \frac{\partial [c_j x_j]}{\partial x_j} = c_j = \lambda_i \frac{\partial [G^*_{Aij} x_j + b^*_{Ai}]}{\partial x_j} = \lambda_i G^*_{Aij}$$

For LPP stationarity condition reduces to a set of **linear equations** in unknown λ_i's

$$\lambda_i G^*_{Aij} = c_j$$

The stationarity condition is satisfied if this set of linear equations is **consistent** so:

$$\rho^* = Rank[G^*_{Aij}] = Rank[G^*_{Aij} | c_j]$$

There are four ways this can occur:

1). **Corner solution**: x^* lies at intersection of n linearly independent constraints.

$$\rho^* = m^*_A = n\,.$$

2). **Trivial interior solution**: occurs only if $c_j = 0$.

$$\rho^* = m^*_A = 0$$

3). **Non-corner boundary solution**: x^* lies along a boundary but not at a corner.

$$\rho^* = m^*_A < n$$

4). **Degenerate solution**: Constraints are linearly dependent (i.e. number of constraints exceeds rank of G^*_{Aij}).

$$\rho^* < m^*_A$$

3. Inequality Lagrange multiplier

If x^* is a local maximum then:

$$\lambda_i \geq 0 \qquad\qquad i \in \mathcal{C}(x^*)$$

In case 1) above there will be only one solution that satisfies this condition.

4. Curvature

In LLP curvature condition **applies for any x^*** since Lagrangian Hessian is **always zero**.

$$W_{kl} = \frac{\partial^2 L(x^*, \lambda)}{\partial x_j \partial x_k} Z_{ik} Z_{lj} = \left. \frac{\partial^2 [c_j x_j - \lambda_i (G^*_{Aij} x_j - b^*_{Ai})]}{\partial x_j \partial x_k} \right|_{x=x^*} Z_{ik} Z_{lj} = 0$$

Crop Allocation Example

Problem is to maximize revenue from two crops, given constraints on available land and water and on minimum total crop grown.

Decision variables:

x_1 = mass of crop 1 grown (tonnes = 10^3 kg)

x_2 = mass of crop 2 grown (tonnes = 10^3 kg)

$$\underset{x_1, x_2}{\text{Maximize}} \; 6x_1 + 11x_2$$

such that :

$2x_1 + x_2 \leq 104$	Water constraint (10^3 m^3/season)
$x_1 + 2x_2 \leq 76$	Land constraint (ha)
$-x_1 - x_2 \leq -25$	Minimum production constraint (tonnes)
$-x_1 \leq 0$	x_1 non-negativity constraint
$-x_2 \leq 0$	x_2 non-negativity constraint

Objective, right-hand side, and technological coefficients:

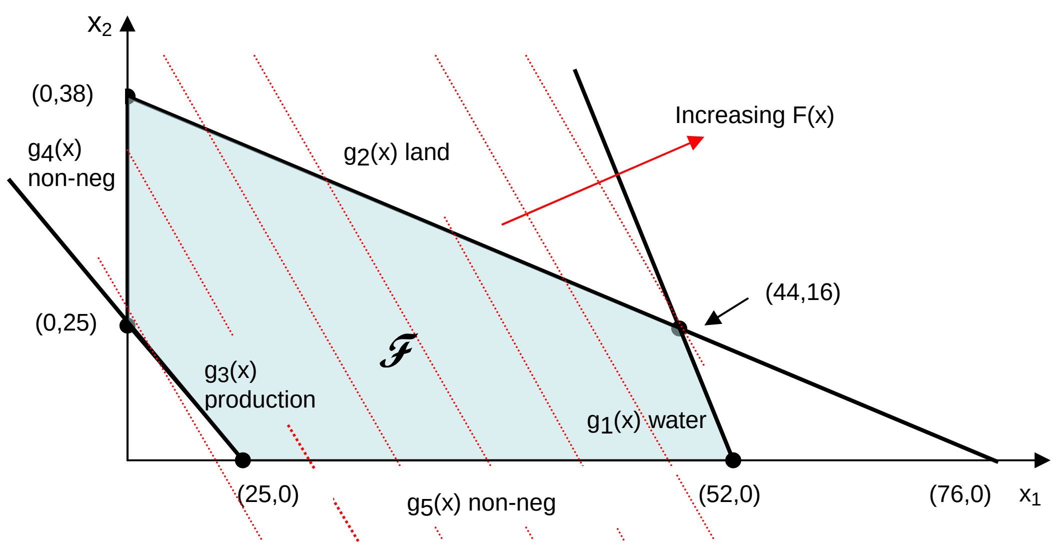
c_j -- **Crop values** (\$/tonne)

b_1 -- **Water available** (m3/season)

b_2 -- **Land available** (ha)

A_{1j} -- **Water requirements** (10^3 m^3/(season tonne)) = (unit water requirement in 10^{-1} m/season)/(yield in tonnes/ha)

A_{2j} -- **Land requirement** (ha/tonne) = (yield in tonnes/ha)$^{-1}$



Pairs of constraints active at the 5 corners of the feasible solution are all linearly independent (i.e. corresponding G^*_{Aij} have rank $\rho^* = m^*_A = n = 2$).

So $\lambda_i G^*_{Aij} = c_j$ is consistent and **stationarity condition is satisfied** at each of these corner points.

There are no interior, non-corner boundary, or degenerate solutions for this example.
So we need only consider the Lagrange multipliers at the 5 corner solutions:

Candidate Solution	**Active Constraints**	**Lagrange Multipliers**	
(25, 0)	3, 5	$\lambda_3 = -6$	$\lambda_5 = -5$
(52, 0)	5, 1	$\lambda_1 = +3$	$\lambda_5 = -8$
(44, 16)	1, 2	$\lambda_1 = +1/3$	$\lambda_2 = +16/3$
(0, 38)	2, 4	$\lambda_2 = +11/2$	$\lambda_4 = -1/2$
(0, 25)	4, 3	$\lambda_4 = -11$	$\lambda_3 = +5$

$x^* = (44, 16)$ is the **local/global maximum** since it is the only corner solution with positive Lagrange multipliers for **all** active constraints.

In this problem the optimum crop allocation mixes Crop 1 and Crop 2 in a way that uses all available land and water while giving maximum revenue.

It is possible to generate a **non-corner (non-unique) boundary solution** to this problem by changing the objective function to $F(x) = 6x_1 + 12x_2$. Then $\rho^* = m^*_A < n$ ($\rho^* = 1, m^*_A = 1, n = 2$). The objective function contours are parallel to $g_1(x)$ and any feasible solution along $g_1(x)$ is local/global maximum.

It is possible to generate a **degenerate solution** to this problem by changing the water constraint to $g_1(x) = 2x_1 + x_2 \leq 152$. Then $\rho^* < m^*_A$ ($\rho^* = 2, n = 2, m^*_A = 3$) at the new corner (76, 0).
There is not a unique set of λ_i's satisfying stationarity condition at this corner.
So the inequality Lagrange multiplier condition cannot be checked.