MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Civil and Environmental Engineering

1.731 Water Resource Systems

Problem Set 1 – Linear Algebra Review
Solutions

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1. Put the following matrix into echelon form by using elementary row operations (indicate the operations explicitly). What are the rank and determinant of this matrix?

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix}$$

Reduced row echelon form:

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 4 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3, determinant = 0

2. Determine the existence and uniqueness of the solutions of the following systems of equations by evaluating the ranks of the appropriate matrices. Also, give the solution (if there is one).

a)
$$\begin{aligned} 2x + y &= 3 \\ x + 3y &= 1 \\ x - 2y &= 2 \end{aligned}$$

b)
$$\begin{aligned} x + y + 2z &= 1 \\ 2x - y + z &= 2 \end{aligned}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 1 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 3 & 1 \\ 1 & -2 & 2 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1.6 \\ 0 & 1 & -0.2 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1.6 \\ -0.2 \end{bmatrix}$$

n = 2, Rank(A) = Rank($A|b$) = 2, system is consistent with unique solution

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 2 & -1 & 1 & 2 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ a-1 \\ 1-a \end{bmatrix}$$

n = 3, Rank(A) = Rank($A|b$) = 2, system is consistent with non-unique solution

3. Find all nontrivial solutions to the following homogeneous equations. Why are they unique or non-unique?

a)

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 2x_1 + 5x_2 + 6x_3 &= 0 \end{aligned}$$

b)

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 0 \\ 3x_1 + 2x_2 + x_3 &= 0 \\ x_1 - 4x_2 + 5x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 5 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3a \\ 0 \\ a \end{bmatrix}$$

n = 3, Rank(A) = 2 < n, solution is nontrivial, nonunique

$$\begin{bmatrix} 2 & -1 & 3 \\ 3 & 2 & 1 \\ 1 & -4 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -a \\ a \\ a \end{bmatrix}$$

n = 3, Rank(A) = 2 < n, solution is nontrivial, nonunique

4. Determine whether the following vectors are linearly dependent or linearly independent:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1/2 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1/2 \\ -1/2 \\ 1/4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1/2 \\ 0 & 1 & -1/2 \\ 1/2 & 0 & 1/4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

n = 4, Rank(A) = 3 < n, system has a nontrivial, nonunique solution
vectors are linearly dependent

5. Find the eigenvalues and eigenvectors of the following matrix:

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalues: 1, 3, 2

Eigenvectors: $\begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ a \\ 0 \end{bmatrix} \quad \begin{bmatrix} -a \\ -a \\ a \end{bmatrix}$

6. Write the following quadratic form as a product of the form $x_{i1} A_{ij} x_j$, where A_{ij} is a symmetric matrix. Compute the eigenvalues of A_{ij} and determine the definiteness of the quadratic form:

$$f(x_1, x_2, x_3) = x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2 x_3$$

Identify a symmetric *A* that is compatible with quadratic form:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix} \rightarrow$$ Eigenvalues are 1, 2 ,4, all > 0, so *A* is positive definite